CSE 332: Spanning Trees

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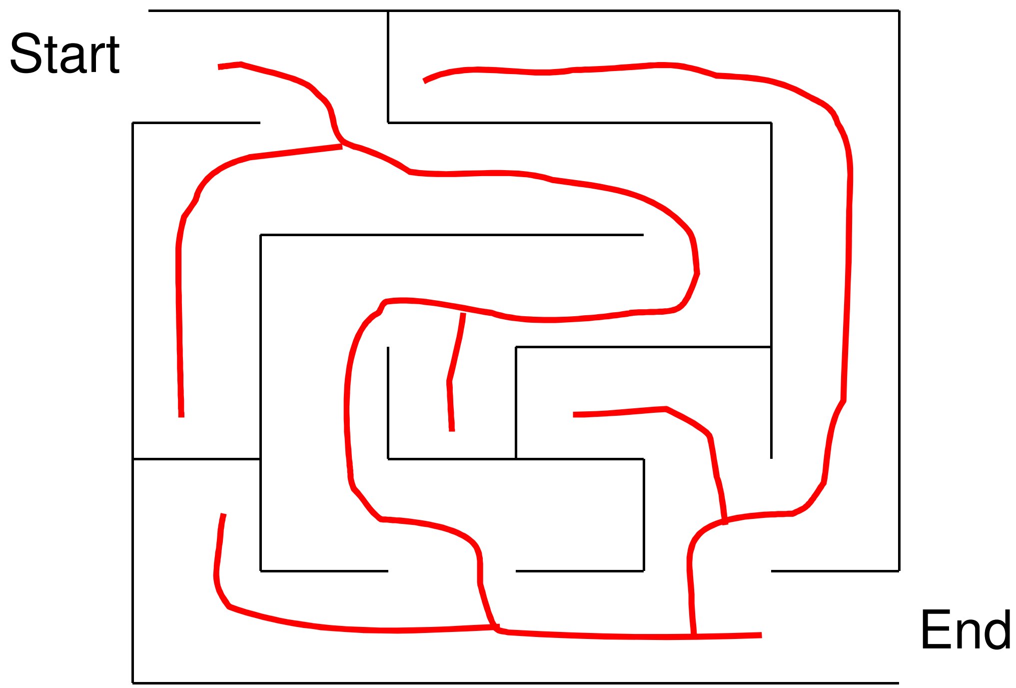
Winter 2014

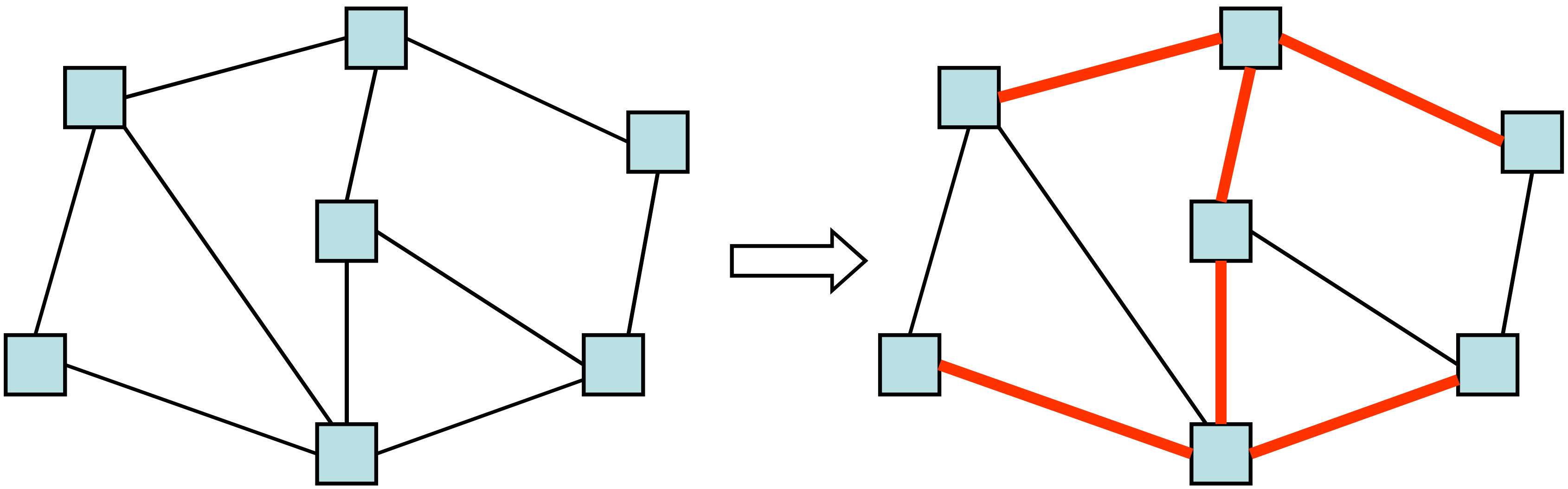
Announcements

- HW3 part 3 due Thursday night
- Final exam topics posted online
  - also sample final
  - covers everything except NP-completeness
  - closed book, notes
  - 4:30 or 6:30 on Monday (attend either one)

A Hidden Tree



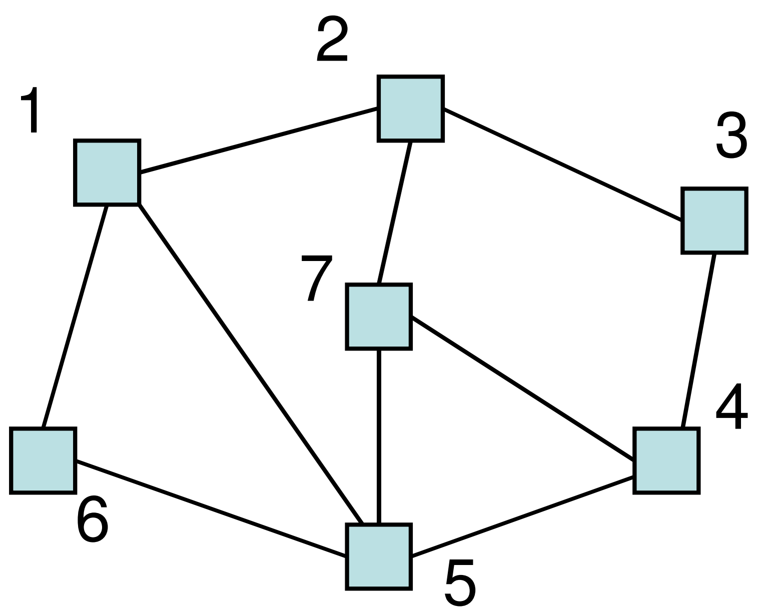
Spanning Tree in a Graph



Spanning tree
- Connects all the vertices
- No cycles

Undirected Graph

- G = (V,E)
  - V is a set of vertices (or nodes)
  - E is a set of unordered pairs of vertices

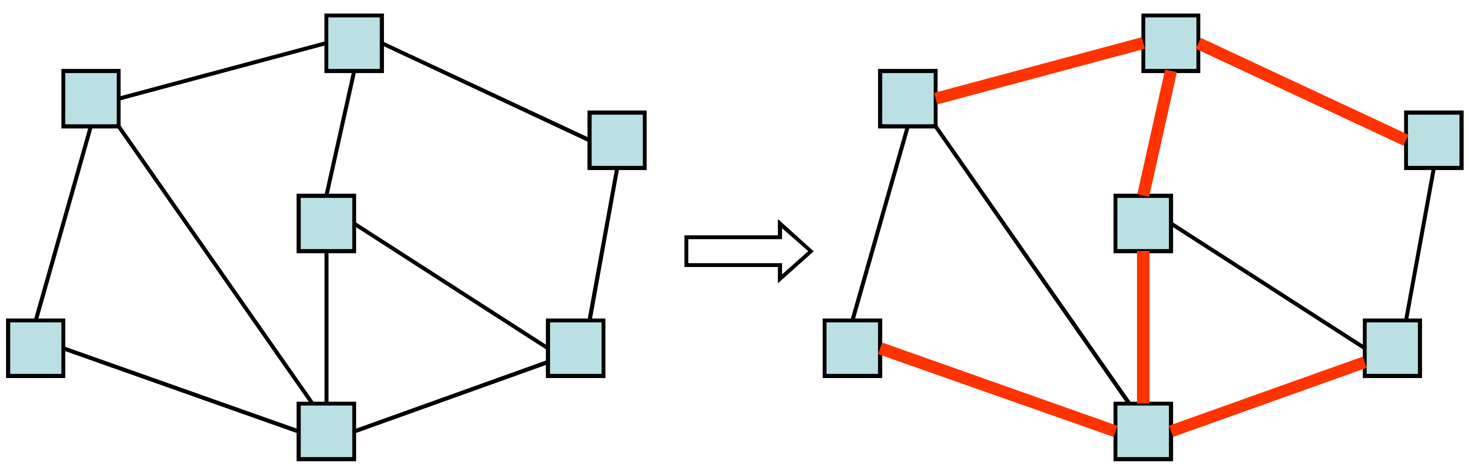


V = {1,2,3,4,5,6,7}
E = {(1,2),(1,6),(1,5),(2,7),(2,3),
(3,4),(4,7),(4,5),(5,6)}

2 and 3 are adjacent
2 is incident to edge (2,3)

Spanning Tree Problem

- Input: An undirected graph G = (V,E). G is connected.
- Output: **T** ⊂ E such that
  – (V,**T**) is a connected graph
  – (V,**T**) has no cycles



Spanning Tree Algorithm

first vertex

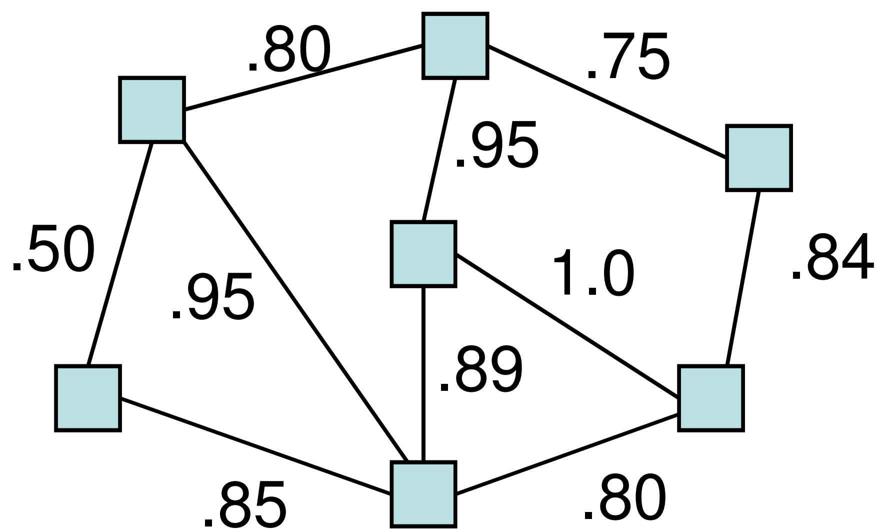
```
ST(Vertex i) {
    mark i;
    for each j adjacent to i  {
        if (j is unmarked) {
           Add (i,j) to T;
           ST(j);
        }
    }
}
```

```
Main( ) {
  T = empty set;
  ST(1);
}
```

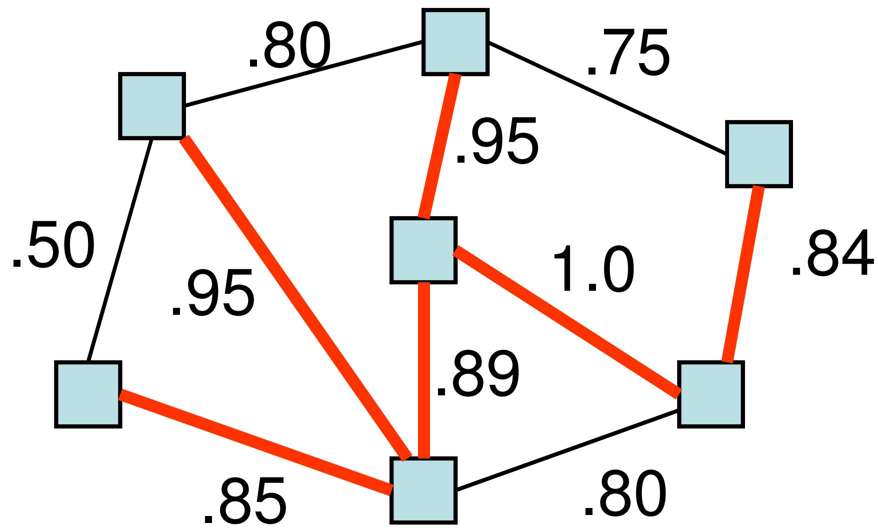
Best Spanning Tree

Finding a reliable routing subnetwork:

- edge cost = probability that it won't fail
- Find the spanning tree that is least likely to fail



Example of a Spanning Tree



Probability of success = .85 x .95 x .89 x .95 x 1.0 x .84
= .5735

Minimum Spanning Trees

Given an undirected graph **G**=(**V**,**E**), find a graph **G'=(V, E')** such that:

- E' is a subset of E
- |E'| = |V| - 1
- G' is connected
- $\sum_{(u,v)\in E'} c_{uv}$ is minimal

G' is a **minimum spanning tree**.

Applications: wiring a house, power grids, Internet connections

Minimum Spanning Tree Problem

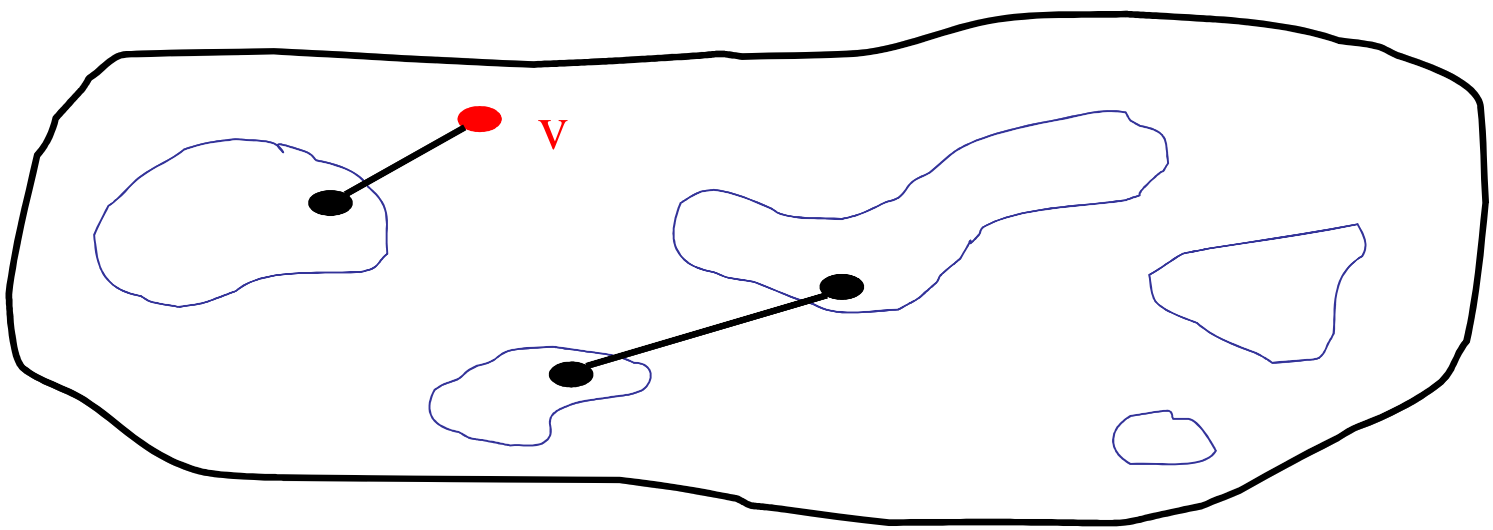
- Input: Undirected Graph G = (V,E) and C(e) is the cost of edge e.
- Output: A spanning tree T with minimum total cost. That is: T that minimizes

$$C(T) = \sum_{e \in T} C(e)$$

Kruskal's MST Algorithm

Idea: Grow a forest out of edges that do not create a cycle. Pick an edge with the smallest weight.

G=(V,E)



Kruskal's Algorithm for MST

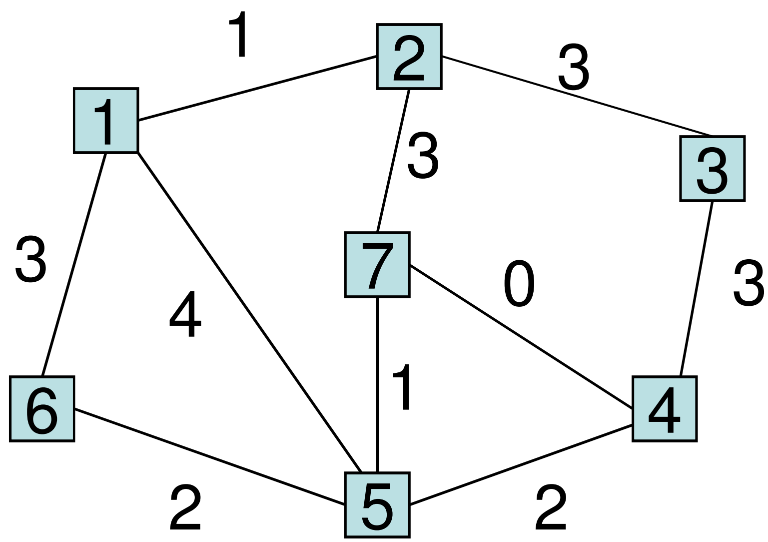
An *edge-based* greedy algorithm

Builds MST by greedily adding edges

1. Initialize with
   - empty MST
   - all vertices marked unconnected
   - all edges unmarked
2. While there are still unmarked edges
   a. Pick the <u>lowest cost edge</u> **`(u,v)`** and mark it
   b. If **`u`** and **`v`** are not already connected, add **`(u,v)`** to the MST and mark **`u`** and **`v`** as connected to each other

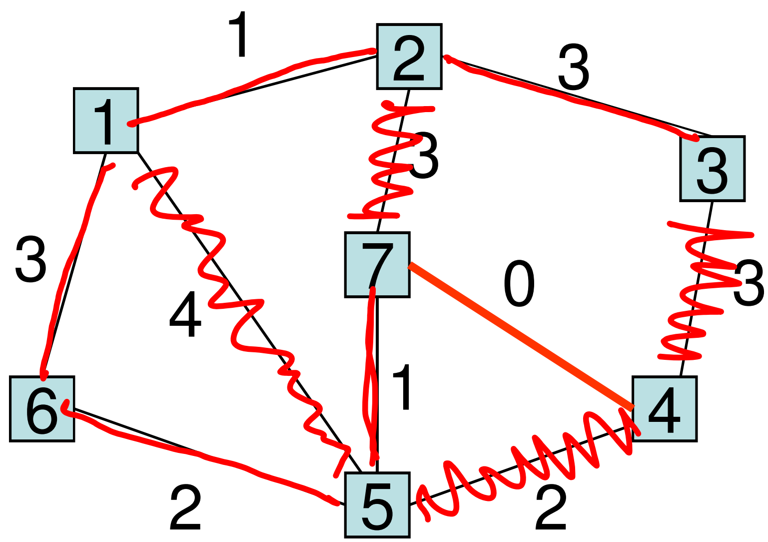
Sound familiar?

Example of Kruskal 1



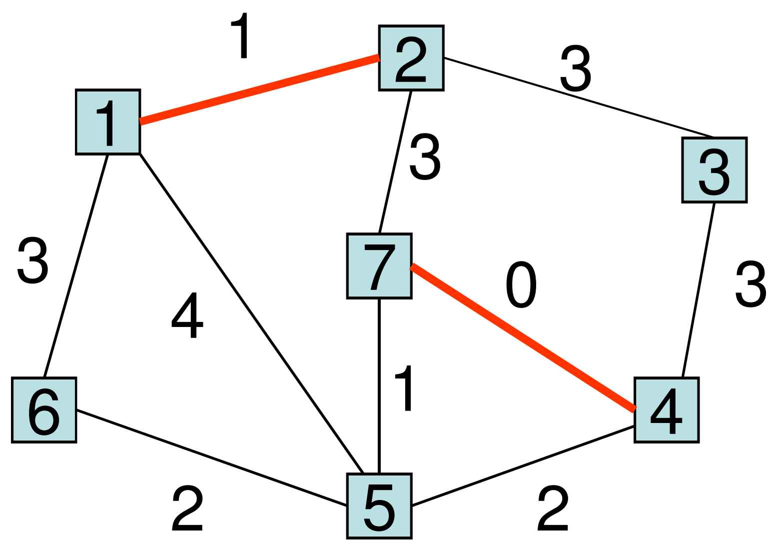
(7,4)	(2,1)	(7,5)	(5,6)	(5,4)	(1,6)	(2,7)	(2,3)	(3,4)	(1,5)
0	1	1	2	2	3	3	3	3	4

Example of Kruskal 2



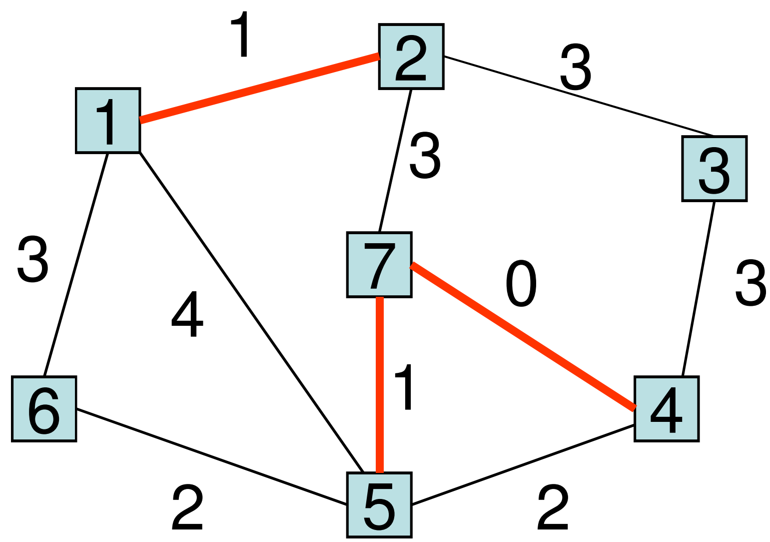
~~(7,4)~~	(2,1)	(7,5)	(5,6)	(5,4)	(1,6)	(2,7)	(2,3)	(3,4)	(1,5)
~~0~~	1	1	2	2	3	3	3	3	4

Example of Kruskal 2



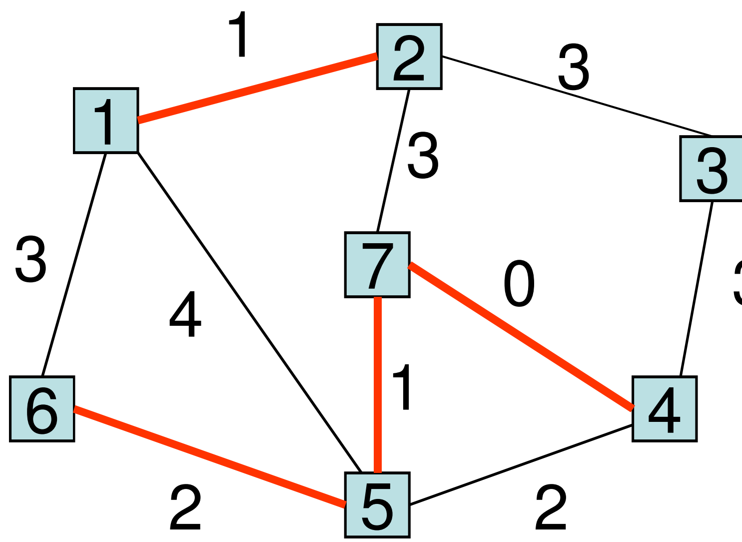
~~(7,4)~~	~~(2,1)~~	(7,5)	(5,6)	(5,4)	(1,6)	(2,7)	(2,3)	(3,4)	(1,5)
~~0~~	~~1~~	1	2	2	3	3	3	3	4

Example of Kruskal 3



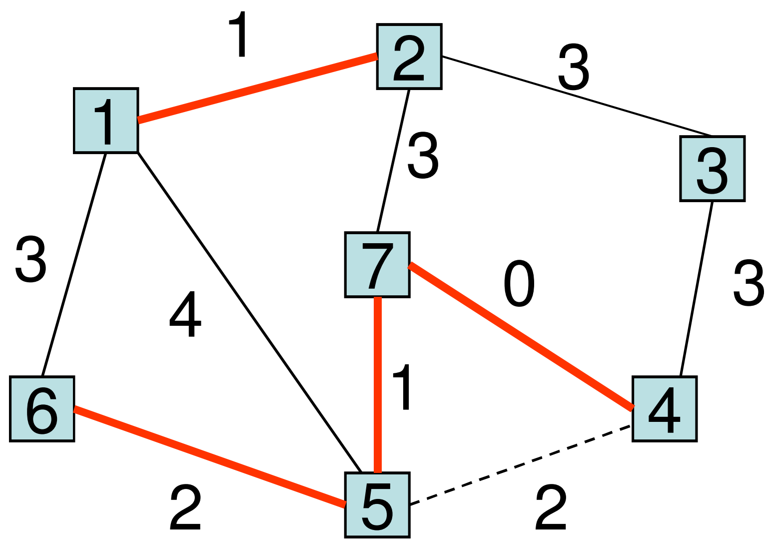
~~(7,4)~~	~~(2,1)~~	~~(7,5)~~	(5,6)	(5,4)	(1,6)	(2,7)	(2,3)	(3,4)	(1,5)
~~0~~	~~1~~	~~1~~	2	2	3	3	3	3	4

Example of Kruskal 4



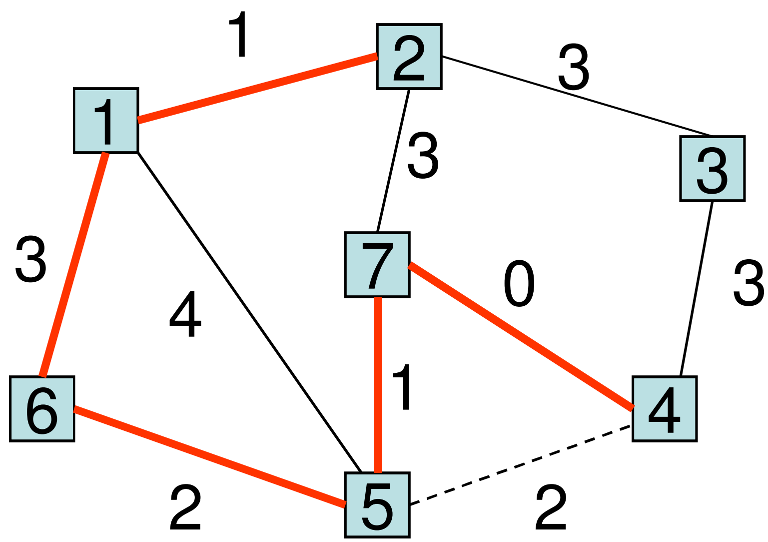
~~(7,4)~~	~~(2,1)~~	~~(7,5)~~	~~(5,6)~~	(5,4)	(1,6)	(2,7)	(2,3)	(3,4)	(1,5)
~~0~~	~~1~~	~~1~~	~~2~~	2	3	3	3	3	4

Example of Kruskal 5



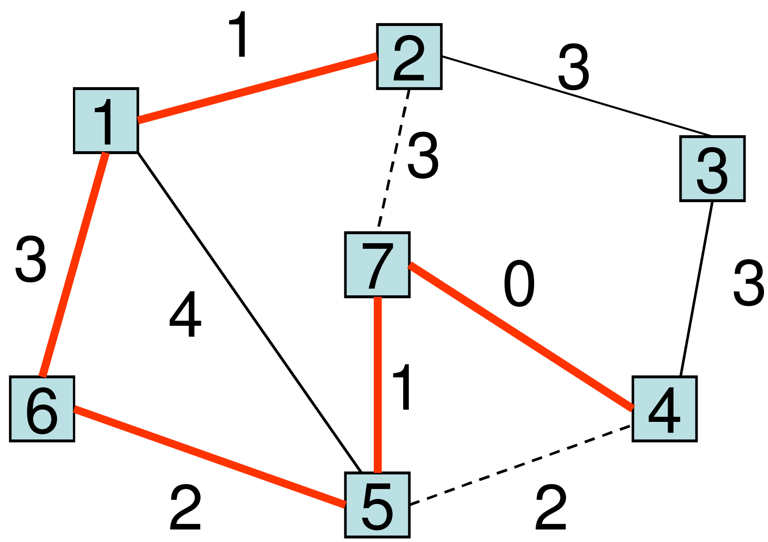
~~(7,4)~~	~~(2,1)~~	~~(7,5)~~	~~(5,6)~~	~~(5,4)~~	(1,6)	(2,7)	(2,3)	(3,4)	(1,5)
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	3	3	3	3	4

Example of Kruskal 6



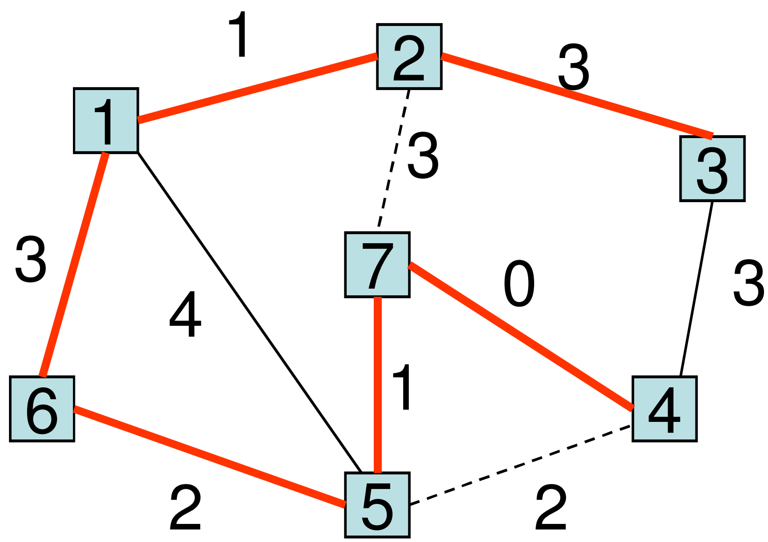
~~(7,4)~~	~~(2,1)~~	~~(7,5)~~	~~(5,6)~~	~~(5,4)~~	~~(1,6)~~	(2,7)	(2,3)	(3,4)	(1,5)
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	~~3~~	3	3	3	4

Example of Kruskal 7



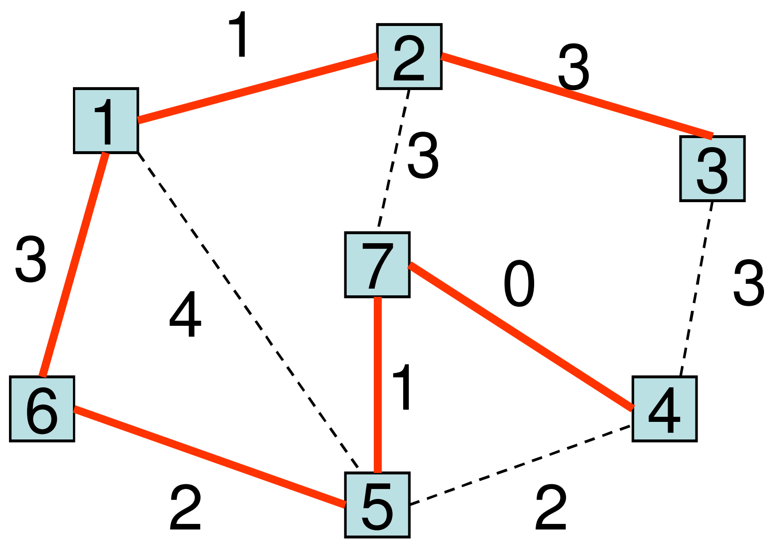
~~(7,4)~~	~~(2,1)~~	~~(7,5)~~	~~(5,6)~~	~~(5,4)~~	~~(1,6)~~	~~(2,7)~~	(2,3)	(3,4)	(1,5)
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	~~3~~	~~3~~	3	3	4

Example of Kruskal 7



~~(7,4)~~	~~(2,1)~~	~~(7,5)~~	~~(5,6)~~	~~(5,4)~~	~~(1,6)~~	~~(2,7)~~	~~(2,3)~~	(3,4)	(1,5)
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	~~3~~	~~3~~	~~3~~	3	4

Example of Kruskal 8,9



~~(7,4)~~	~~(2,1)~~	~~(7,5)~~	~~(5,6)~~	~~(5,4)~~	~~(1,6)~~	~~(2,7)~~	~~(2,3)~~	~~(3,4)~~	~~(1,5)~~
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	~~3~~	~~3~~	~~3~~	~~3~~	~~4~~

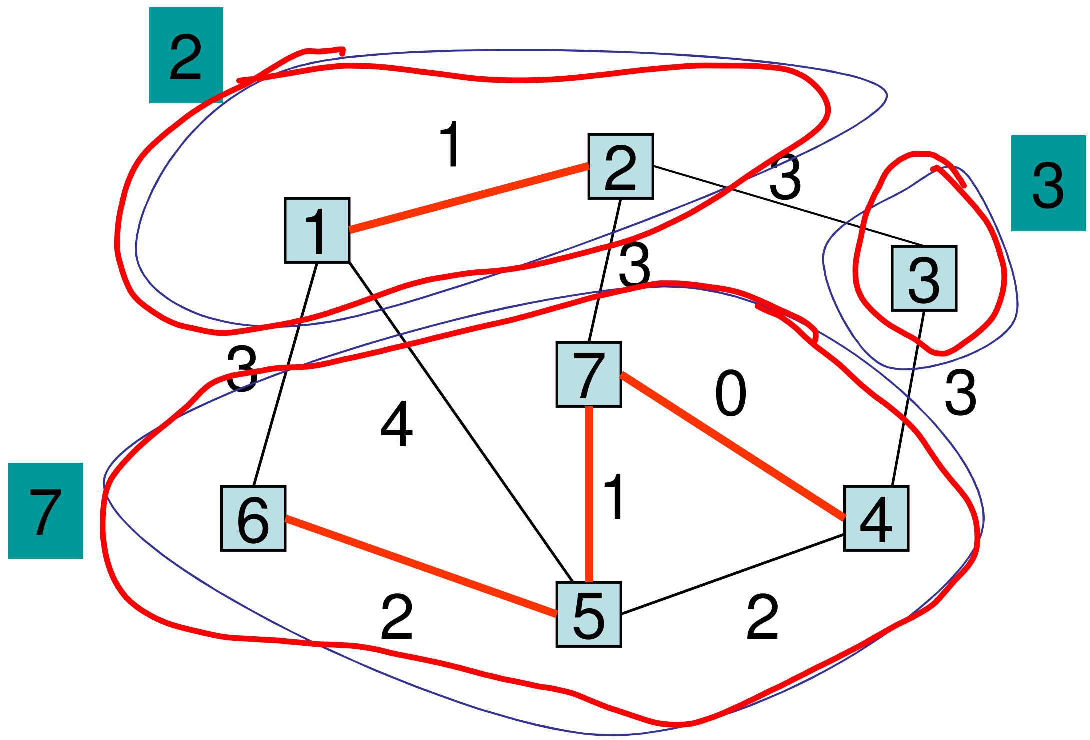
Data Structures for Kruskal

- Sorted edge list

(7,4)	(2,1)	(7,5)	(5,6)	(5,4)	(1,6)	(2,7)	(2,3)	(3,4)	(1,5)
0	1	1	2	2	3	3	3	3	4

- Disjoint Union / Find
  - Union(*a,b*) - merge the disjoint sets named by *a* and *b*
  - Find(a) returns the name of the set containing *a*

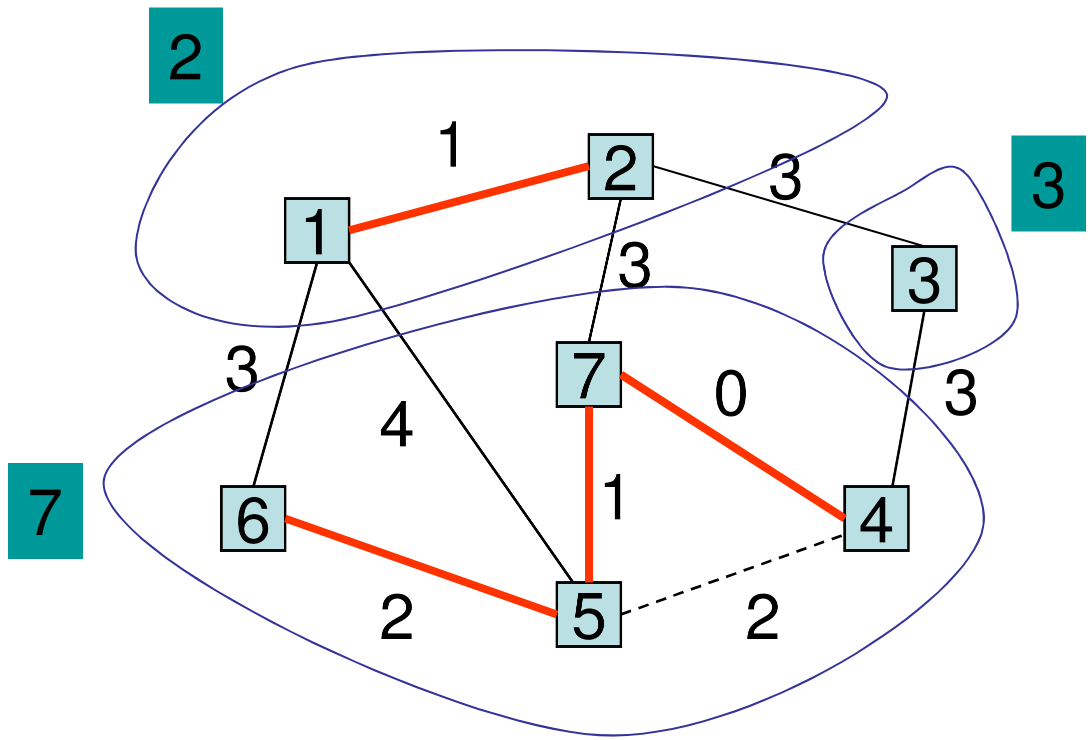
Example of DU/F 1



Find(5) = 7
Find(4) = 7

~~(7,4)~~	~~(2,1)~~	~~(7,5)~~	~~(5,6)~~	(5,4)	(1,6)	(2,7)	(2,3)	(3,4)	(1,5)
~~0~~	~~1~~	~~1~~	~~2~~	2	3	3	3	3	4

Example of DU/F 2



Find(1) = 2
Find(6) = 7

~~(7,4)~~	~~(2,1)~~	~~(7,5)~~	~~(5,6)~~	~~(5,4)~~	(1,6)	(2,7)	(2,3)	(3,4)	(1,5)
~~0~~	~~1~~	~~1~~	~~2~~	~~2~~	3	3	3	3	4

Kruskal's Algorithm with DU / F

```
Sort the edges by increasing cost;
Initialize A to be empty;
for each edge (i,j) chosen in increasing order do
    u := Find(i);
    v := Find(j);
    if not(u = v) then
        add (i,j) to A;
        Union(u,v);
```

This algorithm will work, but it goes through all the edges.

Is this always necessary?

Kruskal code

```
void Graph::kruskal(){
  int edgesAccepted = 0;
  DisjSet s(NUM_VERTICES);

  while (edgesAccepted < NUM_VERTICES – 1){
    e = smallest weight edge not deleted yet;
    // edge e = (u, v)
    uset = s.find(u);
    vset = s.find(v);
    if (uset != vset){
      edgesAccepted++;
      s.unionSets(uset, vset);
    }
  }
}
```

|V| ops to init. sets

|E| heap ops

$|E| \log |E|$

2|E| finds

$O(|E|)$

|V| unions

$O(|V|)$

Total Cost: $O(|E| \log |E| + |E| + |V|) = O(|E| \log |E|)$

Kruskal's Algorithm: Correctness

It clearly generates a spanning tree. Call it T_K.

Suppose T_K is *not* minimum:

- Pick another spanning tree T_{min} with *lower cost* than T_K
- Pick the smallest edge $e_1=(u,v)$ in T_K that is not in T_{min}
- T_{min} already has a path p in T_{min} from u to v
  - $\Rightarrow$ Adding e_1 to T_{min} will create a cycle in T_{min}
- Pick an edge e_2 in p that Kruskal's algorithm considered *after* adding e_1 (must exist: u and v unconnected when e_1 considered)
  - $\Rightarrow$ $\text{cost}(e_2) \geq \text{cost}(e_1)$
  - $\Rightarrow$ can replace e_2 with e_1 in T_{min} without increasing cost!
- Keep doing this until T_{min} is identical to T_K
  - $\Rightarrow$ T_K must also be minimal – contradiction!