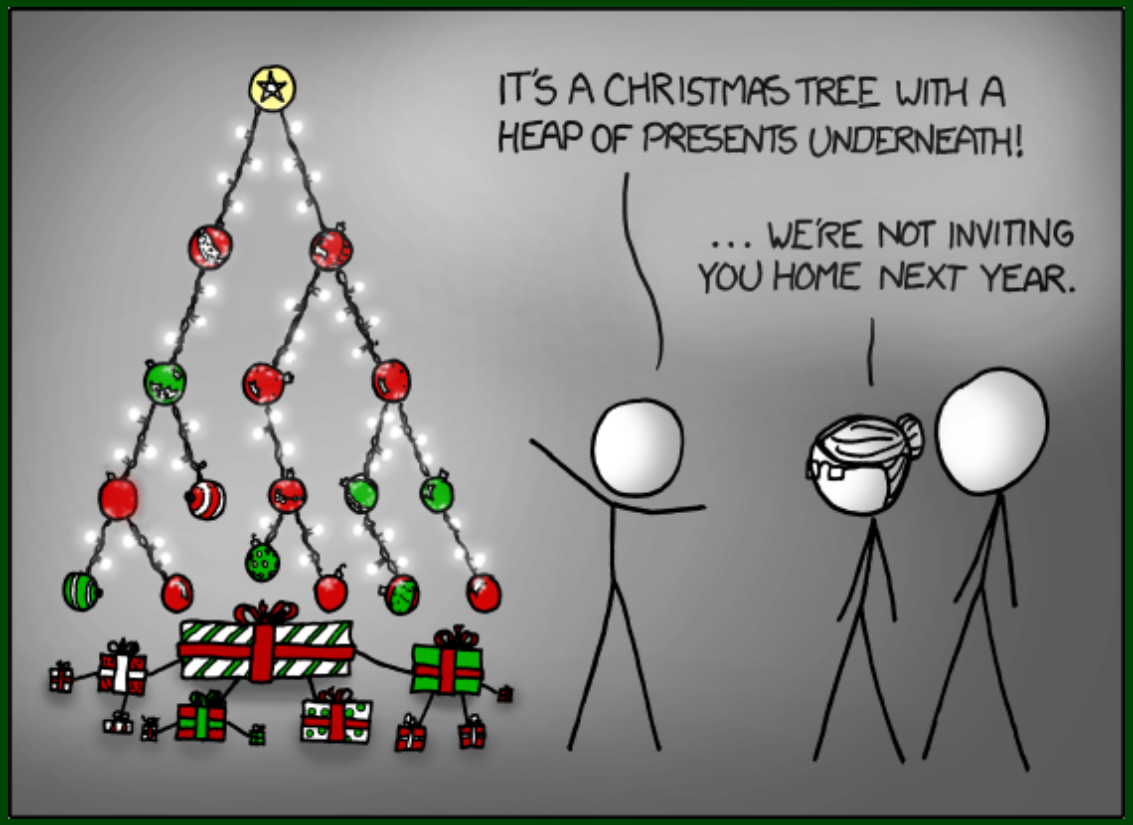
# CSE 332

## Data Abstractions

# Heaps


IT'S A CHRISTMAS TREE WITH A HEAP OF PRESENTS UNDERNEATH!
... WE'RE NOT INVITING YOU HOME NEXT YEAR.

# Outline

1 Reviewing `Heap` Representation

2 `Heap` Operations, Again

3 `buildHeap`

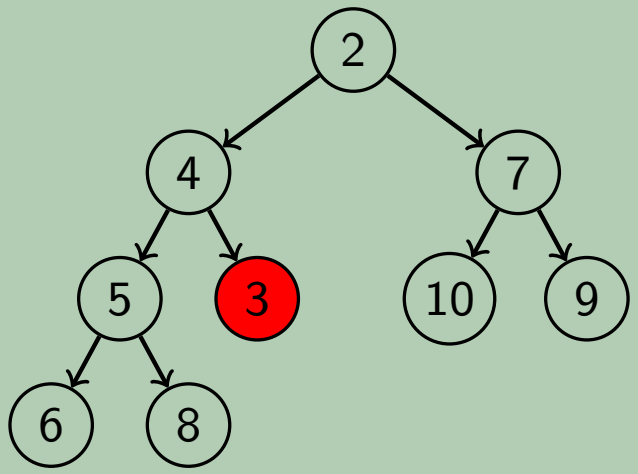
# A New Data Structure: `Heap`

## `PriorityQueue` ADT

| | |
|---|---|
| `insert(`**val**`)` | Adds **val** to the queue. |
| `deleteMin()` | Returns the **highest priority** item not already returned by a `deleteMin`. (Errors if empty.) |
| `findMin()` | Returns the **highest priority** item not already returned by a `deleteMin`. (Errors if empty.) |
| `isEmpty()` | Returns true if all inserted elements have been returned by a `deleteMin`. |

Heaps give us $\mathcal{O}(\lg n)$ `insert` and `deleteMin`:

## And Now, `Heaps`



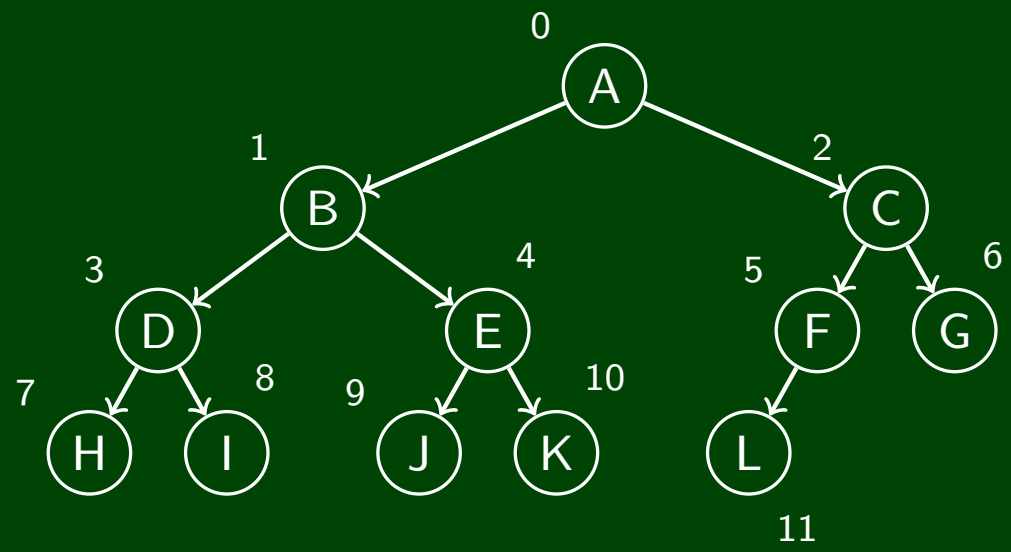

**Heap Property**:
All Children are larger

**Structure Property**:
Insist the tree has no "gaps"

# And. . . how do we implement `Heap`?



We've insisted that the tree be complete to be a valid `Heap`. Why?




Fill in an array in **level-order** of the tree:

| `heap`: | A | B | C | D | E | F | G | H | I | J | K | L | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | `h[0]` | `h[1]` | `h[2]` | `h[3]` | `h[4]` | `h[5]` | `h[6]` | `h[7]` | `h[8]` | `h[9]` | `h[10]` | `h[11]` | `h[12]` | `h[13]` | `h[14]` |

```
parent(n)      = (n – 1) / 2
leftChild(n)   = 2n + 1
rightChild(n) = 2n + 2
```

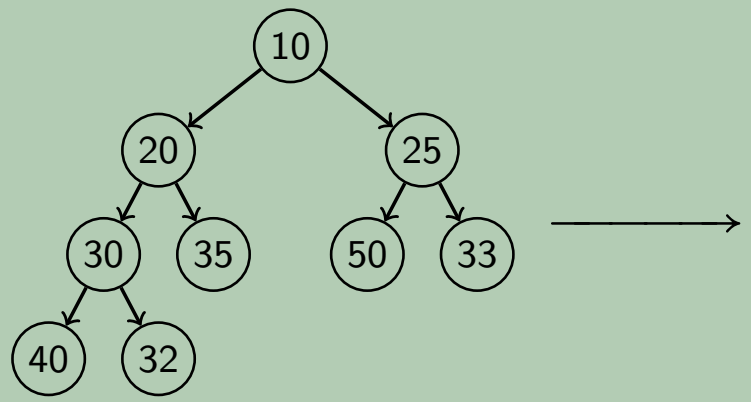
# `insert` Psuedocode



```
1  void insert(val) {
2     if (size == arr.length - 1) {
3        resize();
4     }
5
6     arr[size] = val;
7     percolateUp(size);
8     size++;
9  }
```

```
1  void percolateUp(hole) {
2     while (hole > 0 && arr[hole] < arr[parent(hole)]) {
3        swap(hole, parent(hole));
4        hole = parent(hole);
5     }
6  }
```

## Insert 2 into this Heap

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| 10 | 20 | 25 | 30 | 35 | 50 | 33 | 40 | 32 | 0 | 0 | 0 |

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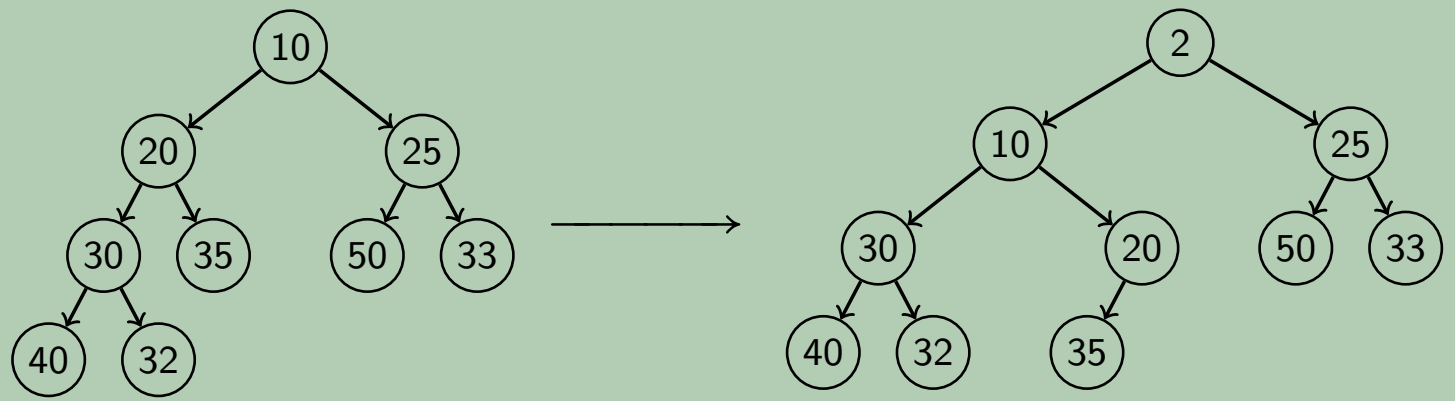


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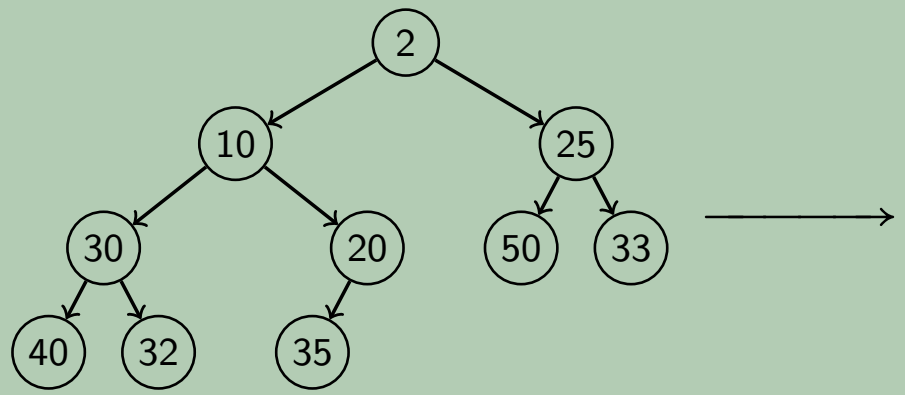
# deleteMin Psuedocode

```
1  int deleteMin() {
2     if (isEmpty()) {
3        throw ...;
4     }
5     ans = arr[0];
6     arr[0] = arr[size - 1];
7     size--;
8     percolateDown(0);
9     return ans;
10 }
```

```
1  void percolateDown(bad) {
2     target = getSmallestChild(bad);
3     while (arr[target] < arr[bad]) {
4        swap(bad, target);
5        bad = target;
6        target = getSmallestChild(bad);
7     }
8  }
```

## Delete Min

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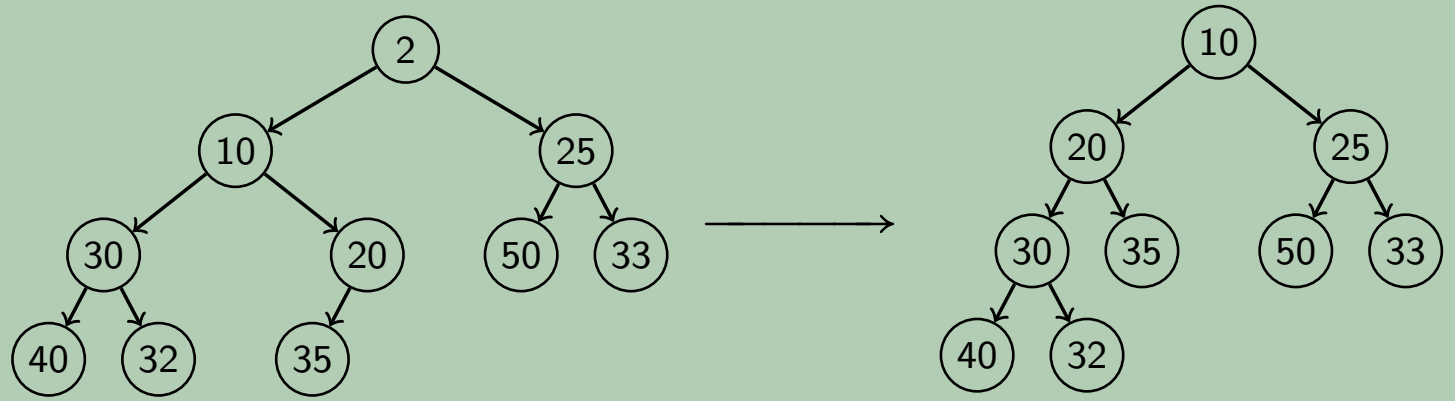


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**We know `insert` is $\mathcal{O}(\lg n)$, but...**

Just like with BSTs, the **order** of insertion makes a big difference.

With **randomly ordered inputs**, we have:

- an average of **2.6** comparisons per insert
- an element moves up 1.6 levels on average

Unfortunately, we're not so lucky on `deleteMin`; we usually have to percolate all the way down.

Suppose a heap has $n$ nodes.

- How many nodes on the bottom level? $\frac{n}{2}$
- And the level above? $\frac{n}{4}$
- etc.

Suppose we have a random value, $x$, in the heap.

- How often is $x$ in the bottom level? $\frac{1}{2}$ **of the time**
- And the level above? $\frac{1}{4}$ `of the time`
- etc.

So, putting these things together, we see that for a random value $x$, there's a $\frac{1}{2}$ probability we compare once, a $\frac{1}{4}$ probability we compare twice, etc.
Taking a weighted average (expected value) gives us:

$$\text{Average \# of Compares} < \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots = \sum_{i=0}^{\infty} \frac{i}{2^i} = 2$$

This is $\mathcal{O}(1)$!

## Advantages

Minimal amount of wasted space:

- Only unused space on right in the array
- No "holes" due to complete tree property
- No wasted space representing tree edges

Fast lookups:

- Benefit of array lookup speed
- Multiplying and dividing by 2 is extremely fast (can be done through bit shifting (see CSE 351)
- Last used position is easily found by using `size - 1` for the index

## Disadvantages

What if the array gets too full (or wastes space by being too empty)? Array will have to be resized.

**Advantages outweigh Disadvantages: This is how it is done!**

# Changing the ADT...

## What else can we do with a heap?

Given a particular index $i$ into the array...

- `decreaseKey(i, newPriority)`: Change priority, percolate up
- `increaseKey(i, newPriority)`: Change priority, percolate down
- `remove(i)`: Call `decreaseKey(i, -∞)`, then `deleteMin`

## What are the running times of these operations?

They're all worst case $\mathcal{O}(\lg n)$, but `decreaseKey` is **average** $\mathcal{O}(1)$.

## The Easy Way...

```
1 void buildHeap(int[] input) {
2    for (int i = 0; i < input.length; i++) {
3       insert(input[i]);
4    }
5 }
```

**What is the time complexity of `buildHeap`?**

The worst case is $\mathcal{O}(n \lg n)$.

## Can we do better?

With our current ADT, no! But if we have access to the internals of the data structure, we can.
In other words, if we **add a new operation to the ADT**, then we can do better.

This is a trade-off: convenience, efficiency, simplicity

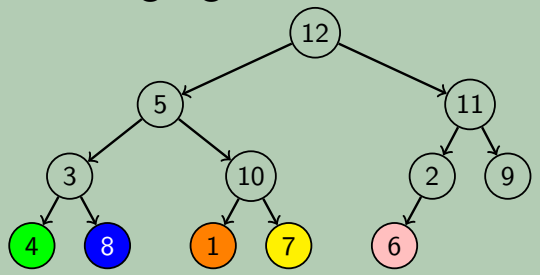
# Building a Heap, Take 2

## Floyd's `buildHeap` Idea

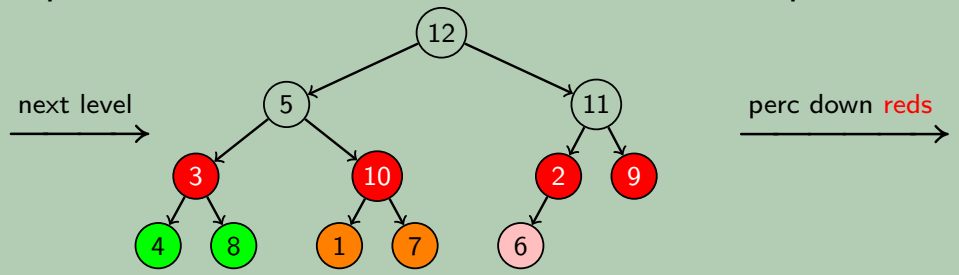
Our previous attempt added a node, then fixed the heap, then added a node, then fixed the heap, etc.
What if we added all the nodes and then **fixed the heap all at once**!

## Floyd's `buildHeap`

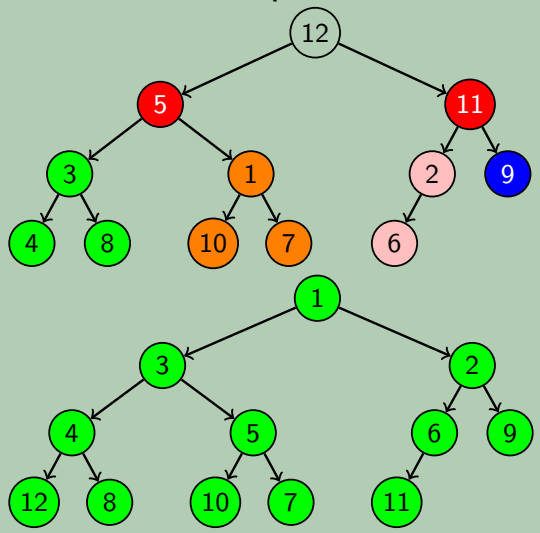
Each highlighted node is a valid heap!

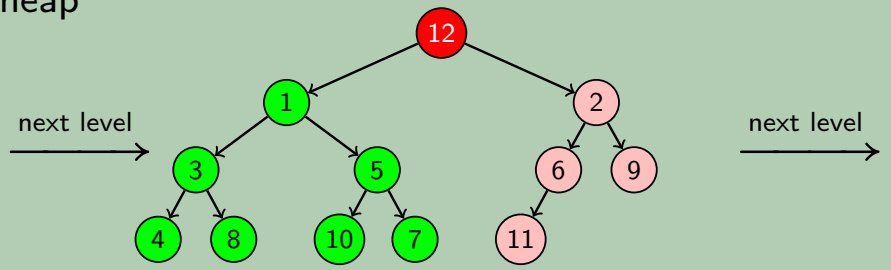



Percolate down red nodes until at top




Each color represents a valid heap

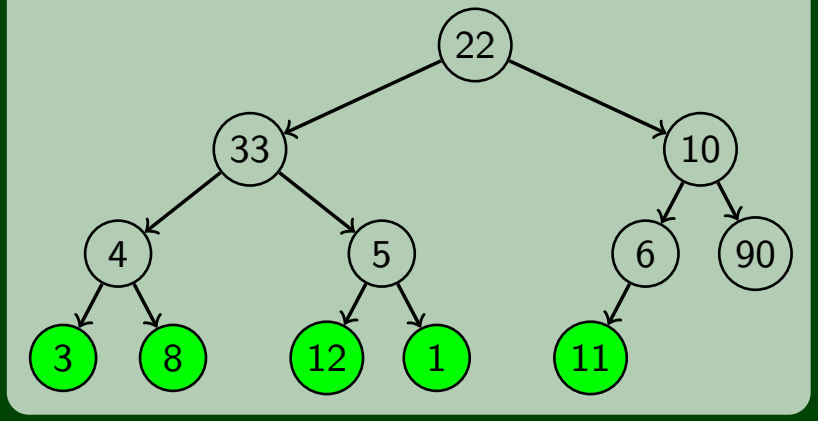


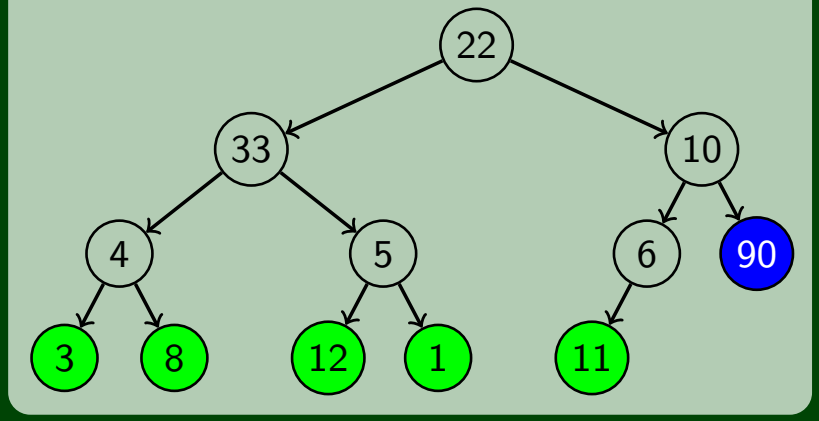
# Floyd's `buildHeap`

```
1  void buildHeap(int[] input) {
2     for (i = (size + 1)/2; i >= 0; i--) {
3        percolateDown(i);
4     }
5  }
```

The leaves begin as valid heaps




Now, we begin percolating down



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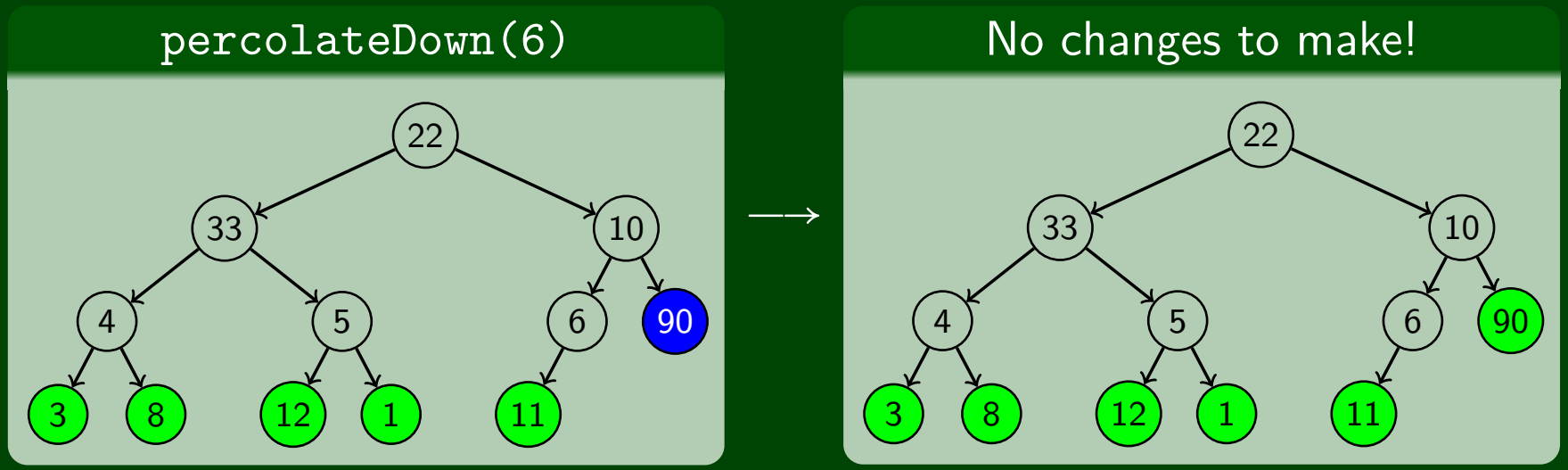

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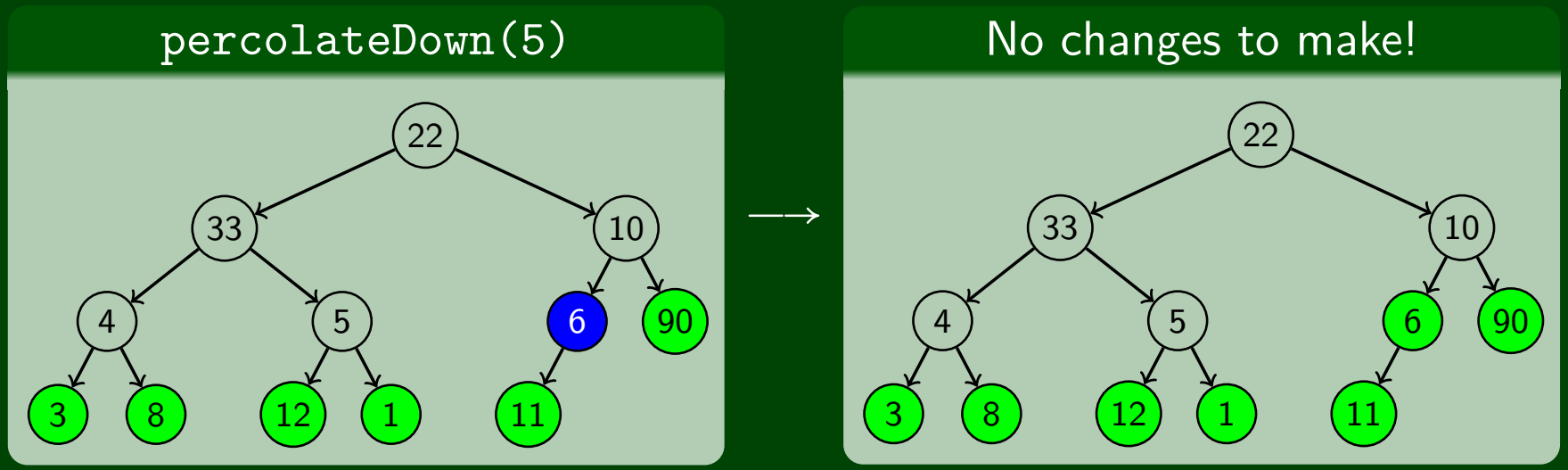

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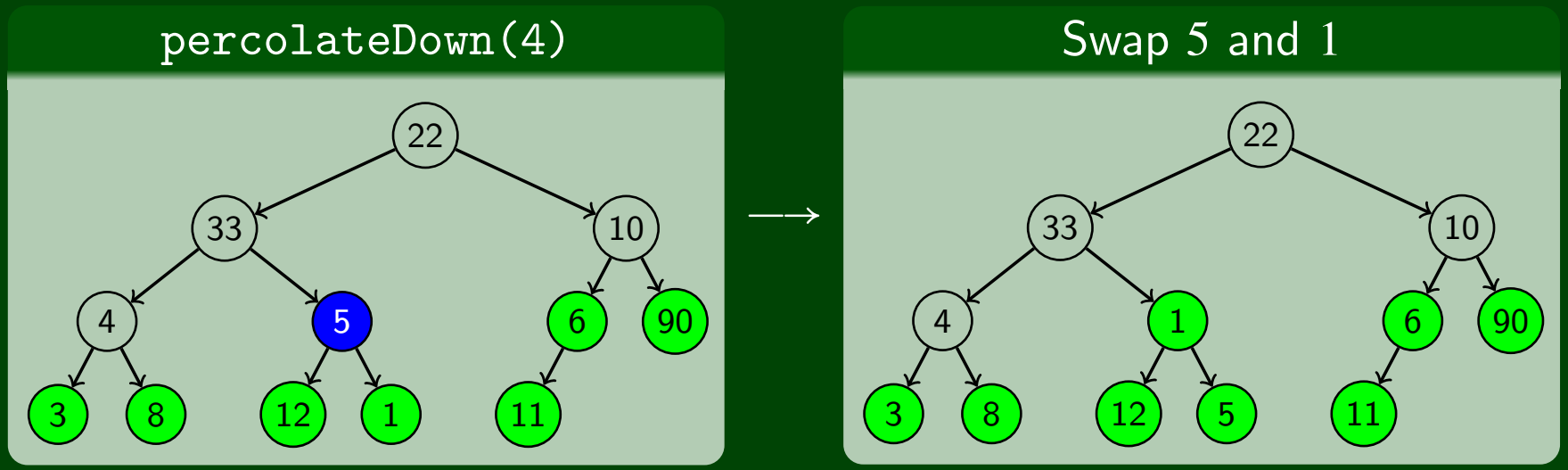

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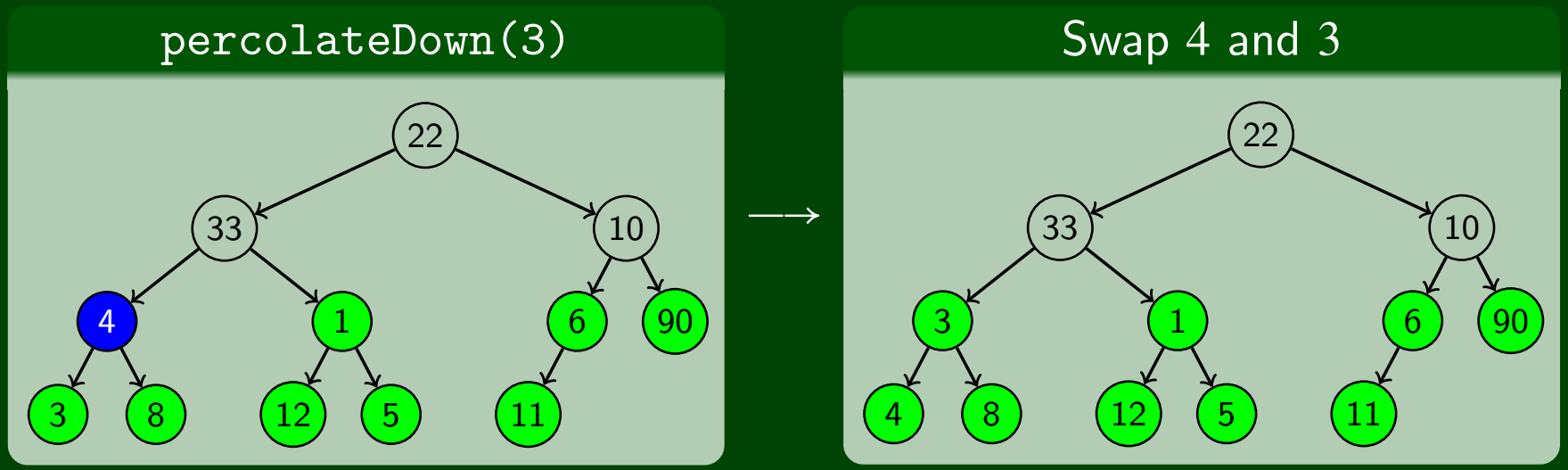

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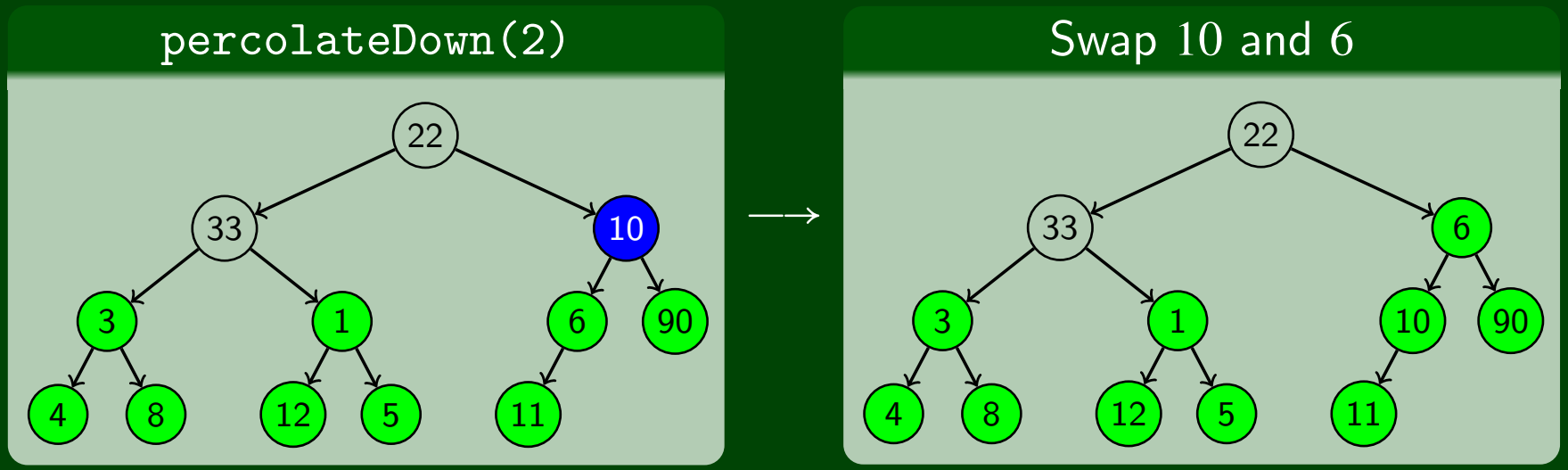

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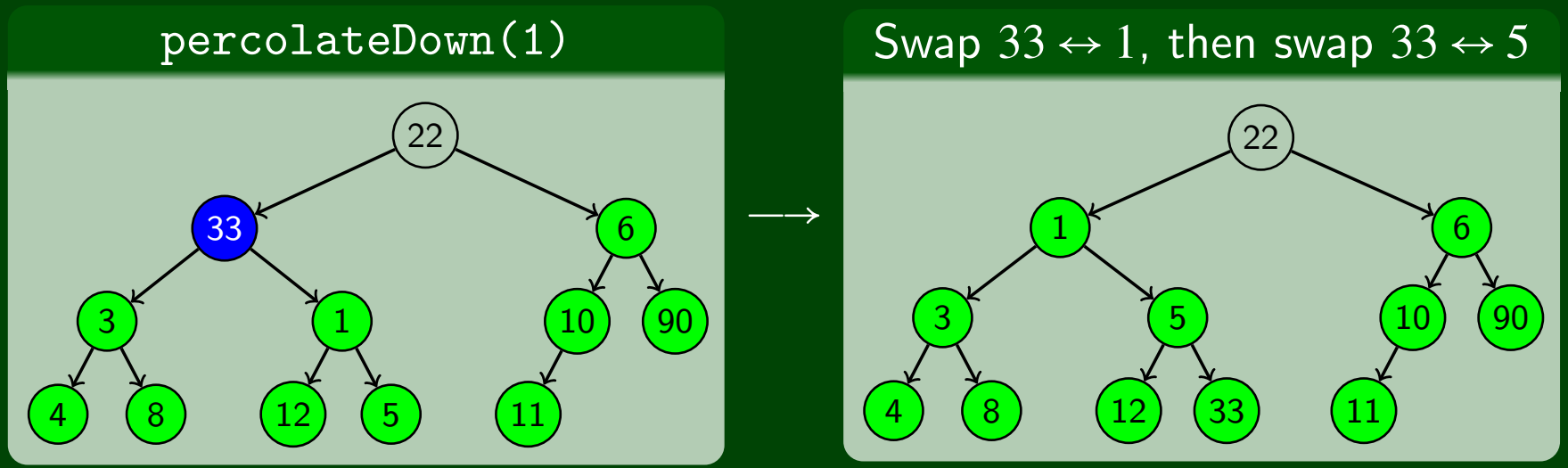

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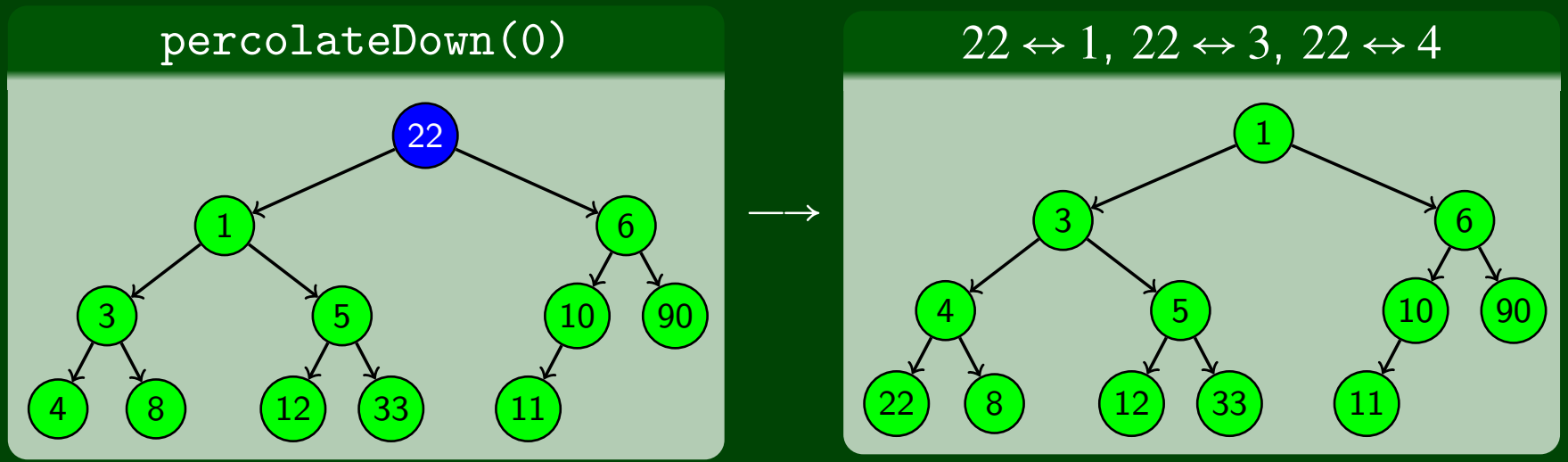

# Correctness of Floyd's `buildHeap`

```
1 void buildHeap(int[] input) {
2    for (i = (size + 1)/2; i >= 0; i--) {
3       percolateDown(i);
4    }
5 }
```

The algorithm seems to work. Let's **prove it**:
To prove that it works, we'll prove the following:

> Before loop iteration $i$, all `arr[j]` where $j > i$ have the heap property

Formally, we'd do this by induction. Here's a sketch of the proof:

- Base Case:
- Induction Step:

So, since the loop ends with index $0$, once we're done all the elements of the array will have the heap property.

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Formally, we'd do this by induction. Here's a sketch of the proof:

- Base Case: All $j >$ `(size + 1) / 2` **have no children**.
- Induction Step:

  We know that `percolateDown` **preserves the heap property** and makes its argument also have the heap property. So, after the $(i+1)$st iteration, we know $i$ is less than all its children and by the IH, we know that all of the children past `arr[i]` already had the heap property (and `percolateDown` didn't break it).

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# Efficiency of Floyd's `buildHeap`

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```

## Was this even worth the effort?

The loop runs $n/2$ iterations and each one is $\mathcal{O}(\lg n)$; so, the algorithm is $\mathcal{O}(n \lg n)$.

This is certainly true, but it's **not** $\Omega(n \lg n)$...

## A Tighter Analysis

- On the second lowest level there are $\frac{n}{2^2}$ elements and each one can percolate **at most** 1 time
- On the third lowest level there are $\frac{n}{2^3}$ elements and each one can percolate **at most** 2 times
- ...

Putting this together, the **largest possible number of swaps is**:

$$\sum_{i=1}^{k} \frac{ni}{2^{i+1}} < \frac{n}{2}\left(\sum_{i=1}^{\infty} \frac{i}{2^i}\right) = \frac{2n}{2} = n$$

# Takeaways from `buildHeap`

## ADT?

- Without `buildHeap`, our ADT already let clients implement their own in $\Omega(n \lg n)$ worst case
- By providing a specialized operation internally (with access to the data structure), we can do $\mathcal{O}(n)$ worst case

## Our Analyses!

- Correctness: Non-trivial inductive proof using loop invariant
- Efficiency:
  - First analysis easily proved it was $\mathcal{O}(n \lg n)$
  - A tighter analysis shows the same algorithm is $\mathcal{O}(n)$

## More Complicated Heaps

- Leftist heaps, skew heaps, binomial queues (Weiss 6.6-6.8)
- Different data structures for priority queues that support a logarithmic time `merge` operation (impossible with binary heaps)
- Intuition: We already saw `merge` for the amortized array dictionary
- `insert` & `deleteMin` defined in terms of `merge`

## $d$-heaps

We can have heaps with $d$ children instead of just 2 (see Weiss 6.5)

- Makes heaps shallower, useful for heaps too big for memory
- How does this affect the asymptotic run-time (for small $d$'s)?