CSE 332: Data Structures
Disjoint Set Union/Find

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Announcements

- Last week of the quarter – lots of deadlines
- Exam Monday

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Disjoint Set ADT

- Data: set of pairwise **disjoint sets**.
- Required operations
  - **Union** – merge two sets to create their union
  - **Find** – determine which set an item appears in
- Each set has a unique name: one of its members (for convenience)
  - {3,5,7} , {4,2,8}, {9}, {1,6}
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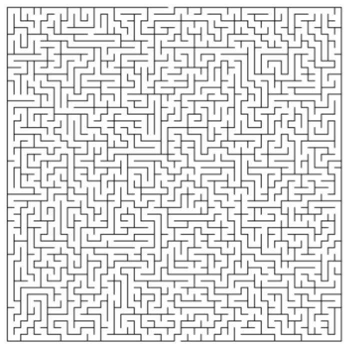
Application: Building Mazes



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Algorithm

- S = set of sets of connected cells
  - Initialize to {{1}, {2}, …, {n}}
- W = set of walls
  - Initialize to set of all walls {{1,2},{1,7}, …}
- Maze = set of walls in maze (initially empty)

```
While there is more than one set in S
  Pick a random non-boundary wall (x,y) and remove from W
  u = Find(x);
  v = Find(y);
  if u ≠ v then
    Union(u,v)
  else
    Add wall (x,y) to Maze
Add remaining members of W to Maze
```

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Tree-based Approach

Each set is a tree

- Root of each tree is the set name.
- Represent: {3,5,7} , {4,2,8}, {9}, {1,6}
- Support: find(x), union(x,y)

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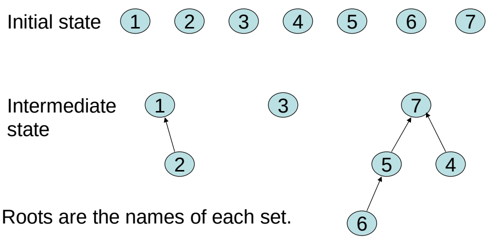
Up-Tree for DS Union/Find

Observation: we will only traverse these trees upward from any given node to find the root.

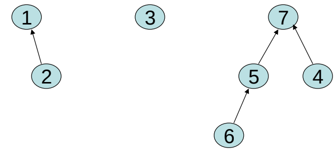
Idea: *reverse* the pointers (make them point up from child to parent). The result is an **up-tree**.



Roots are the names of each set.

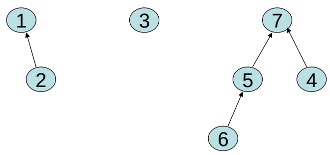
Find Operation

Find(x) follow x to the root and return the root.



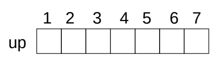
Union Operation

Union(i, j) - assuming i and j roots, point i to j.

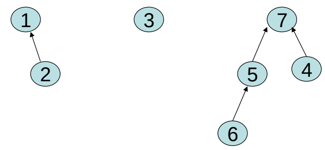


Simple Implementation

- Array of indices



up[x] = -1 means x is a root.



Implementation

```
void Union(int x, int y) {
  assert(up[x]<0 && up[y]<0);
  up[x] = y;
}
```

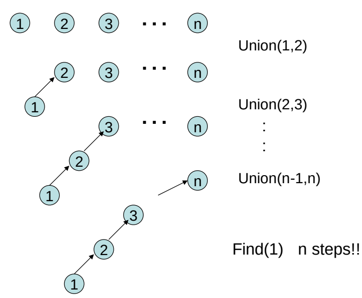
```
int Find(int x) {
  while(up[x] >= 0) {
    x = up[x];
  }
  return x;
}
```

runtime for Union:

runtime for Find:

Amortized complexity is no better.

A Bad Case



Two Big Improvements

Can we do better? *Yes!*

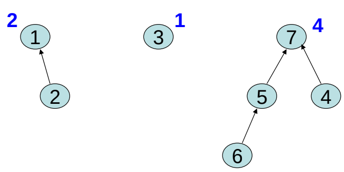
1. Union-by-size
   - Improve Union so that *Find* only takes worst case time of $\Theta(\log n)$.
2. Path compression
   - Improve Find so that, with Union-by-size, Find takes amortized time of <u>almost</u> $\Theta(1)$.

Union-by-Size

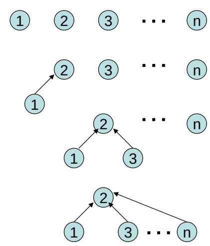
Union-by-size

– Always point the smaller tree to the root of the larger tree

S-Union(7,1)



Example Again



S-Union(1,2)

S-Union(2,3)

:

:

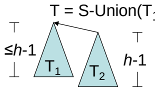
S-Union(n-1,n)

Find(1) constant time

Analysis of Union-by-Size

- Theorem: With union-by-size an up-tree of height h has size at least 2^h.
- Proof by induction
  - Base case: $h = 0$. The up-tree has one node, $2^0 = 1$
  - Inductive hypothesis: Assume true for h-1
  - Observation: tree gets taller only as a result of a union.

$T = \text{S-Union}(T_1,T_2)$



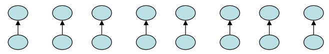
Analysis of Union-by-Size

- What is worst case complexity of Find(x) in an up-tree forest of n nodes?

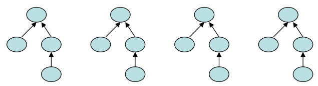
- (Amortized complexity is no better.)

Worst Case for Union-by-Size

n/2 Unions-by-size



n/4 Unions-by-size



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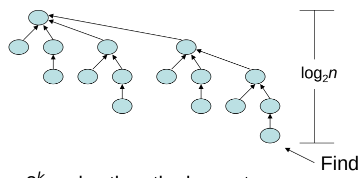
Example of Worst Cast (cont')

After n -1 = n/2 + n/4 + …+ 1 Unions-by-size



If there are $n = 2^k$ nodes then the longest path from leaf to root has length k.

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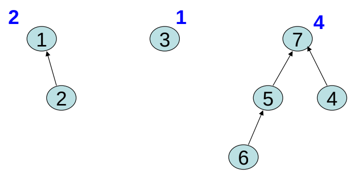
Array Implementation



Can store separate size array:

	1	2	3	4	5	6	7
up	-1	1	-1	7	7	5	-1
size	2		1				4

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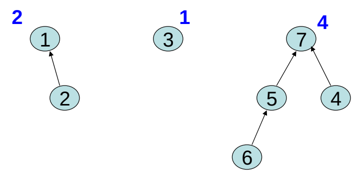
Elegant Array Implementation



Better, store sizes in the up array:

	1	2	3	4	5	6	7
up	-2	1	-1	7	7	5	-4

Negative up-values correspond to sizes of roots.

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Code for Union-by-Size

```
S-Union(i,j){
  // Collect sizes
  si = -up[i];
  sj = -up[j];

  // verify i and j are roots
  assert(si >=0 && sj >=0)
  // point smaller sized tree to
  // root of larger, update size
  if (si < sj) {
    up[i] = j;
    up[j] = -(si + sj);
  else {
    up[j] = i;
    up[i] = -(si + sj);
  }
}
```

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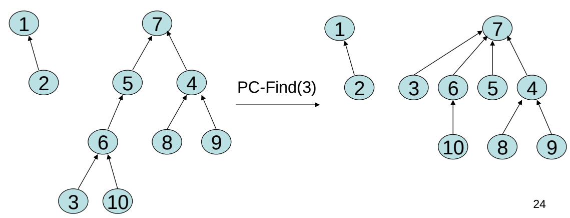
Path Compression

- To improve the amortized complexity, we'll introduce a new idea:
  – When going up the tree, *improve nodes on the path*!
- On a Find operation point all the nodes on the search path directly to the root. This is called "path compression."



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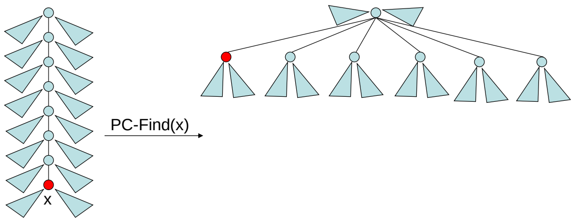
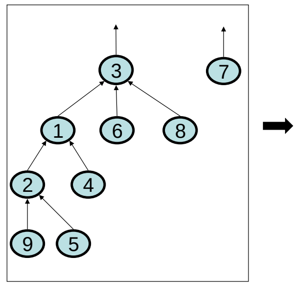
Self-Adjustment Works



Draw the result of Find(5):



Code for Path Compression Find

```
PC-Find(i) {
  //find root
  j = i;
  while (up[j] >= 0) {
    j = up[j];
  root = j;

  //compress path
  if (i != root) {
    parent = up[i];
    while (parent != root) {
      up[i] = root;
      i = parent;
      parent = up[parent];
    }
  }
  return(root)
}
```


Complexity of Union-by-Size + Path Compression

- Worst case time complexity for…
  - …a single Union-by-size is:
  - …a single PC-Find is:
- Time complexity for $m \geq n$ operations on n elements has been shown to be O(m log* n). [See Weiss for proof.]
  - Amortized complexity is then O(log* n)
  - What is log* ?

log* *n*

log* *n* = number of times you need to apply log to bring value down to at most 1

log* 2 = 1

log* 4 = log* 2^2 = 2

log* 16 = log* 2^{2^2} = 3 (log log log 16 = 1)

log* 65536 = log* $2^{2^{2^2}}$ = 4 (log log log log 65536 = 1)

log* 2^{65536} = …………… ≈ log* (2 x $10^{19,728}$) = 5

log * $n \leq 5$ for all reasonable n.

The Tight Bound

In fact, Tarjan showed the time complexity for $m \geq n$ operations on n elements is:

$$\Theta(m\ \alpha(m, n))$$

Amortized complexity is then $\Theta(\alpha(m, n))$.

What is $\alpha(m, n)$?

- Inverse of Ackermann's function.
- For reasonable values of m, n, grows even slower than log * n. So, it's even "more constant."

Proof is beyond scope of this class. A simple algorithm can lead to incredibly hardcore analysis!