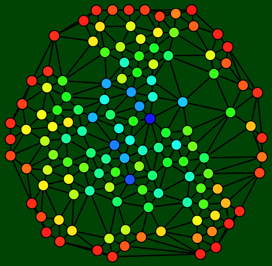
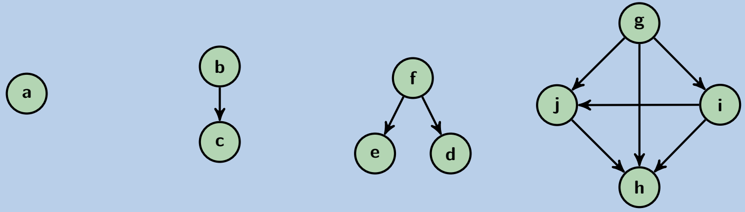
CSE 332

Data Abstractions

Graphs 2: Representing Graphs Topological Sort



A **Directed** Graph is a Thingy...

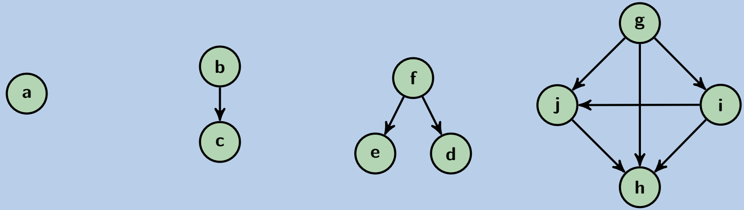


$V = \{a\}$, $E = \varnothing$

$V = \{b, c\}$,
$E = \{(b, c)\}$

Let's extend our terminology for **directed graphs**!

A **Directed** Graph is a Thingy. . .



$V = \{a\}$, $E = \varnothing$

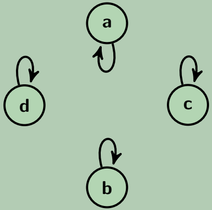
$V = \{b, c\}$,
$E = \{(b, c)\}$

$V = \{d, e, f\}$,
$E = \{(f, e), (f, d)\}$

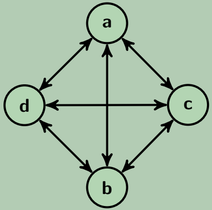
$V = \{g, h, i, j\}$,
$E = \{(g, h), (h, i), (g, j),$
$(i, h), (j, h), (i, j)\}$

Let's extend our terminology for **directed graphs**!

A Lonely Graph



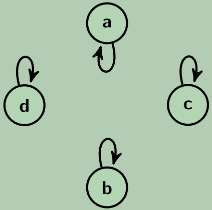
Complete Directed Graph



Some Questions

- How many edges can a **directed** graph with $|V| = n$ have?
- How many edges can a **directed** graph with $|V| = n$ and possible loops have?

A Lonely Graph



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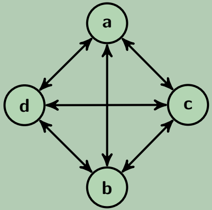

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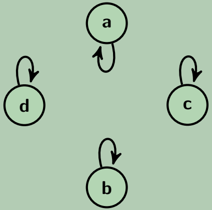
- How many edges can a **directed** graph with $|V| = n$ have?

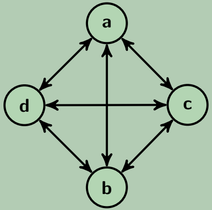
$$|E| = n(n-1)$$

- How many edges can a **directed** graph with $|V| = n$ and possible loops have?

A Lonely Graph



Complete Directed Graph



Some Questions

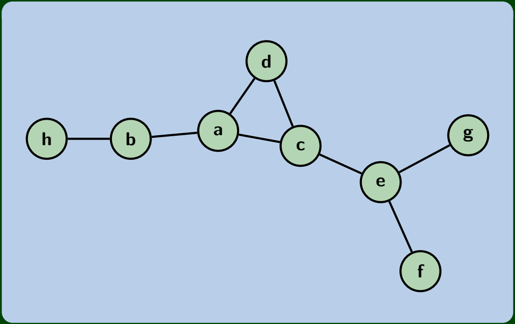
- How many edges can a **directed** graph with $|V| = n$ have?

$$|E| = n(n-1)$$

- How many edges can a **directed** graph with $|V| = n$ and possible loops have?

$$|E| = n^2$$

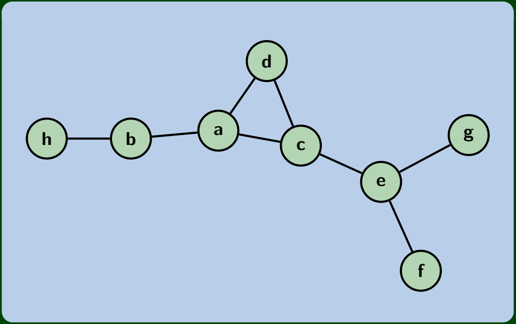
New Terminology: Degree



Definition (Degree)

The **degree** of a vertex in a graph is the number of vertices adjacent to it. In the above graph, we have:

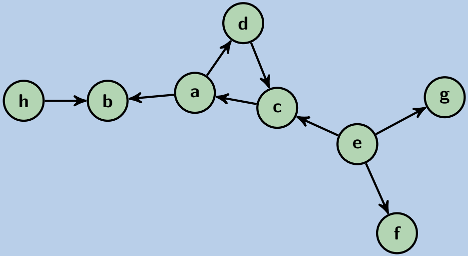
a	b	c	d	e	f	g	h



Definition (Degree)

The **degree** of a vertex in a graph is the number of vertices adjacent to it. In the above graph, we have:

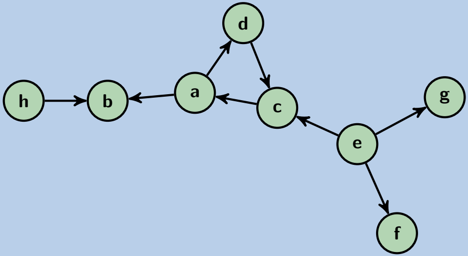
a	b	c	d	e	f	g	h
3	2	3	2	3	1	1	1



Definition (In & Out Degree)

The **in-degree** of a vertex, v, in a graph is $|\{(x,v) \mid (x,v) \in E, x \in V\}|$.
The **out-degree** of a vertex, v, in a graph is $|\{(v,x) \mid (x,v) \in E, x \in V\}|$.

	a	b	c	d	e	f	g	h
In-Degree								
Out-Degree								



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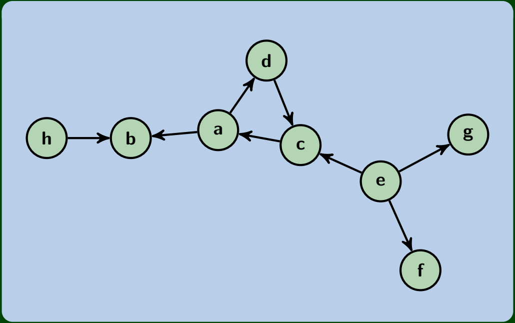
	a	b	c	d	e	f	g	h
In-Degree	1	2	2	1	0	1	1	0
Out-Degree								



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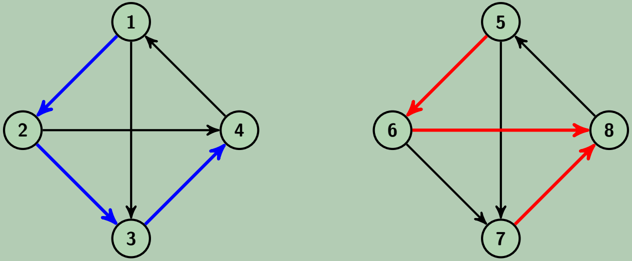
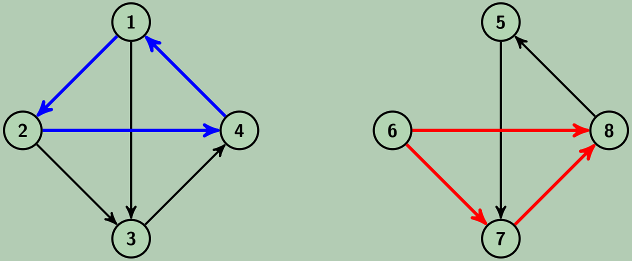
	a	b	c	d	e	f	g	h
In-Degree	1	2	2	1	0	1	1	0
Out-Degree	2	0	1	1	3	0	0	1

Re-examining Paths and Cycles on Directed Graphs

Paths?

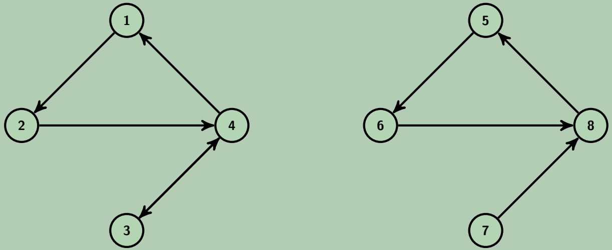


Cycle



Definition (Strongly Connected Directed Graph)

We say a directed graph is **strongly connected** iff for every pair of vertices, $u, v \in V$, there is a path from u to v.



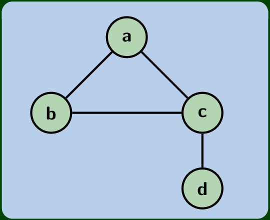
Strongly Connected! Not Strongly Connected!

Definition (Weakly Connected Directed Graph)

We say a directed graph is **weakly connected** iff the underlying undirected graph is connected.

That is, if we "undirected the edges", if the graph is connected, then the digraph is weakly connected.

Graph Data Structures



Adjacency Matrix

	a	b	c	d
a	0	1	1	0
b	1	0	1	0
c	1	1	0	1
d	0	0	1	0

Adjacency List

a: $\boxed{b} \longrightarrow \boxed{c} \longrightarrow$

b: $\boxed{a} \longrightarrow \boxed{c} \longrightarrow$

c: $\boxed{a} \longrightarrow \boxed{b} \longrightarrow \boxed{d} \longrightarrow$

d: $\boxed{c} \longrightarrow$

Adjacency Matrix Analysis

Adjacency Matrix

	a	b	c	d
a	0	1	1	0
b	1	0	1	0
c	1	1	0	1
d	0	0	1	0

Adjacency List

a: $\boxed{b} \longrightarrow \boxed{c} \longrightarrow$

b: $\boxed{a} \longrightarrow \boxed{c} \longrightarrow$

c: $\boxed{a} \longrightarrow \boxed{b} \longrightarrow \boxed{d} \longrightarrow$

d: $\boxed{c} \longrightarrow$

Adjacency Matrix Properties

How long to...

- Get a vertex's out-edges? $\mathcal{O}(|V|)$
- Get a vertex's in-edges? $\mathcal{O}(|V|)$
- Check if an edge exists? $\mathcal{O}(1)$
- Insert an edge? $\mathcal{O}(1)$
- Delete an edge? $\mathcal{O}(1)$

Space Requirements: $\mathcal{O}(|V|^2)$
Adjacency Matrices are reasonable for dense graphs, but not otherwise.

Adjacency List Analysis

Adjacency Matrix

	a	b	c	d
a	0	1	1	0
b	1	0	1	0
c	1	1	0	1
d	0	0	1	0

Adjacency List

a: $\boxed{\text{b}} \longrightarrow \boxed{\text{c}} \longrightarrow$

b: $\boxed{\text{a}} \longrightarrow \boxed{\text{c}} \longrightarrow$

c: $\boxed{\text{a}} \longrightarrow \boxed{\text{b}} \longrightarrow \boxed{\text{d}} \longrightarrow$

d: $\boxed{\text{c}} \longrightarrow$

Adjacency List Properties

How long to...

- Get a vertex's out-edges? $\mathcal{O}(d)$
- Get a vertex's in-edges? $\mathcal{O}(|E|)$
  - To fix this, keep a **second** adjacency list going the other way
- Check if an edge exists? $\mathcal{O}(d)$
- Insert an edge? $\mathcal{O}(1)$
- Delete an edge? $\mathcal{O}(d)$

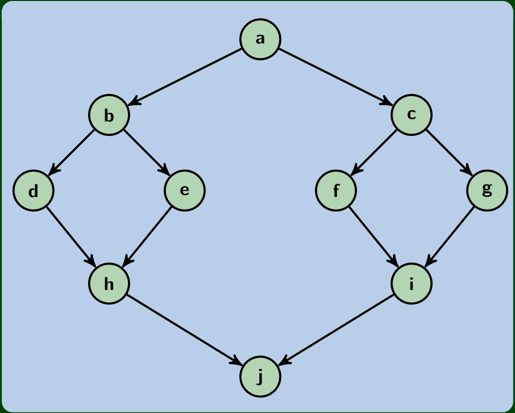
Space Requirements: $\mathcal{O}(|V| + |E|)$

Adjacency Lists should be your goto choice.

Directed Acyclic Graphs: DAGs

Definition (DAG)

A **DAG** is a **directed**, **acyclic** graph.



By "acyclic", we mean in the **directed** sense.

DAGs vs. Trees?

Is there a tree that isn't a DAG?

Is there a DAG that isn't a tree?

DAGs vs. Trees?

All trees are DAGs (remember, trees must be acyclic and connected!).

Not all DAGs are trees. See previous slide. Also, DAGs don't have to be connected!

Why DAGs?

They come up a lot in practice. Cycles can be icky. Examples:

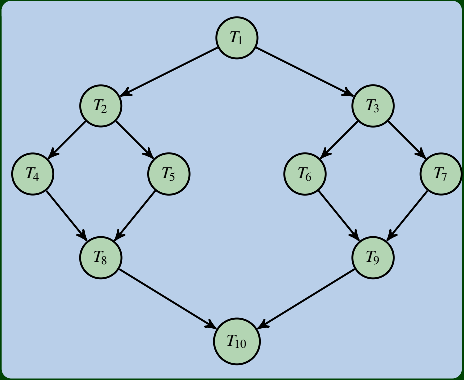
- Any sort of scheduling problem (scheduling your courses, scheduling fork-join threads, . . .)
- Causal Structures (Baysian Networks)
- Genealogy
- . . .

Topological Sort

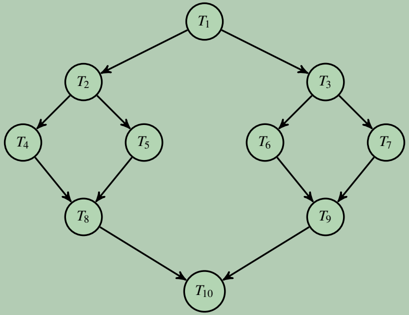
Given a DAG $(G = (V, E))$, output all the vertices in an order such that no vertex appears before any vertex that has an edge to it.

"Output an order to process the graph that meets all dependencies"

This is how we can allocate work in the `ForkJoin` model!



How Many Valid Topological Sorts?



- $T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}$
- $T_1, T_2, T_4, T_3, T_5, T_6, T_7, T_8, T_9, T_{10}$
- $T_1, T_2, T_5, T_4, T_3, T_6, T_7, T_8, T_9, T_{10}$
- $T_1, T_3, T_6, T_7, T_9, T_2, T_5, T_4, T_8, T_{10}$
- ...

Implementing Topological Sort

Throw all the **in-degrees** in a priority queue. `removeMin()` repeatedly.

- This works, but it's **too slow**.
- Insight: `PriorityQueues` must deal with negative numbers; indegree will never be negative!
- Instead: Split ready vs. not ready (0 vs. non-zero) sets
- The "ready set" is a worklist!

Setup

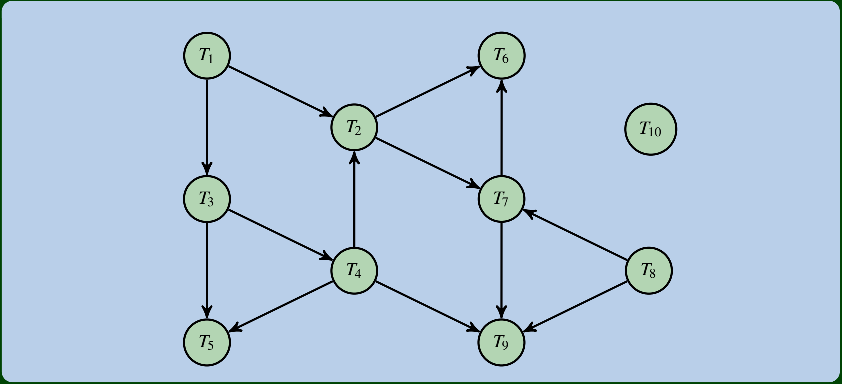
```
output = []
deps = {}
worklist = []
for (v : vertices) {
   deps[v] = in-degree(v);
   if (deps[v] == 0) {
      worklist.add(v);
   }
}
```

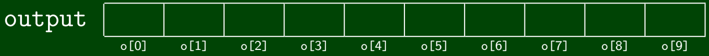
Do Work

```
while (worklist.hasWork()) {
   v = worklist.next();
   output.add(v);
   for (w : neighbors(v)) {
      deps[w] -= 1
      if (deps[w] == 0) {
         worklist.add(w);
      }
   }
}
```

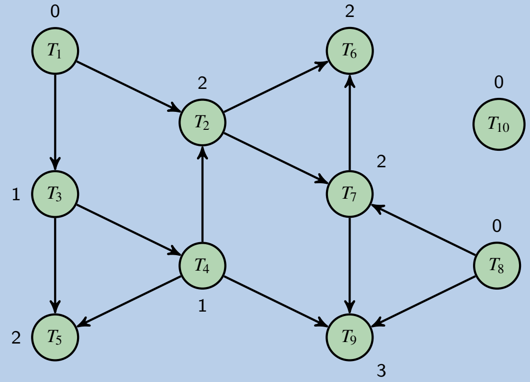
Topologically Sorting A DAG (**with a `Queue`**)

worklist ←





worklist ←

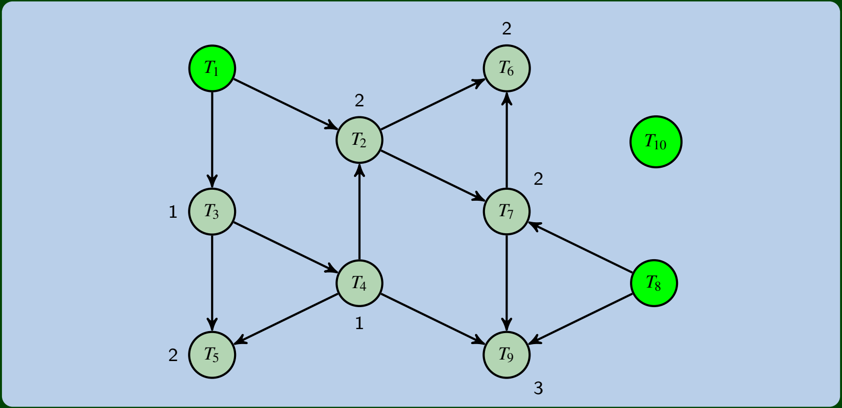


output

o[0]	o[1]	o[2]	o[3]	o[4]	o[5]	o[6]	o[7]	o[8]	o[9]

Topologically Sorting A DAG (**with a `Queue`**)

worklist $\leftarrow$ | T_1 | T_8 | T_{10} | $\leftarrow$

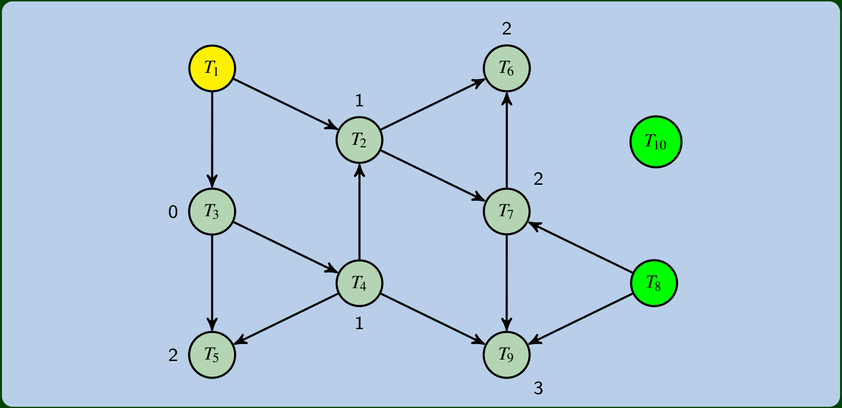


output

o[0]	o[1]	o[2]	o[3]	o[4]	o[5]	o[6]	o[7]	o[8]	o[9]

Topologically Sorting A DAG (with a `Queue`)

worklist ← | T_8 | T_{10} | ←

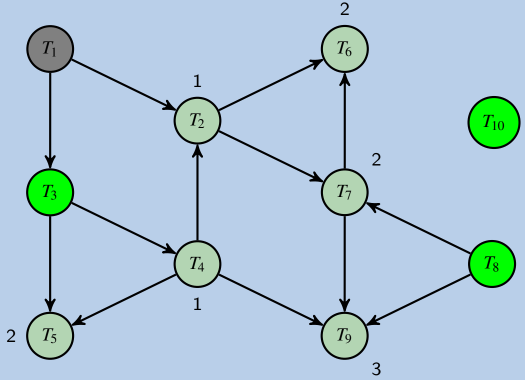


`output`

T_1									
`o[0]`	`o[1]`	`o[2]`	`o[3]`	`o[4]`	`o[5]`	`o[6]`	`o[7]`	`o[8]`	`o[9]`

Topologically Sorting A DAG (with a `Queue`)

worklist $\leftarrow$ | T_8 | T_{10} | T_3 | $\leftarrow$

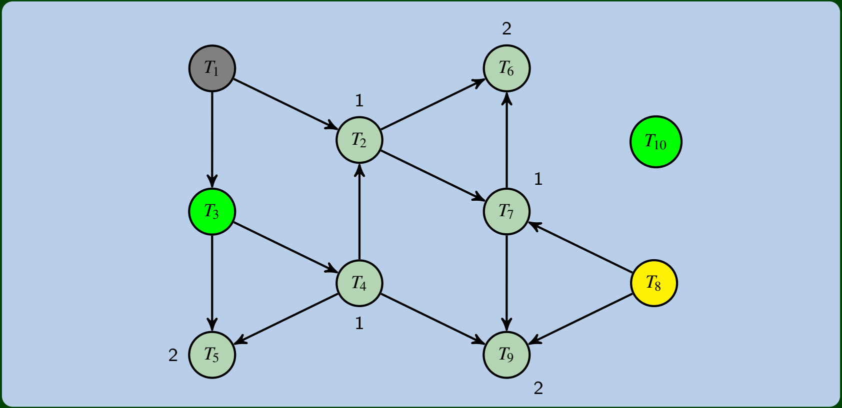


`output`

T_1									
o[0]	o[1]	o[2]	o[3]	o[4]	o[5]	o[6]	o[7]	o[8]	o[9]

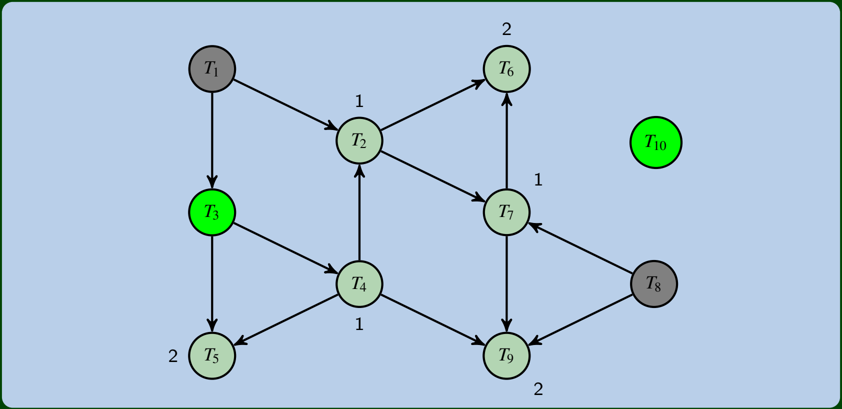
Topologically Sorting A DAG (with a `Queue`)

worklist ← | T_{10} | T_3 | ←



output	T_1	T_8								
	o[0]	o[1]	o[2]	o[3]	o[4]	o[5]	o[6]	o[7]	o[8]	o[9]

worklist ← | T_{10} | T_3 | ←

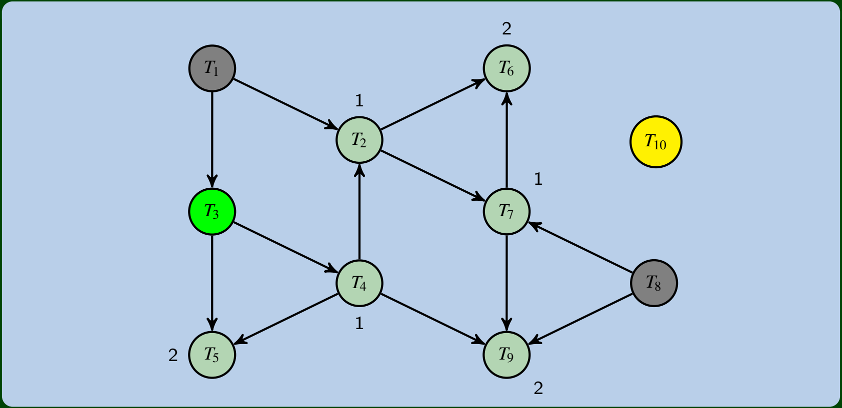


`output`

T_1	T_8								
`o[0]`	`o[1]`	`o[2]`	`o[3]`	`o[4]`	`o[5]`	`o[6]`	`o[7]`	`o[8]`	`o[9]`

Topologically Sorting A DAG (with a `Queue`)

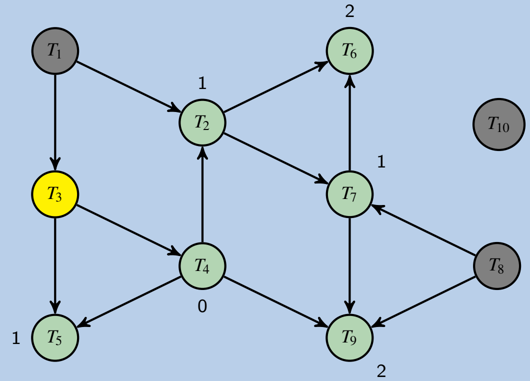
worklist ← T_3 ←



`output`

T_1	T_8	T_{10}							
o[0]	o[1]	o[2]	o[3]	o[4]	o[5]	o[6]	o[7]	o[8]	o[9]

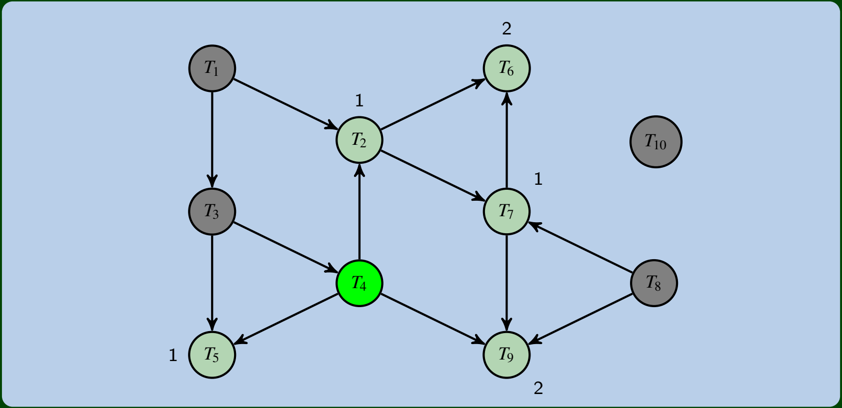
worklist ←



`output`	T_1	T_8	T_{10}	T_3						
	`o[0]`	`o[1]`	`o[2]`	`o[3]`	`o[4]`	`o[5]`	`o[6]`	`o[7]`	`o[8]`	`o[9]`

Topologically Sorting A DAG (with a `Queue`)

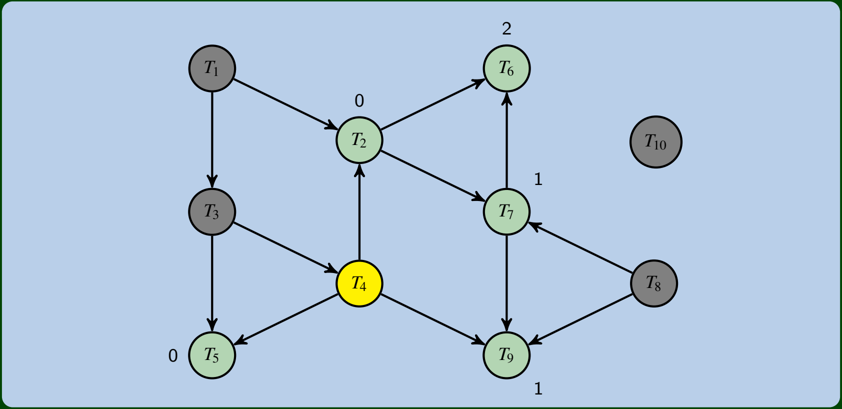
worklist $\leftarrow \boxed{T_4} \leftarrow$



output

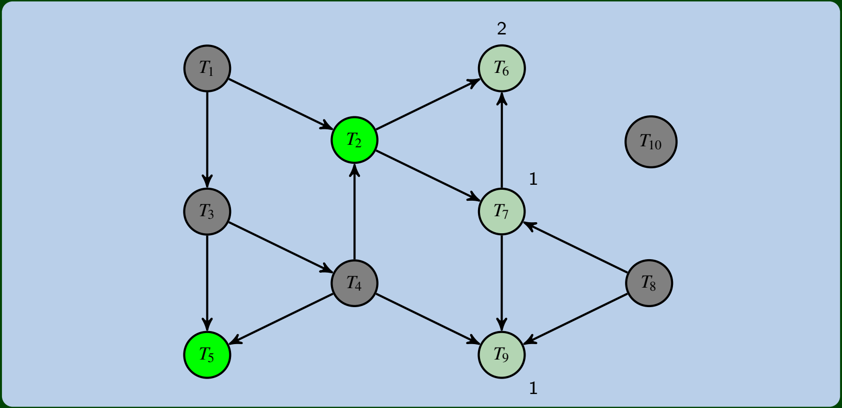
T_1	T_8	T_{10}	T_3						
o[0]	o[1]	o[2]	o[3]	o[4]	o[5]	o[6]	o[7]	o[8]	o[9]

worklist ←



`output`	T_1	T_8	T_{10}	T_3	T_4					
	`o[0]`	`o[1]`	`o[2]`	`o[3]`	`o[4]`	`o[5]`	`o[6]`	`o[7]`	`o[8]`	`o[9]`

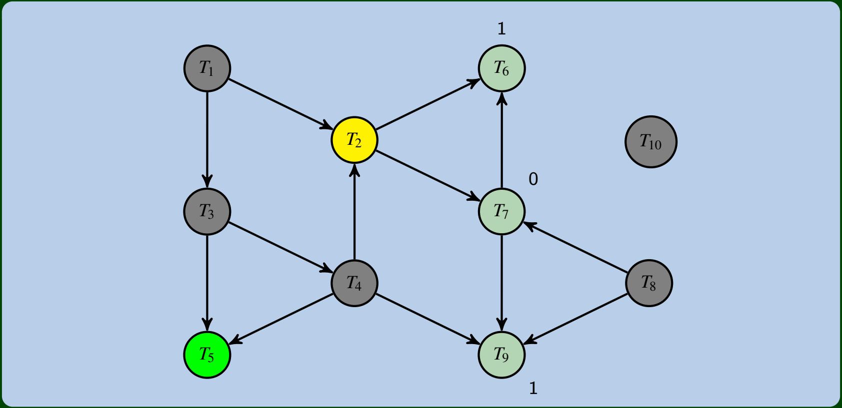
worklist ← | T_2 | T_5 | ←



output	T_1	T_8	T_{10}	T_3	T_4					
	o[0]	o[1]	o[2]	o[3]	o[4]	o[5]	o[6]	o[7]	o[8]	o[9]

Topologically Sorting A DAG (with a `Queue`)

worklist ← T_5 ←

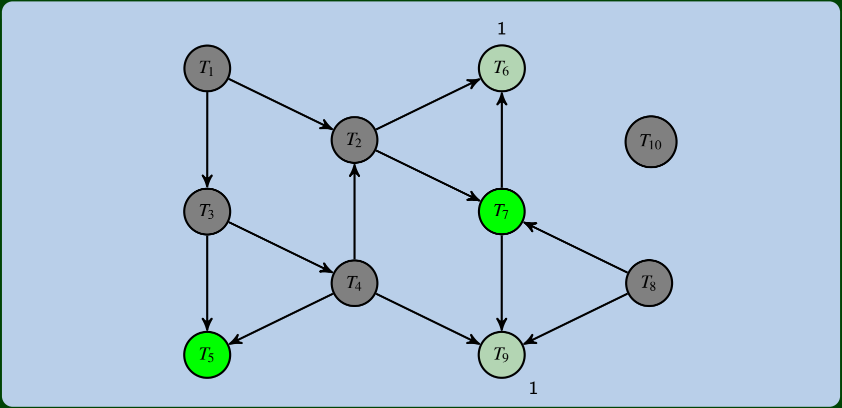


output

o[0]	o[1]	o[2]	o[3]	o[4]	o[5]	o[6]	o[7]	o[8]	o[9]
T_1	T_8	T_{10}	T_3	T_4	T_2				

Topologically Sorting A DAG (with a `Queue`)

worklist ← T_5 | T_7 ←

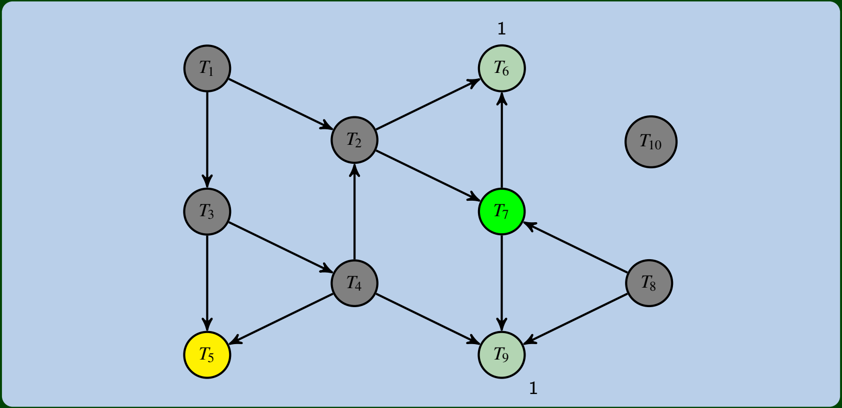


output

T_1	T_8	T_{10}	T_3	T_4	T_2				
o[0]	o[1]	o[2]	o[3]	o[4]	o[5]	o[6]	o[7]	o[8]	o[9]

Topologically Sorting A DAG (with a `Queue`)

worklist ← $\boxed{T_7}$ ←

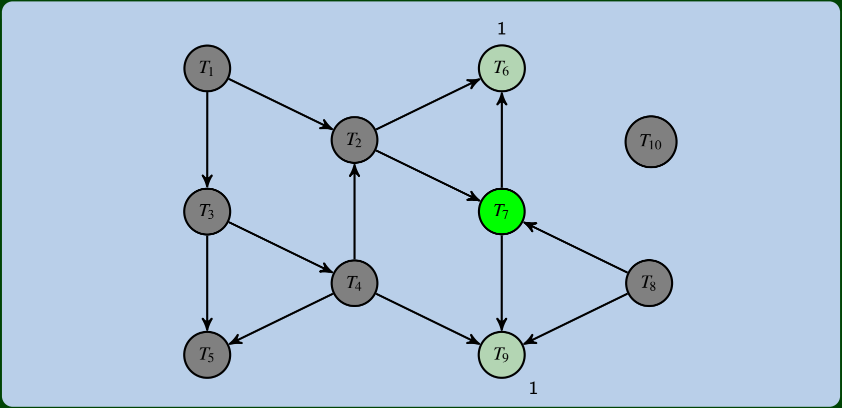


`output`

o[0]	o[1]	o[2]	o[3]	o[4]	o[5]	o[6]	o[7]	o[8]	o[9]
T_1	T_8	T_{10}	T_3	T_4	T_2	T_5			

Topologically Sorting A DAG (with a Queue)

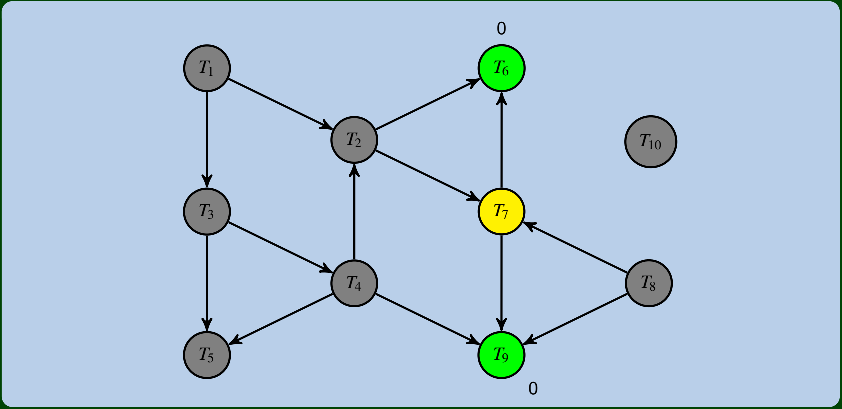
worklist ← $\boxed{T_7}$ ←



output

o[0]	o[1]	o[2]	o[3]	o[4]	o[5]	o[6]	o[7]	o[8]	o[9]
T_1	T_8	T_{10}	T_3	T_4	T_2	T_5			

worklist ←

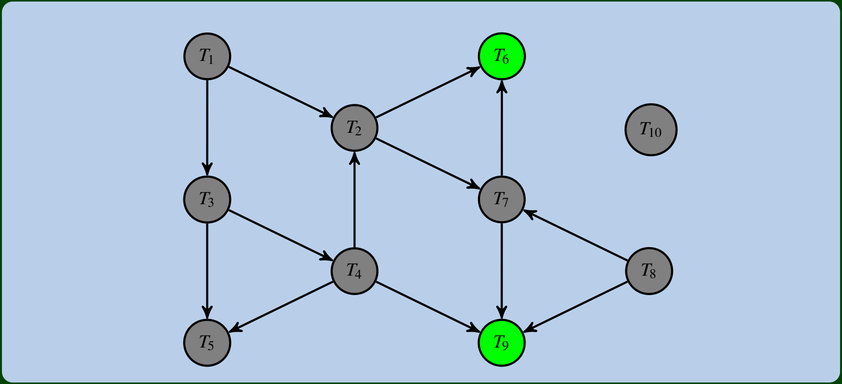


`output`

o[0]	o[1]	o[2]	o[3]	o[4]	o[5]	o[6]	o[7]	o[8]	o[9]
T_1	T_8	T_{10}	T_3	T_4	T_2	T_5	T_7		

Topologically Sorting A DAG (with a `Queue`)

worklist ← | T_6 | T_9 | ←

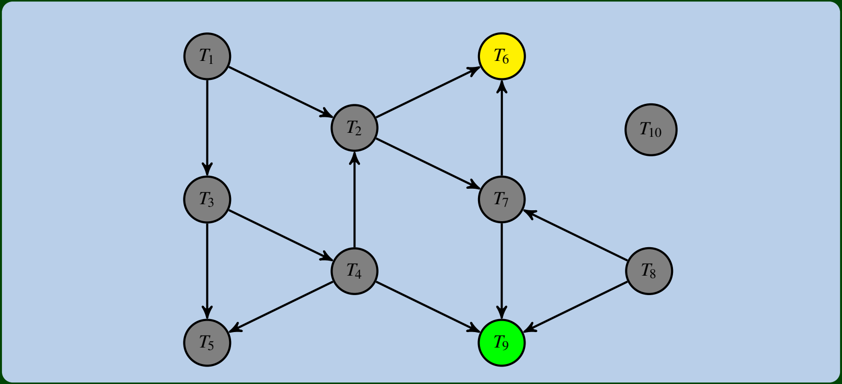


`output`

T_1	T_8	T_{10}	T_3	T_4	T_2	T_5	T_7		
o[0]	o[1]	o[2]	o[3]	o[4]	o[5]	o[6]	o[7]	o[8]	o[9]

Topologically Sorting A DAG (with a `Queue`)

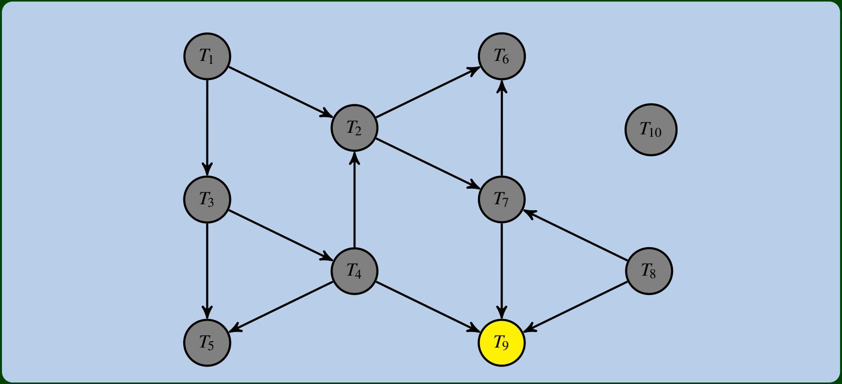
worklist ← T_9 ←



`output`

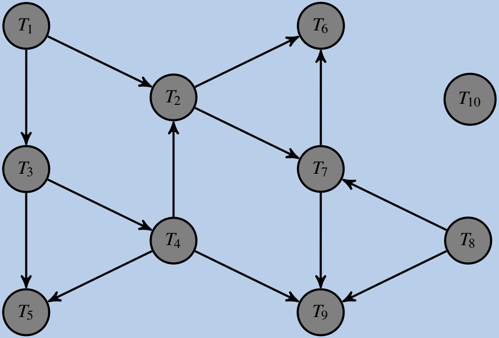
T_1	T_8	T_{10}	T_3	T_4	T_2	T_5	T_7	T_6	
o[0]	o[1]	o[2]	o[3]	o[4]	o[5]	o[6]	o[7]	o[8]	o[9]

worklist ←



`output`

T_1	T_8	T_{10}	T_3	T_4	T_2	T_5	T_7	T_6	T_9
`o[0]`	`o[1]`	`o[2]`	`o[3]`	`o[4]`	`o[5]`	`o[6]`	`o[7]`	`o[8]`	`o[9]`



T_1
T_2
T_3
T_4
T_5
T_6
T_7
T_8
T_9
T_{10}

Analyzing Topological Sort

What happens if there is a cycle?

Our worklist will be empty before we've processed all of the vertices. (e.g., "there are no nodes ready to print next, but we haven't gone through all of them)
In this case: our algorithm should throw a "not a DAG exception".

Runtime?

- Setup: We follow every edge for every vertex: $\mathcal{O}(|V|+|E|)$
- We add/remove each vertex from the work list once: $\mathcal{O}(|V|)$
- We decrement each indegree until zero (once for each edge): $\mathcal{O}(|E|)$
- So, overall, it's graph linear!