CSE 332: Parallel Sorting

Richard Anderson, Steve Seitz

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Announcements

- Project 3 PartA due Thursday night

Recap

Last week

- simple parallel programs
- common patterns: map, reduce
- analysis tools (work, span, parallelism)
- Amdahl's Law

Now

- parallel quicksort, merge sort
- useful building blocks: prefix, pack

Parallelizable?

Fibonacci (N)

$$\text{Fib}(N) = \text{fib}(N-1) + \text{Fib}(N-2)$$
$$= 1 \quad \text{if } N = 1, 2$$

Not really* because of sequential dependencies

naive approach spawns 2^n threads

→ exponential time

Parallelizable?

Prefix-sum:

`input`	6	3	11	10	8	2	7	8
`output`	6	9	20	30	38	40	47	55

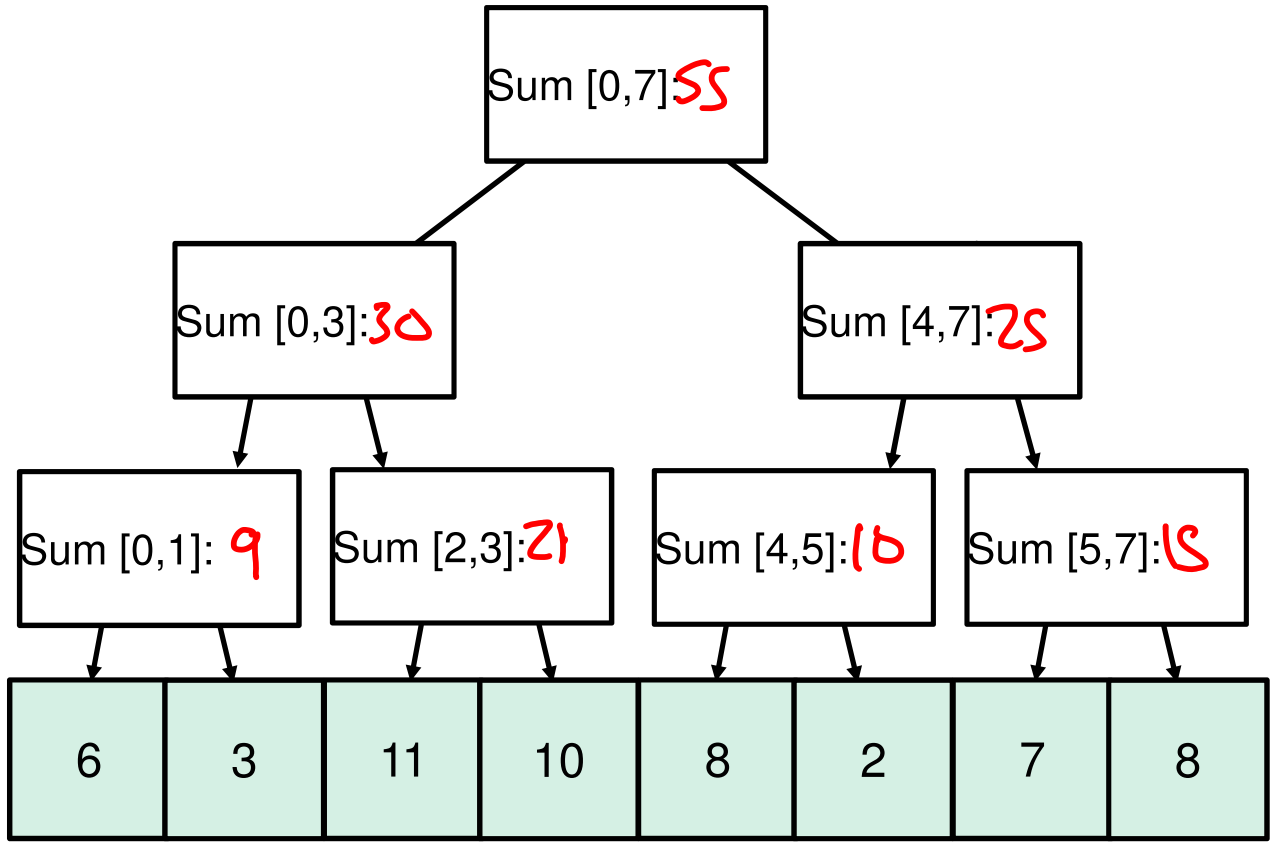
$$output[i] = \sum_{0}^{i-1} input[i]$$

First Pass: Sum

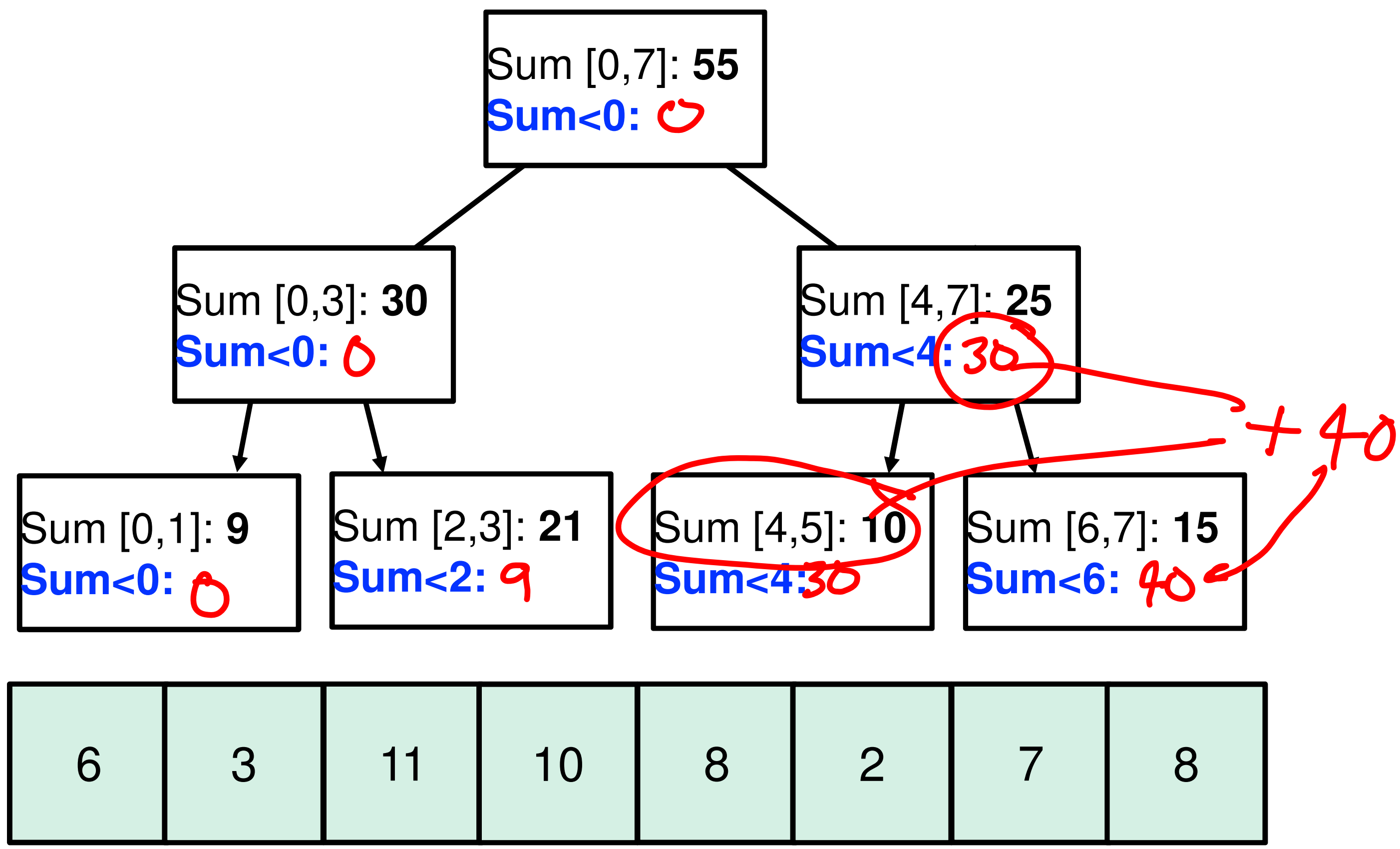
Sum [0,7]: 55

6	3	11	10	8	2	7	8

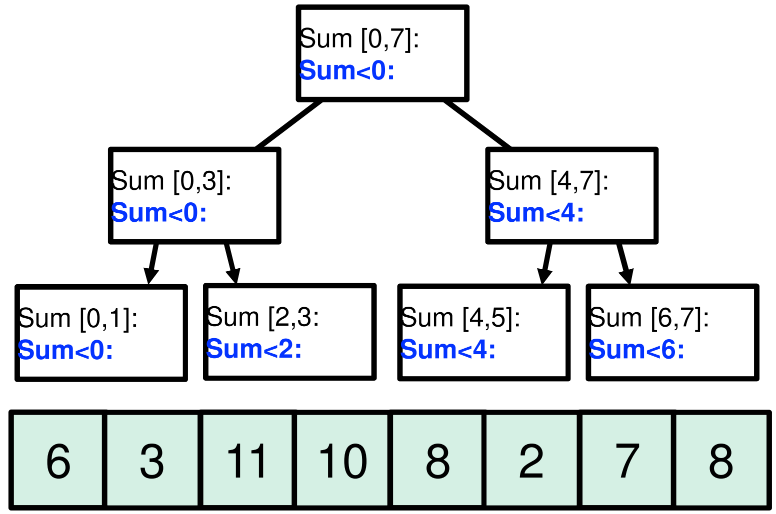
First Pass: Sum



2nd Pass: Use Sum for Prefix-Sum



2nd Pass: Use Sum for Prefix-Sum



Go from root down to leaves

Root

- sum<0 = 0

Left-child

- sum<K = parent's sum<K

Right-child

- sum<K = parent's sum<K + siblings sum[J, K]

Prefix-Sum Analysis

- First Pass (Sum):
  - span = $O(\log n)$
- Second Pass:
  - single pass from root down to leaves
    - update children's sum<K value based on parent and sibling
  - span = $O(\log n)$
- Total
  - span = $O(\log n)$

Parallel Prefix, Generalized

Prefix-sum is another common pattern (prefix problems)

- maximum element to the left of i
- is there an element to the left of i i satisfying some property?
- count of elements to the left of i satisfying some property
- …

We can solve all of these problems in the same way

Pack

Pack:

`input`	6	3	11	10	8	2	7	8
`output`	6	3	2	7				

`test:   X < 8?`

Output array of elements satisfying **`test`**, in original order

Parallel Pack?

Pack

input	6	3	11	10	8	2	7	8
output	6	3	2	7				

`test:  X < 8?`

- Determining **which** elements to include is **easy**
- Determining **where** each element goes in output is **hard**
  - seems to depend on previous results

Parallel Pack

1. map test input, output [0,1] bit vector

input	6	3	11	10	8	2	7	8
test	1	1	0	0	0	1	1	0

test: X < 8?

Parallel Pack

1. map test input, output [0,1] bit vector

input	6	3	11	10	8	2	7	8
test	1	1	0	0	0	1	1	0

test: X < 8?

2. transform bit vector into array of indices into result array

pos	1	2	2	2	2	3	4	4

Parallel Pack

1. map test input, output [0,1] bit vector

input	6	3	11	10	8	2	7	8
test	1	1	0	0	0	1	1	0

`test:   X < 8?`

2. prefix-sum on bit vector

pos	**1**	**2**	2	2	2	**3**	**4**	4

3. map input to corresponding positions in output

output	6	3	2	7				

– `if (test[i] == 1) output[pos[i]] = input[i]`

Parallel Pack Analysis

- Parallel Pack
  1. map: O( ) span
  2. sum-prefix: O( ) span
  3. map: O( ) span

- Total: O( ) span

Sequential Quicksort

Quicksort (review):

1. Pick a pivot — $O(1)$
2. Partition into two sub-arrays — $O(n)$
   A. values less than pivot
   B. values greater than pivot
3. Recursively sort A and B — $2T(n/2)$, avg

Complexity (avg case)

- $T(n) = n + 2T(n/2)$ $T(0) = T(1) = 1$
- $O(n \log n)$

How to parallelize?

Parallel Quicksort

Quicksort

1. Pick a pivot — $O(1)$
2. Partition into two sub-arrays — $O(n)$
   A. values less than pivot
   B. values greater than pivot
3. Recursively sort A and B in parallel — $T(n/2)$, avg

Complexity (avg case)

- $T(n) = n + T(n/2)$ $\qquad T(0) = T(1) = 1$
- Span: O( )
- Parallelism (work/span) = O( )

Taking it to the next level…

- O(`log n`) speed-up with infinite processors is okay, but a bit underwhelming
  - Sort 10^9 elements 30x faster

- Bottleneck:

Parallel Partition

Partition into sub-arrays

A. values less than pivot
B. values greater than pivot

What parallel operation can we use for this?

Parallel Partition

- Pick pivot

8	1	4	9	0	3	5	2	7	6

- Pack (test: **`<6`**)

1	4	0	3	5	2				

- Right pack (test: **`>=6`**)

1	4	0	3	5	2	6	8	9	7

Parallel Quicksort

Quicksort

1. Pick a pivot — O(1)
2. Partition into two sub-arrays — O(   ) span
    A. values less than pivot
    B. values greater than pivot
3. Recursively sort A and B in parallel — T(n/2), avg

Complexity (avg case)

- T(n) = O(   ) + T(n/2)   T(0) = T(1) = 1
- Span: O(   )
- Parallelism (work/span) = O(   )

Sequential Mergesort

Mergesort (review):

1. Sort left and right halves   2T(n/2)
2. Merge results   O(n)

Complexity (worst case)

– $T(n) = n + 2T(n/2)$ $T(0) = T(1) = 1$
– $O(n \log n)$

How to parallelize?

– Do left + right in parallel, improves to O(n)
– To do better, we need to…

Parallel Merge

0	**4**	**6**	**8**	**9**

1	**2**	**3**	**5**	**7**

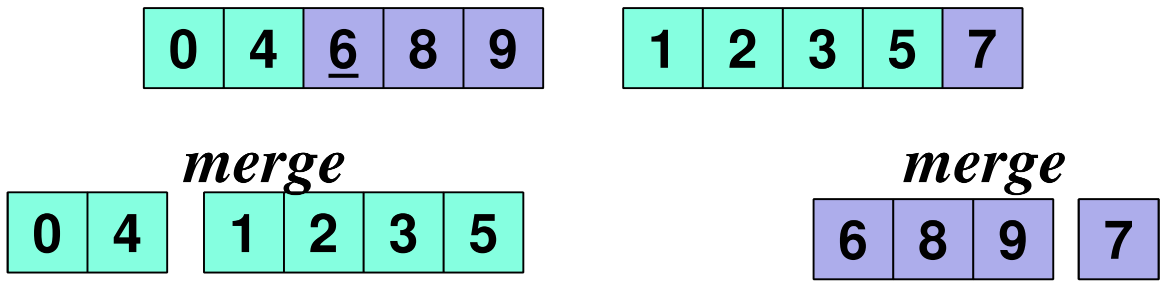
How to merge two sorted lists in parallel?

Parallel Merge

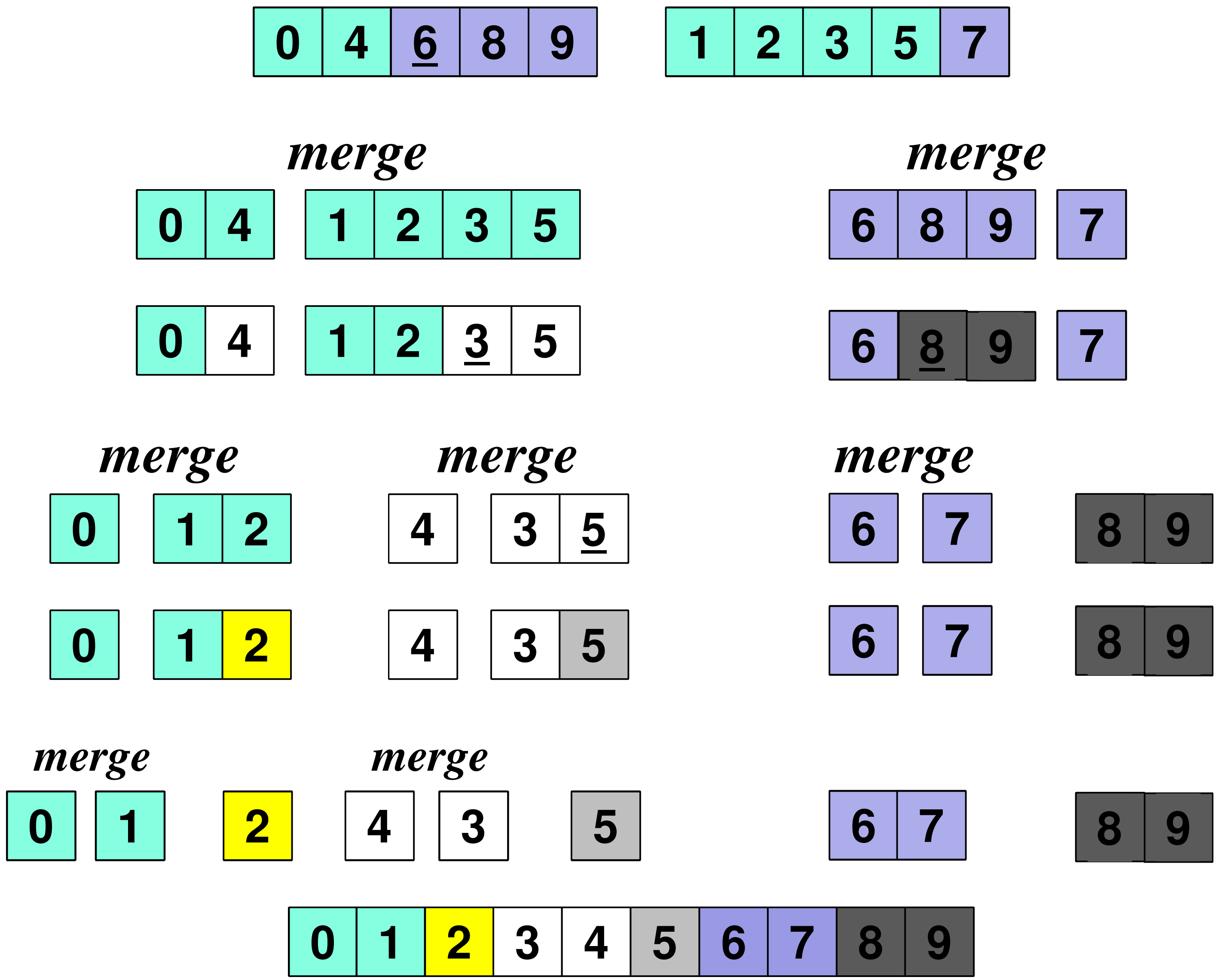
0	4	6	8	9		1	2	3	5	7
		M								

1. Choose median M of left half O( )
2. Split both arrays into < M, >=M O( )
   - how?

Parallel Merge



1. Choose median M of left half
2. Split both arrays into < M, >=M
   – how?
3. Do two submerges in parallel


0 4 6 8 9
1 2 3 5 7
merge
merge
0 4
1 2 3 5
6 8 9
7
merge
merge
merge
0
1 2
4
3 5
6
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8 9
merge
merge
0
1
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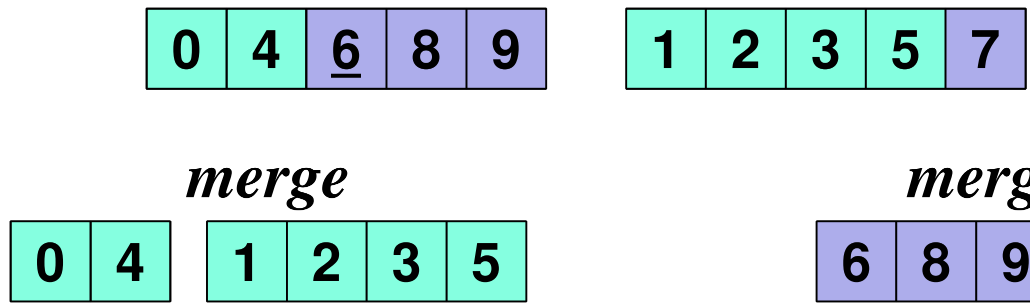
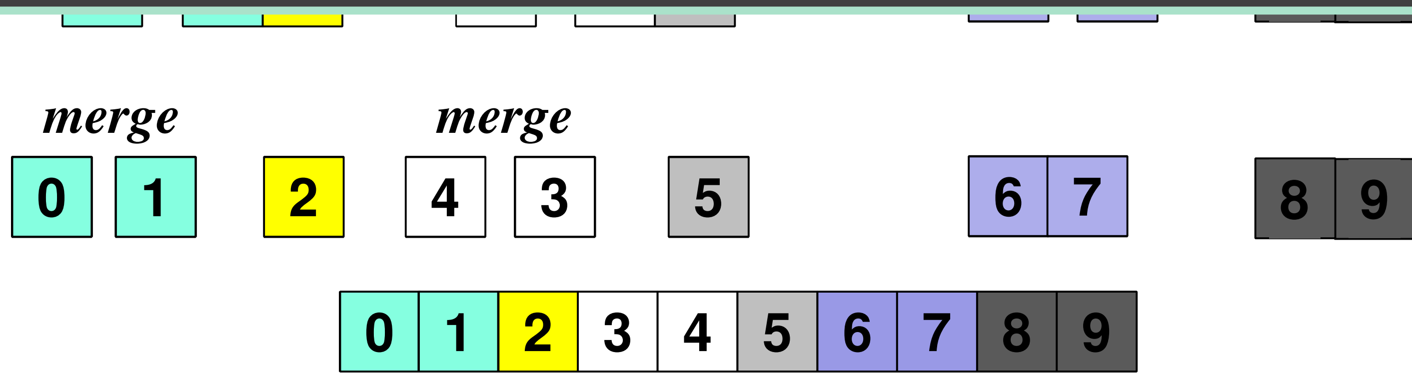
6 7
8 9
0 1 2 3 4 5 6 7 8 9



When we do each merge in parallel:

- we split the bigger array in half
- use binary search to split the smaller array
- And in base case we copy to the output array



Parallel Mergesort Pseudocode

```
Merge(arr[], left_1, left_2, right_1, right_2, out[], out_1, out_2 )
    int leftSize = left_2 – left_1
    int rightSize = right_2 – right_1
    // Assert: out_2 – out_1 = leftSize + rightSize
    // We will assume leftSize > rightSize without loss of generality

    if (leftSize + rightSize < CUTOFF)
        sequential merge and copy into out[out1..out2]

    int mid = (left_2 – left_1)/2
    binarySearch arr[right1..right2] to find j such that
        arr[j] ≤ arr[mid] ≤ arr[j+1]

    Merge(arr[], left_1, mid, right_1, j, out[], out_1, out_1+mid+j)
    Merge(arr[], mid+1, left_2, j+1, right_2, out[], out_1+mid+j+1, out_2)
```

Analysis

Parallel Merge (worst case)

- Height of partition call tree with n elements: O(   )
- Complexity of each thread (ignoring recursive call): O(   )
- Span: O(   )

Parallel Mergesort (worst case)

- Span: O(   )
- Parallelism (work / span): O(   )

Subtlety: uneven splits

0	4	6	8

1	2	3	5

- but even in worst case, get a 3/4 to 1/4 split
  - still gives O(log n) height

Parallel Quicksort vs. Mergesort

Parallelism (work / span)

- quicksort: $O(n / \log n)$ avg case
- mergesort: $O(n / \log^2 n)$ worst case