CSE 332: Disjoint Set Union/Find (and finishing Dijkstra's algorithm)

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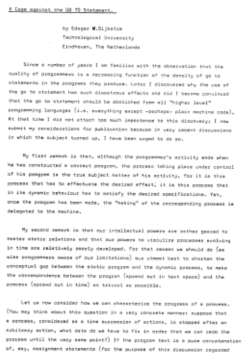
Announcements

- Reading for this lecture: Chapter 8.

http://www.cs.utexas.edu/users/EWD/

- Edsger Wybe Dijkstra was one of the most influential members of computing science's founding generation. Among the domains in which his scientific contributions are fundamental are
  - algorithm design
  - programming languages
  - program design
  - operating systems
  - distributed processing
  - formal specification and verification
  - design of mathematical arguments

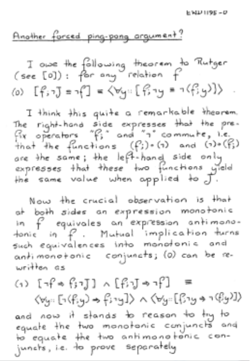


A Case against the GO TO Statement.

by Edsger W.Dijkstra
Technological University
Eindhoven, The Netherlands

Since a number of years I am familiar with the observation that the quality of programmers is a decreasing function of the density of go to statements in the programs they produce. Later I discovered why the use of the go to statement has such disastrous effects and did I become convinced that the go to statement should be abolished from all "higher level" programming languages (i.e. everything except -perhaps- plain machine code). At that time I did not attach too much importance to this discovery; I now submit my considerations for publication because in very recent discussions in which the subject turned up, I have been urged to do so.

My first remark is that, although the programmer's activity ends when he has constructed a correct program, the process taking place under control of his program is the true subject matter of his activity, for it is this process that has to effectuate the desired effect; it is this process that in its dynamic behaviour has to satisfy the desired specifications. Yet, once the program has been made, the "making" of the corresponding process is delegated to the machine.



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Another forced ping-pong argument?

I owe the following theorem to Rutger (see [0]): for any relation f

(0) $[f;\neg J \equiv \neg f] \equiv \langle \forall y :: [f;\neg y \equiv \neg(f;y)]\rangle$.

I think this quite a remarkable theorem. The right-hand side expresses that the prefix operators "f;" and "¬" commute, i.e. that the functions (f;)∘(¬) and (¬)∘(f;) are the same; the left-hand side only expresses that these two functions yield the same value when applied to J.

Now the crucial observation is that at both sides an expression monotonic in f equivales an expression antimonotonic in f. Mutual implication turns such equivalences into monotonic and antimonotonic conjuncts; (0) can be rewritten as

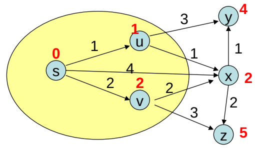
(1) $[\neg f \Rightarrow f;\neg J] \wedge [f;\neg J \Rightarrow \neg f] \equiv$
$\langle \forall y :: [\neg(f;y) \Rightarrow f;\neg y]\rangle \wedge \langle \forall y :: [f;\neg y \Rightarrow \neg(f;y)]\rangle$

and now it stands to reason to try to equate the two monotonic conjuncts and to equate the two antimonotonic conjuncts, i.e. to prove separately

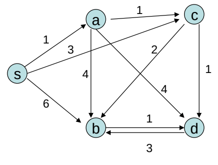
Assume all edges have non-negative cost

Dijkstra's Algorithm

S = {}; d[s] = 0; d[v] = infinity for v != s
While S != V
  Choose v in V-S with minimum d[v]
  Add v to S
  For each w in the neighborhood of v
    d[w] = min(d[w], d[v] + c(v, w))



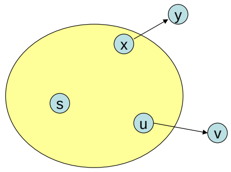
Simulate Dijkstra's algorithm (strarting from s) on the graph



Round	Vertex Added	s	a	b	c	d
1						
2						
3						
4						
5						

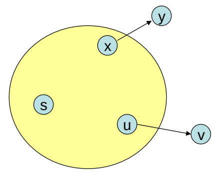
Correctness Proof

- Elements in S have the correct label
- Key to proof: when v is added to S, it has the correct distance label.



Proof

- Let v be a vertex in V-S with minimum d[v]
- Let P_v be a path of length d[v], with an edge (u,v)
- Let P be some other path to v. Suppose P first leaves S on the edge (x, y)
  - $P = P_{sx} + c(x,y) + P_{yv}$
  - $Len(P_{sx}) + c(x,y) >= d[y]$
  - $Len(P_{yv}) >= 0$
  - $Len(P) >= d[y] + 0 >= d[v]$



Union-Find Data Structure

- ADT Definition
- How it's implemented with pointers
- Optimizations
- Results of analysis
  - (Some of the strangest mathematics in CS)

Making Connections

You have a set of nodes (numbered 1-9) on a network. You are given a sequence of pairwise connections between them:

3-5
4-2
1-6
5-7
4-8
3-7

Q: Are nodes 2 and 4 (indirectly) connected?
Q: How about nodes 3 and 8?
Q: Are any of the paired connections redundant due to indirect connections?
Q: How many sub-networks do you have?

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Making Connections

Answering these questions is much easier if we create disjoint sets of nodes that are connected:

Start: {1} {2} {3} {4} {5} {6} {7} {8} {9}
3-5
4-2
1-6
5-7
4-8
3-7

Q: Are nodes 2 and 4 (indirectly) connected?
Q: How about nodes 3 and 8?
Q: Are any of the paired connections redundant due to indirect connections?
Q: How many sub-networks do you have?

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Applications of Disjoint Sets

Maintaining disjoint sets in this manner arises in a number of areas, including:
- Networks
- Transistor interconnects
- Compilers
- Image segmentation
- Building mazes (this lecture)
- Graph problems
  - Minimum Spanning Trees (upcoming topic in this class)

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Disjoint Set ADT

- Data: set of pairwise **disjoint sets**.
- Required operations
  - **Union** – merge two sets to create their union
  - **Find** – determine which set an item appears in
- A common operation sequence:
  - Connect two elements if not already connected:
    if (Find(x) != Find(y)) then Union(x,y)

Disjoint Sets and Naming

- Maintain a set of pairwise disjoint sets.
  - {3,5,7} , {4,2,8}, {9}, {1,6}
- Each set has a unique name: one of its members (for convenience)
  - {3,5,7} , {4,2,8}, {9}, {1,6}

Union

- Union(x,y) – take the union of two sets named x and y
  - {3,5,7} , {4,2,8}, {9}, {1,6}
  - Union(5,1)
    {3,5,7,1,6}, {4,2,8}, {9},

Find

- Find(x) – return the name of the set containing x.
  - {3,5,7,1,6}, {4,2,8}, {9},
  - Find(1) = 5
  - Find(4) = 8

Example

S
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}

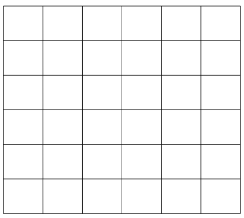
Find(8) = 7
Find(14) = 20

Union(7,20)

S
{1,2,7,8,9,13,19,14,20 26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}

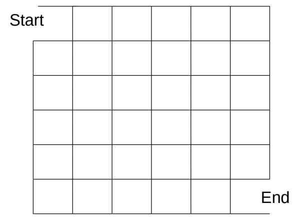
Nifty Application: Building Mazes

Idea: Build a random maze by erasing walls.



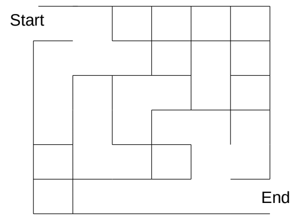
Building Mazes

- Pick Start and End



Building Mazes

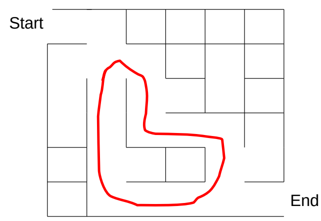
- Repeatedly pick random walls to delete.



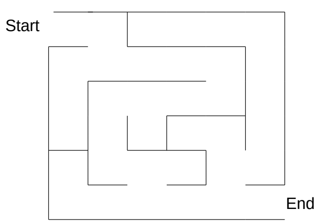
Desired Properties

- None of the boundary is deleted (except at “start” and “end”).
- Every cell is reachable from every other cell.
- There are no cycles – no cell can reach itself by a path unless it retraces some part of the path.

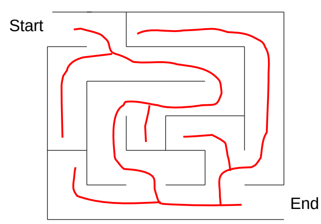
A Cycle



A Good Solution



A Hidden Tree



Number the Cells

We start with disjoint sets S ={ {1}, {2}, {3}, {4},… {36} }.
We have all possible walls between neighbors
W ={ (1,2), (1,7), (2,8), (2,3), … } 60 walls total.

Start

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

End

Idea: Union-find operations will be done on cells.

Maze Building with Disjoint Union/Find

Algorithm sketch:

1. Choose wall at random.
   → *Boundary walls are not in wall list, so left alone*
2. Erase wall if the neighbors are in disjoint sets.
   → *Avoids cycles*
3. Take union of those sets.
4. Go to 1, iterate until there is only one set.
   → *Every cell reachable from every other cell.*

Pseudocode

- S = set of sets of connected cells
  - Initialize to {{1}, {2}, …, {n}}
- W = set of walls
  - Initialize to set of all walls {{1,2},{1,7}, …}
- Maze = set of walls in maze (initially empty)

```
While there is more than one set in S
  Pick a random non-boundary wall (x,y) and remove from W
  u = Find(x);
  v = Find(y);
  if u ≠ v then
    Union(u,v)
  else
    Add wall (x,y) to Maze
Add remaining members of W to Maze
```


Example Step

Pick (8,14)

Start

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

End

S
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
.
.
{22,23,24,29,30,32
33,34,35,36}

Example

S
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}

Find(8) = 7
Find(14) = 20

Union(7,20)

S
{1,2,7,8,9,13,19,14,20 26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}

Example

Pick (19,20)

Start

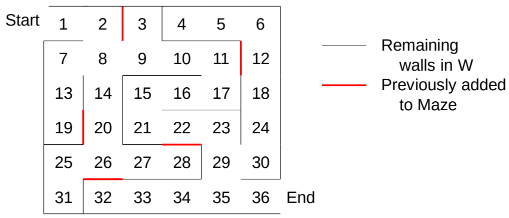
1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

End

S
{1,2,7,8,9,13,19
14,20,26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}

Example at the End

S
{1,2,3,4,5,6,7,… 36}



Data structure for disjoint sets?

- Represent: {3,5,7} , {4,2,8}, {9}, {1,6}
- Support: find(x), union(x,y)

Union/Find Trade-off

- Known result:
  – Find and Union cannot *both* be done in worst-case *O*(1) time with any data structure.
- We will instead aim for good *amortized* complexity.
- For *m* operations on *n* elements:
  – Target complexity: *O*(*m*) *i.e.* *O*(1) amortized

Tree-based Approach

Each set is a tree

- Root of each tree is the set name.
- Allow large fanout (why?)

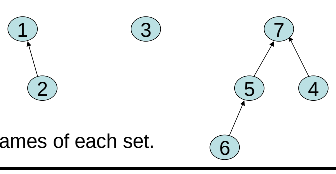
Up-Tree for DS Union/Find

Observation: we will only traverse these trees upward from any given node to find the root.

Idea: *reverse* the pointers (make them point up from child to parent). The result is an **up-tree**.

Initial state

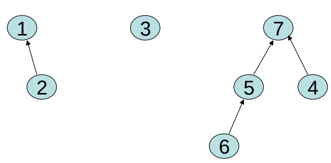
Intermediate state



Roots are the names of each set.

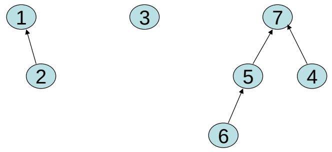
Find Operation

Find(x) follow x to the root and return the root.



Union Operation

Union(i, j) - assuming i and j roots, point i to j.



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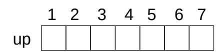
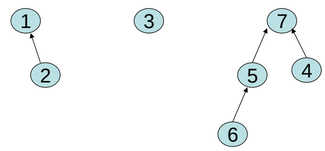
Simple Implementation

- Array of indices



up[x] = -1 means
x is a root.



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Implementation

```
void Union(int x, int y) {
  assert(up[x]<0 && up[y]<0);
  up[x] = y;
}
```

```
int Find(int x) {
  while(up[x] >= 0) {
    x = up[x];
  }
  return x;
}
```

runtime for Union:

runtime for Find:

Amortized complexity is no better.

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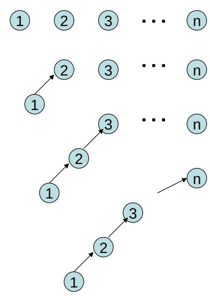
A Bad Case



Union(1,2)

Union(2,3)

:

:

Union(n-1,n)

Find(1) n steps!!

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Two Big Improvements

Can we do better? *Yes!*

1. Union-by-size
   - Improve Union so that *Find* only takes worst case time of Θ(log *n*).

2. Path compression
   - Improve Find so that, with Union-by-size, Find takes amortized time of <u>almost</u> Θ(1).

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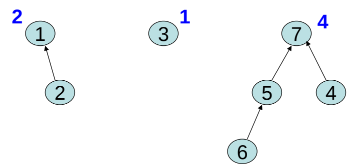
Union-by-Size

Union-by-size

– Always point the smaller tree to the root of the larger tree

S-Union(7,1)



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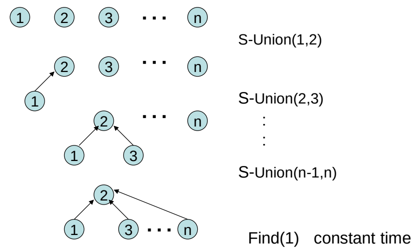
Example Again



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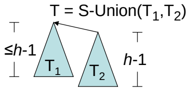
Analysis of Union-by-Size

- Theorem: With union-by-size an up-tree of height h has size at least 2^h.
- Proof by induction
  - Base case: $h = 0$. The up-tree has one node, $2^0 = 1$
  - Inductive hypothesis: Assume true for h-1
  - Observation: tree gets taller only as a result of a union.



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Analysis of Union-by-Size

- What is worst case complexity of Find(x) in an up-tree forest of n nodes?
- (Amortized complexity is no better.)

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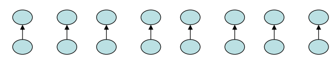
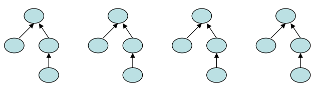
Worst Case for Union-by-Size

n/2 Unions-by-size



n/4 Unions-by-size



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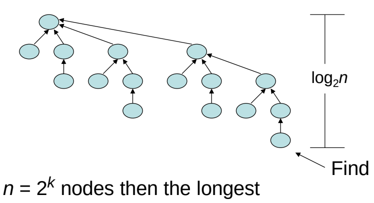
Example of Worst Cast (cont')

After n -1 = n/2 + n/4 + …+ 1 Unions-by-size



If there are $n = 2^k$ nodes then the longest path from leaf to root has length k.

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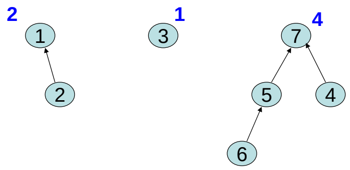
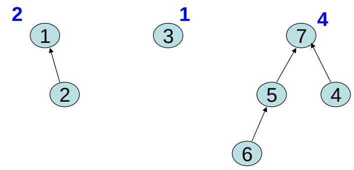
Array Implementation



Can store separate size array:

	1	2	3	4	5	6	7
up	-1	1	-1	7	7	5	-1
size	2		1				4

Elegant Array Implementation



Better, store sizes in the up array:

	1	2	3	4	5	6	7
up	-2	1	-1	7	7	5	-4

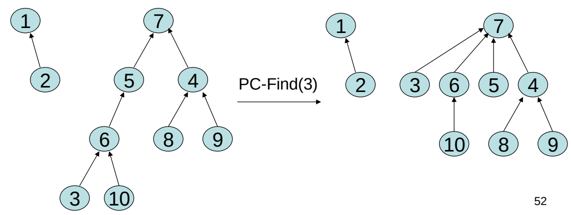
Negative up-values correspond to sizes of roots.

Code for Union-by-Size

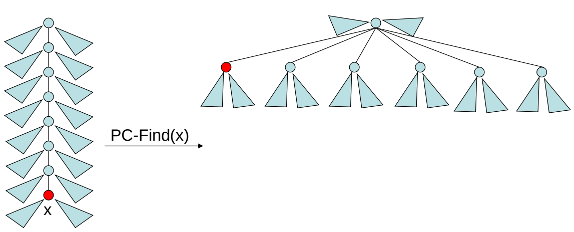
```
S-Union(i,j){
  // Collect sizes
  si = -up[i];
  sj = -up[j];

  // verify i and j are roots
  assert(si >=0 && sj >=0)
  // point smaller sized tree to
  // root of larger, update size
  if (si < sj) {
    up[i] = j;
    up[j] = -(si + sj);
  else {
    up[j] = i;
    up[i] = -(si + sj);
  }
}
```


Path Compression

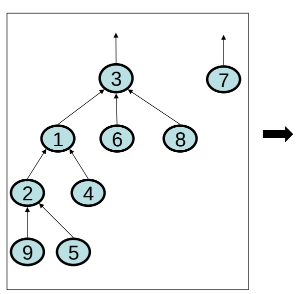
- To improve the amortized complexity, we'll borrow an idea from splay trees:
  - When going up the tree, *improve nodes on the path*!
- On a Find operation point all the nodes on the search path directly to the root. This is called "path compression."



Self-Adjustment Works



Draw the result of Find(5):



Code for Path Compression Find

```
PC-Find(i) {
  //find root
  j = i;
  while (up[j] >= 0) {
    j = up[j];
  root = j;

  //compress path
  if (i != root) {
    parent = up[i];
    while (parent != root) {
      up[i] = root;
      i = parent;
      parent = up[parent];
    }
  }
  return(root)
}
```

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Complexity of Union-by-Size + Path Compression

- Worst case time complexity for…
  - …a single Union-by-size is:
  - …a single PC-Find is:
- Time complexity for $m \geq n$ operations on n elements has been shown to be $O(m \log^* n)$.
  [See Weiss for proof.]
  - Amortized complexity is then $O(\log^* n)$
  - What is log* ?

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log* *n*

log* *n* = number of times you need to apply log to bring value down to at most 1

$\log^* 2 = 1$
$\log^* 4 = \log^* 2^2 = 2$
$\log^* 16 = \log^* 2^{2^2} = 3$ (log log log 16 = 1)
$\log^* 65536 = \log^* 2^{2^{2^2}} = 4$ (log log log log 65536 = 1)
$\log^* 2^{65536} = \ldots\ldots\ldots\ldots\ldots \approx \log^* (2 \times 10^{19{,}728}) = 5$

$\log^* n \leq 5$ for all reasonable n.

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The Tight Bound

In fact, Tarjan showed the time complexity for $m \geq n$ operations on n elements is:

$$\Theta(m\ \alpha(m, n))$$

Amortized complexity is then $\Theta(\alpha(m, n))$.

What is $\alpha(m, n)$?

- Inverse of Ackermann's function.
- For reasonable values of m, n, grows even slower than $\log^* n$. So, it's even "more constant."

Proof is beyond scope of this class. A simple algorithm can lead to incredibly hardcore analysis!

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