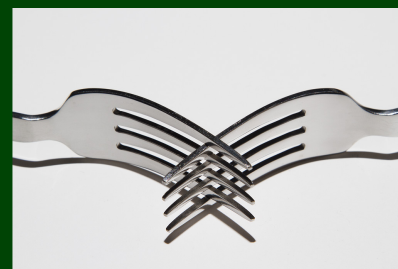
CSE 332

Data Abstractions

Analysis of Parallel Programs



Outline

More Parallel Primes-ish

Largest Factors

Last time, we found the number of primes in a range. This time, let's find the largest factors for each number in an input array.

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```java
protected void compute() {
   if (hi − lo <= CUTOFF) {
      seqReplaceWithLargestFactor(arr, lo, hi);
      return;
   }

   int mid = lo + (hi − lo) / 2;
   LargestFactorTask left = new LargestFactorTask(arr, lo, mid);
   LargestFactorTask right = new LargestFactorTask(arr, mid, hi);

   left.fork();
   right.compute();
   left.join();
}
```

This problem was different than the previous ones. The goal was to apply a function to every element of an array rather than to return a result.

Maps and Reductions

Reductions

Last time, we saw several problems of the form:
INPUT: An array
OUTPUT: A combination of the array by an associative operation
The general name for this type of problem is a **reduction**. Examples include: `max`, `min`, `has-a`, `first`, `count`, `sorted`

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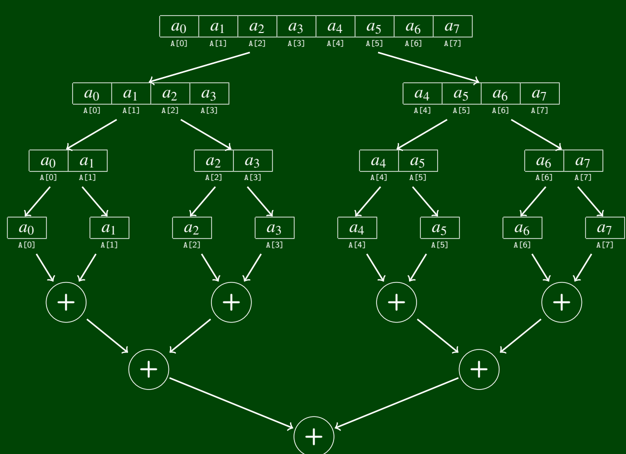
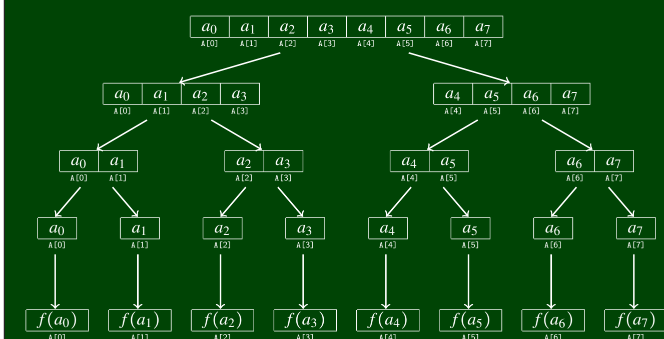
These two types of problems are "**parallel primitives**" in the same way loops and if statements are "programming primitives". Next lecture, we'll add two more primitives.

A Reduction
3



A Map
4



Google `MapReduce` and `Hadoop`
5

You may have heard of Googles `MapReduce` (or the open-source version `Hadoop`).

- Idea: Perform maps/reduces on data using many machines
  - The system takes care of distributing the data and managing fault tolerance
  - You just write code to map one element and reduce elements to a combined result

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  - The system takes care of distributing the data and managing fault tolerance
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- Separates how to do recursive divide-and-conquer from what computation to perform
  - Old idea in higher-order functional programming transferred to large-scale distributed computing
  - Complementary approach to declarative queries for databases

Parallelism on Other Data Structures

So far, we've only tried to apply parallelism to an `Array` (or, equivalently, an `ArrayList`). What about the other data structures we know? In particular, how does `ForkJoin` do on:

- `LinkedLists`?
- `BinaryTrees`?
- (Balanced) `BinaryTrees`?
- n-ary `Trees`?

Let's think about this with our toy problem of "sum up all the elements of the input".

Parallelism on `LinkedLists`

We wrote code that treated the array like a `LinkedList` last lecture.

```
compute() {
   if (not the end of the list) {
      fork a thread to do the rest of the elements;
   }

   do my work

   join with the thread after me
}
```


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The only gain we're going to get with `LinkedLists` is if the **map function** is very expensive. Then we'll at least get most of those going at once.

Naturally, as with standard algorithms on unbalanced trees, since they degenerate to linked lists, we have the same problem.

Parallelism on Balanced `Trees`

The idea here is to divide-and-conquer **each child** instead of array sub-ranges:

```
compute() {
   left.fork();     // Handles the entire left subtree
   right.compute(); // Handles the entire right subtree

   return left.join() + rightResult;
}
```

But what about the sequential cut-off?

Either store the **number of nodes** in each subtree or approximate it with **the height**

Consider the **MAXIMUM** problem from a few lectures ago.

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Consider the **MAXIMUM** problem from a few lectures ago.

The best we could do in sequential-land was $\Omega(n)$, but with parallelism, we can find the maximum element in $\Theta(\lg n)$ time (with enough processors...)!

Work and Span

With sequential algorithms, we often considered $T(n)$ (the runtime of the algorithm). Now, we'll consider a more general notion:

> Let $T_P(n)$ be the runtime of an algorithm **using P processors**.

There are two important runtime quantities for a parallel algorithm:

- How long it would take if it were fully sequential (work)
- How long it would take if it were as parallel as possible (span)

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We say $\text{work}(n) = T_1(n) = T(n)$ is the culmulative work that **all** processors must complete.

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Definition (Span)

We say $\text{span}(n) = T_\infty(n)$ is the largest amount of work **some** processor must complete.

Analyzing a Parallel Algorithm

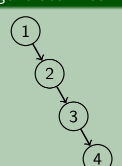
For each "type" of tree, figure out work(−) and span(−) of `findMin` in terms of the number of nodes, n.

A (Parallel) Algorithm

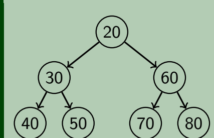
```
int findMin(Node current) {
   if (current is a leaf) {
      return current.data;
   }

   return min(current.data, findMin(left), findMin(right));
}
```

Degenerate Tree



Perfect Tree



Analyzing a Parallel Algorithm: Work of Degenerate Tree

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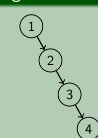

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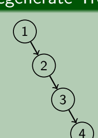

To calculate work, we just do our standard analysis. First, we make a recurrence:

$$\text{work}(n) = \begin{cases} 0 & \text{if } n = 0 \\ \mathcal{O}(1) & \text{if } n = 1 \\ \text{work}(0) + \text{work}(n-1) + \mathcal{O}(1) & \text{otherwise} \end{cases}$$

Solving this recurrence gives us:

$$\text{work}(n) = \sum_{i=0}^{n} 1 = \Theta(n)$$

Analyzing a Parallel Algorithm: Span of Degenerate Tree

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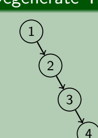

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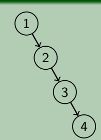

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This ends up being the same recurrence as for work(−). Notice for the degenerate tree work(n) = span(n). This proves our intution that we don't get much of a (any!) speed-up with parallelism for linked lists!

Analyzing a Parallel Algorithm: Work of Perfect Tree

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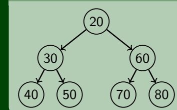

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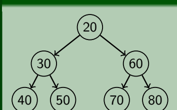

To calculate work, we just do our standard analysis. First, we make a recurrence:

$$\text{work}(n) = \begin{cases} \mathcal{O}(1) & \text{if } n = 1 \\ 2 \times \text{work}(n/2) + \mathcal{O}(1) & \text{otherwise} \end{cases}$$

Master Theorem says this recurrence is $\Theta(n)$.

Analyzing a Parallel Algorithm: Span of Perfect Tree

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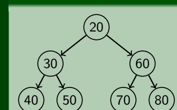

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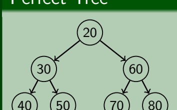

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Master Theorem says this recurrence is $\Theta(\lg n)$.

Again, this proves our intuition that parallelizing tree algorithms helps.

But what does it mean for work to be $\Theta(n)$ and span to be $\Theta(\lg n)$?

Speed-up, Parallelism, and T_P

Definition (Speed-Up)

The **speed-up** given P processors is $\frac{T_1}{T_P}$.

If the speed-up is P as we vary P, it's called a **perfect linear speed-up**.

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We want to decrease span without increasing work!

Okay, but we don't have ∞ processors...

Consider T_P. We know the following:

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The Good News!

The ForkJoin Framework gives an expected-time guarantee of asymptotically optimal! (Want to know how? Take an advanced course!)
But this is only true given some assumptions about your code:

- The program splits up the work into small and approximately equal pieces
- The program combines the pieces efficiently

Applying Our Asymptotic Bound

Minimum in a Perfect Tree

When calculating the minimum element in a tree, we had:

- work $(n) \in \Theta(n)$
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So, we expect the algorithm to take $\mathcal{O}\left(\frac{n}{P} + \lg n\right)$

Another Example

Suppose we have the following work and span:

- work $(n) \in \Theta(n^2)$
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Amdahl's Law

Every program has:

- parts that parallelize easily/well
- parts that don't parallelize at all

For example, we **can't** parallelize reading a linked list.

The non-parallelizable parts of a program are a huge bottleneck

Amdahl's Law

Split the work up into two pieces: the "parallelizable" piece and the "non-parallelizable" piece. Let S be the inherently sequential work.

$$T_1 = S \times \text{work}(n) + (1 - S) \times \text{work}(n)$$

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So, the speed-up is:

$$\frac{T_1}{T_P} = \frac{1}{S + \frac{1-S}{P}}$$

The Bad News

Suppose 33% of a program is sequential. Then, the **absolute best speed-up** we can get is:

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That means infinitely many processors won't help us get more than a 3 times speed-up!

So, Let's Give Up?

Amdahl tells us that if a **particular algorithm** has too many sequential computations, it's better to find a **more parallelizable** algorithm than to just add more processors.

We'll see next time that unexpected problems can be solved in parallel!

Moore and Amdahl

Moore's "Law" is an observation about the progress of the semiconductor industry:

Transistor density doubles roughly every 18 months

Amdahls Law is a mathematical theorem:

Diminishing returns of adding more processors

Both are incredibly important in designing computer systems