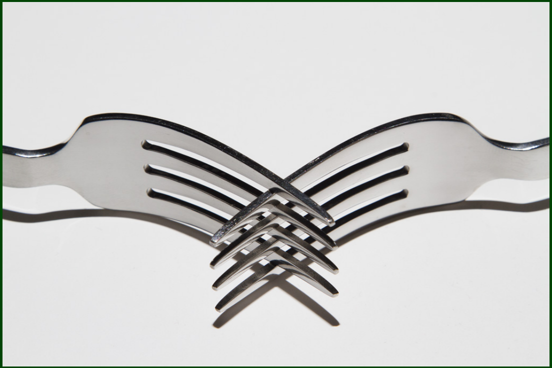
# CSE 332

## Data Abstractions

# More Parallel Primitives and Parallel Sorting



# Outline

### Reductions

**INPUT:** An array
**OUTPUT:** A combination of the array by an associative operation
The general name for this type of problem is a **reduction**. Examples include: `max`, `min`, `has-a`, `first`, `count`, `sorted`

### Maps

**INPUT:** An array
**OUTPUT:** Apply a function to every element of that array
The general name for this type of problem is a **map**. You can do this with any function, because the array elements are independent.

Today, we'll add in two more:

- Scan
- Pack (or filter)

As we'll see, both of these are quite a bit less intuitive **in parallel** than map and reduce.

## Scan

Suppose we have an associative operation $\oplus$ and an array `a`:

`a`:

| $a_0$ | $a_1$ | $a_2$ | $a_3$ |
|---|---|---|---|
| `a[0]` | `a[1]` | `a[2]` | `a[3]` |

Then, `scan(a)` returns an array of "partial sums" (using $\oplus$):

`scan(a)`:

| $a_0$ | $a_0 \oplus a_1$ | $a_0 \oplus a_1 \oplus a_2$ | $a_0 \oplus a_1 \oplus a_2 \oplus a_3$ |
|---|---|---|---|
| `b[0]` | `b[1]` | `b[2]` | `b[3]` |

It's hard to see at first, but this is actually a really powerful tool. It gives us a "partial trace" of the operation as we apply it to the array (for free).

## No Seriously

splitting, load balancing, quicksort, line drawing, radix sort, designing binary adders, polynomial interpolation, decoding gray codes

# Sequential Scan (with $\oplus = +$)



For the sake of being clear, we'll discuss scan with $\oplus = +$.
That is, "prefix sums" of an array":

## Example (Prefix Sum)

| | a[0] | a[1] | a[2] | a[3] | a[4] |
|---|---|---|---|---|---|
| a: | 5 | 1 | 3 | 4 | 2 |

| | b[0] | b[1] | b[2] | b[3] | b[4] |
|---|---|---|---|---|---|
| scan(a): | 5 | 6 | 9 | 13 | 15 |

## Sequential Code

```java
int[] prefixSum(int[] input) {
   int[] output = new int[input.length];
   int sum = 0;
   for (int i = 0; i < input.length; i++) {
      sum += input[i];
      output[i] = sum;
   }
   return output;
}
```

If you have a really good memory, you'll remember that on the **very first day of lecture**, we discussed a very similar problem.

# Sequential Prefix-Sum



### Sequential Code

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int[] prefixSum(int[] input) {
   int[] output = new int[input.length];
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   for (int i = 0; i < input.length; i++) {
      sum += input[i];
      output[i] = sum;
   }
   return output;
}
```

### Bad News

This **algorithm** does not parallelize well. Step $i$ needs the outputs from all the previous steps. This might as well be an algorithm on a linked list.

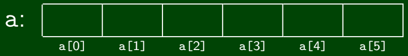
So, what do we do?

### Come Up With A Better Algorithm!

The solution here will be to add a “pre-processing step”. This is essentially what we did in the first lecture.

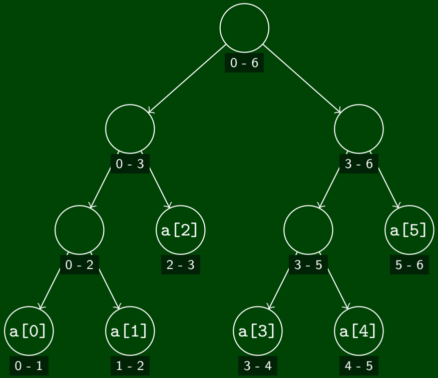
# Better Prefix-Sum

We begin with an array as usual:

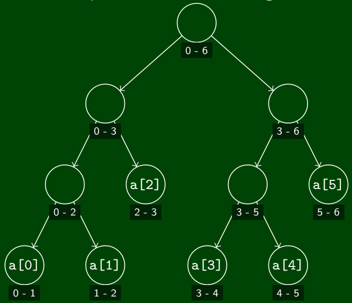



Then, transform it into a **balanced tree**, because $\lg n$ height will allow us to get a span of $\lg n$, eventually:

```
PSTNode {
   int lo, hi;
   int sum;
   PSTNode left, right;
}
```



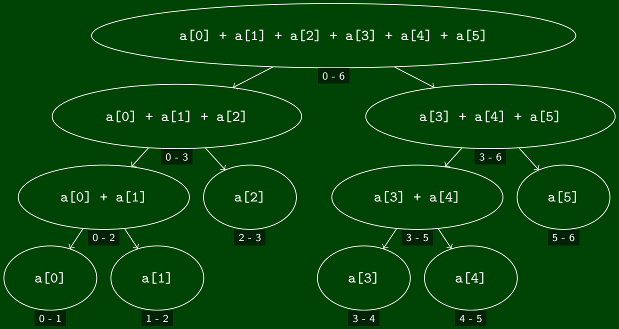
Creating the tree is a standard divide-and-conquer recursive algorithm:




```
PSTNode {
   int lo, hi;
   int sum;
   PSTNode left, right;
}
```

```
PSTNode processInput(int[] input, int lo, int hi) {
   if (hi − lo == 1) {
      return new PSTNode(lo, hi, input[lo]);
   }
   else {
      mid = lo + (hi − lo)/2;
      PSTNode left = processInput(lo, mid);
      PSTNode right = processInput(mid, hi);
      return new PSTNode(lo, hi, left.sum + right.sum, left, right);
   }
}
```

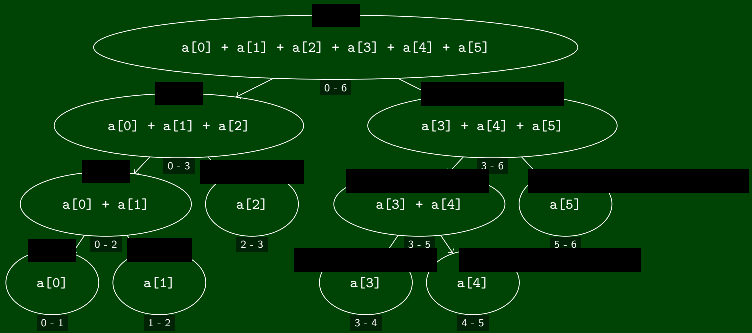
Now, we have the entire tree filled out:




To fill in all the prefix sums, we recursively fill them in down the tree. Since the non-leaf nodes don't have access to the elements of the array, we fill in a **pre-scan** (everything up to, but not including the range).

# Better Prefix-Sum: Constructing the Output

To fill in all the **pre-scans**, we recursively fill them in down the tree:




```
void makeOutput(int[] output, PSTNode current, int prescan) {
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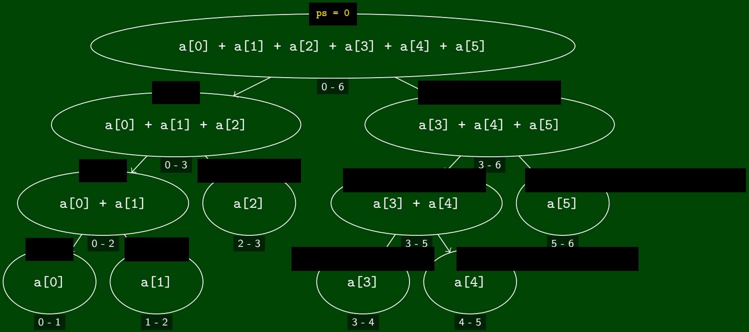


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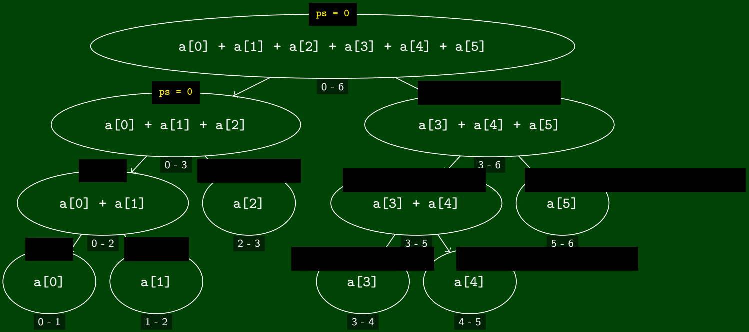


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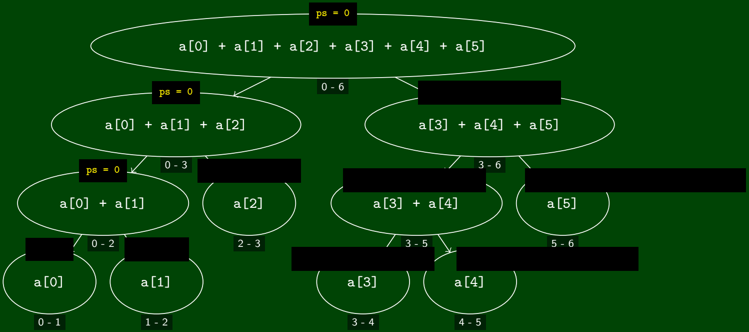


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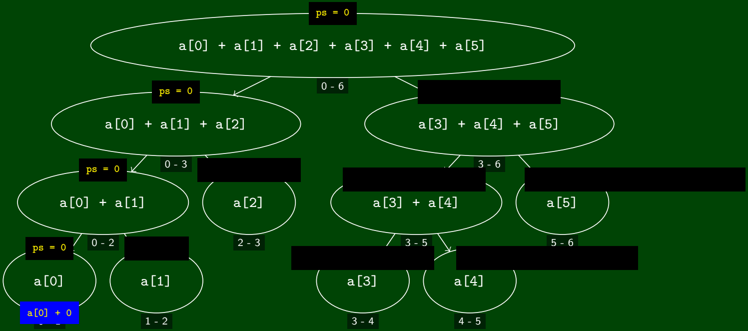


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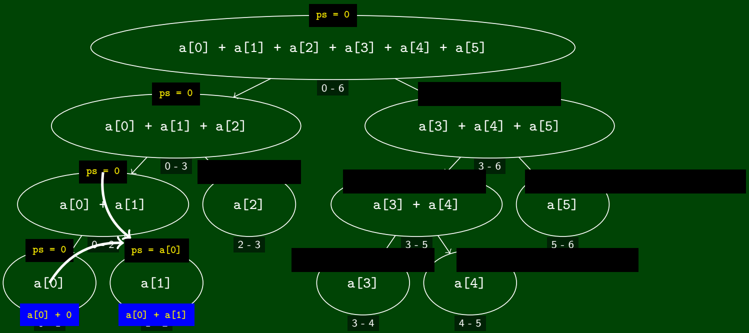


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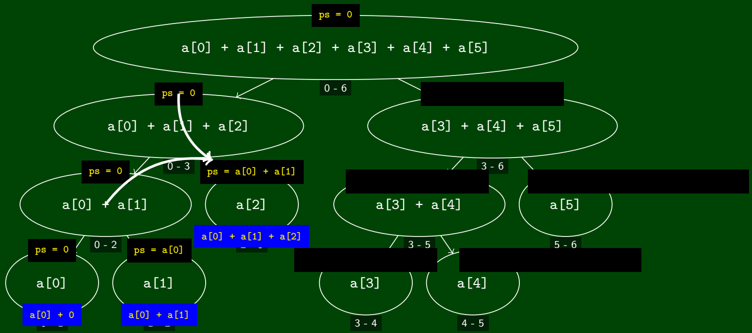


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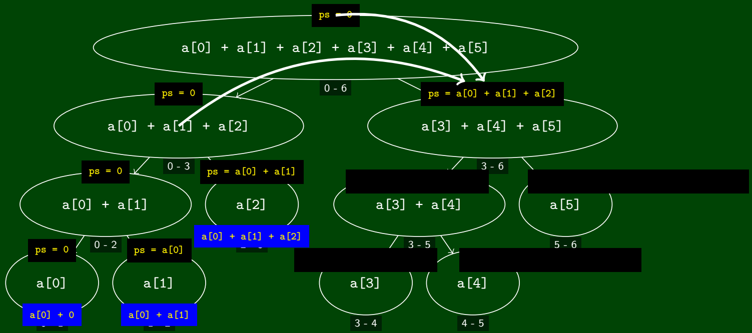


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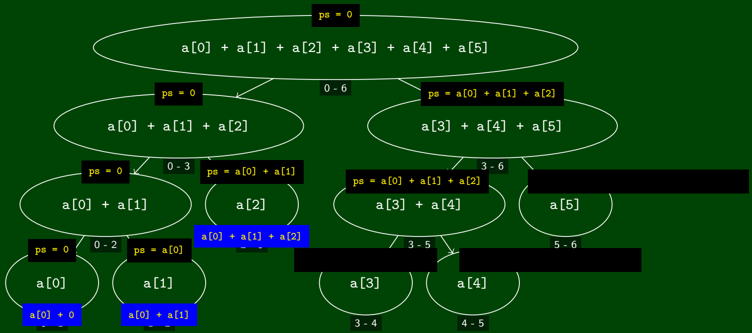


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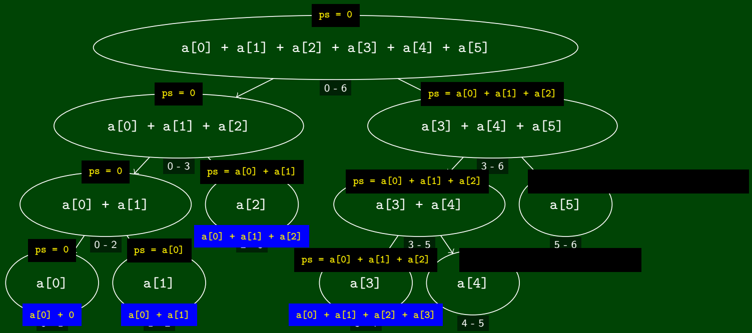


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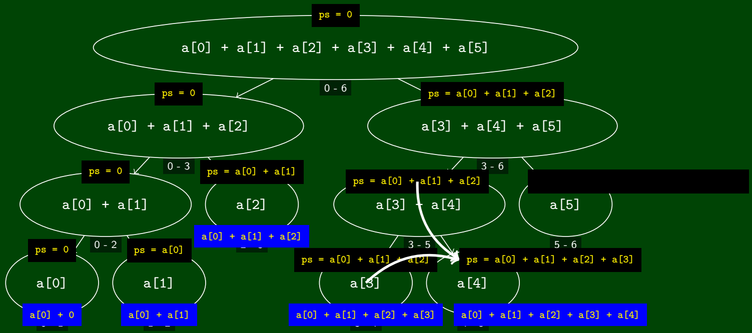


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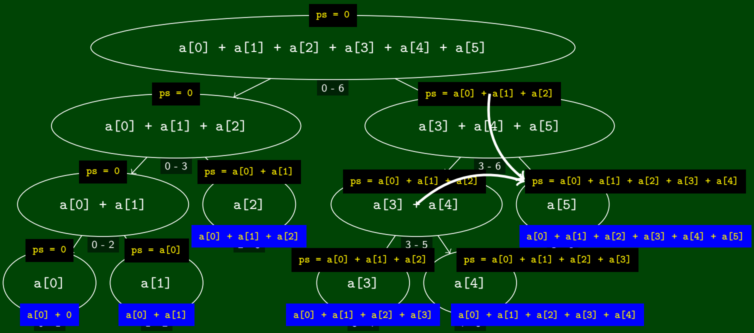


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}
```

# Sequential Cut-off



Adding a sequential cut-off isn't too bad:

## Processing the Input

This is just a normal sequential cut-off. The leaves end up being cutoff size ranges instead of ranges of one.

## Constructing the Output

We must sequentially compute the prefix sum at our leaves as well:

```
output[lo] = prescan + input[lo];
for (i = lo + 1; i < hi; i++) {
   output[i] = output[i−1] + input[i]
}
```

Notice that this means we must pass the `input` array to this phase now.

# Another Primitive: Parallel Pack (or "filter")



Here the idea is that we'd like to filter the array given some predicate (e.g., $\leq 7$). More specifically:

## Pack/Filter

Suppose we have a function $f : \texttt{E} \rightarrow \texttt{boolean}$ and an array a of type `E`:

a:

| $a_0$ | $a_1$ | $a_2$ | $a_3$ |
|---|---|---|---|
| a[0] | a[1] | a[2] | a[3] |

Then, `pack(a)` returns an array of elements $x$ for which $f(x) = \texttt{true}$.
For example, if `arr = [1, 3, 8, 6, 7, 2, 4, 9]` and
`f(x) = x % 2 == 0`, then `pack(arr) = [8, 6, 2, 4]`.

**The key to doing this in parallel is scan!**

# Another Primitive: Parallel Pack (or "filter")

Let `f(x) = x % 2 == 0`.

## Parallel Pack

input:

| a[0] | a[1] | a[2] | a[3] | a[4] | a[5] | a[6] | a[7] |
|---|---|---|---|---|---|---|---|
| 1 | 3 | 8 | 6 | 7 | 2 | 4 | 9 |

1. Use a **map** to compute a bitset for $f(x)$ applied to each element

bitset:

| b[0] | b[1] | b[2] | b[3] | b[4] | b[5] | b[6] | b[7] |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 |

2. Do a **scan on the bit vector** with $\oplus = +$:

bitsum:

| c[0] | c[1] | c[2] | c[3] | c[4] | c[5] | c[6] | c[7] |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 2 | 3 | 4 | 4 |

3. Do a **map on the bit sum** to produce the output:

output:

| d[0] | d[1] | d[2] | d[3] |
|---|---|---|---|
| 8 | 6 | 2 | 4 |

```
output = new E[bitsum[n-1]];
for (i=0; i < input.length; i++) {
   if (bitset[i] == 1) {
      output[bitsum[i] - 1] = input[i];
   }
}
```

# More on Pack



- We can combine the first two passes into one (just use a different base case for prefix sum)
- We can also combine the third step into the second part of prefix sum
- Overall: $\mathcal{O}(n)$ work and $\mathcal{O}(\lg n)$ span. (Why?)

**We can use scan and pack in all kinds of situations!**

```
int[] quicksort(int[] arr) {
   int pivot = choosePivot();
   int[] left = filterLessThan(arr, pivot);
   int[] right = filterGreaterThan(arr, pivot);
   return quicksort(left) + quicksort(right);
}
```

## Do The Recursive Calls in Parallel

Assuming a good pivot, we have:

$$\text{work}(n) = \begin{cases} \mathcal{O}(1) & \text{if } n = 1 \\ 2\text{work}(n/2) + \mathcal{O}(n) & \text{otherwise} \end{cases}$$

and

$$\text{span}(n) = \begin{cases} \mathcal{O}(1) & \text{if } n = 1 \\ \max(\text{span}(n/2), \text{span}(n/2)) + \mathcal{O}(n) & \text{otherwise} \end{cases}$$

These solve to $\mathcal{O}(n \lg n)$ and $\mathcal{O}(n)$. So, the parallelism is $\mathcal{O}(\lg n)$.

```
int[] quicksort(int[] arr) {
   int pivot = choosePivot();
   int[] left = filterLessThan(arr, pivot);
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   return quicksort(left) + quicksort(right);
}
```

### Do The Partition in Parallel

The partition step is just two filters or packs. Each pack is $\mathcal{O}(n)$ work, but $\mathcal{O}(\lg n)$ span! So, our new span recurrence is:

$$\text{span}(n) = \begin{cases} \mathcal{O}(1) & \text{if } n = 1 \\ \max(\text{span}(n/2), \text{span}(n/2)) + \mathcal{O}(\lg n) & \text{otherwise} \end{cases}$$

Master Theorem says this is $\mathcal{O}(\lg^2 n)$ which is neat!

```
int[] mergesort(int[] arr) {
   int[] left = getLeftHalf();
   int[] right = getRightHalf();
   return merge(mergesort(left), mergesort(right));
}
```

**Do The Recursive Calls in Parallel**

This will get us the same work and span we got for quicksort when we did this:

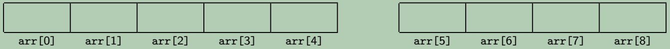
- work $(n) = \mathcal{O}(n \lg n)$
- span $(n) = \mathcal{O}(n)$
- Parallelism is $\mathcal{O}(\lg n)$

Now, let's try to parallelize the `merge` part.

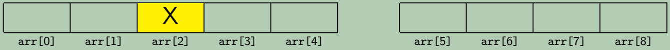
As always, when we want to parallelize something, we can turn it into a divide-and-conquer algorithm.

## Do The Merge in Parallel

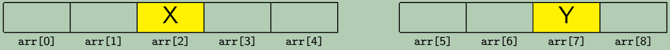
Merge takes as input two arrays:



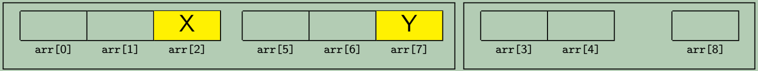

1. Find the median of the **larger** array (just the middle index):




2. Partition the **smaller** array using $X$ as a pivot. To do this, **binary search** the smaller array:




3. Now, we have four pieces $\boxed{\leq X}$, $\boxed{> X}$, $\boxed{\leq Y}$, and $\boxed{> Y}$. In the sorted array, the $\leq$ pieces will be entirely before the $>$ pieces.




4. Recursively apply the merge algorithm (until some cut-off)!

# Parallel Mergesort Analysis



First, we analyze **just the parallel merge**:

## Parallel Merge Analysis

The non-recursive work is $\mathcal{O}(1) + \mathcal{O}(\lg n)$ to find the splits.

The **worst case** is when we split the bigger array in half and the smaller array is all on the left (or all on the right). In other words:

$$\text{work}(n) \le \begin{cases} \mathcal{O}(1) & \text{if } n = 1 \\ \text{work}(3n/4) + \text{work}(n/4) + \mathcal{O}(\lg n) & \text{otherwise} \end{cases}$$

and

$$\text{span}(n) \le \begin{cases} \mathcal{O}(1) & \text{if } n = 1 \\ \max(\text{span}(3n/4) + \text{span}(n/4)) + \mathcal{O}(\lg n) & \text{otherwise} \end{cases}$$

These solve to $\text{work}(n) = \mathcal{O}(n)$ and $\text{span}(n) = \mathcal{O}(\lg^2 n)$.

Now, we calculate the work and span of **the entire parallel mergesort**.

**Putting It Together**

$$\text{work}(n) = \mathcal{O}(n \lg n)$$

$$\text{span}(n) \leq \begin{cases} \mathcal{O}(1) & \text{if } n = 1 \\ \text{span}(n/2) + \mathcal{O}(\lg^2 n) & \text{otherwise} \end{cases}$$

This works out to $\text{span}(n) = \mathcal{O}(\lg^3 n)$.

This isn't quite as much parallelism as quicksort, but **this one is a worst case guarantee**!