CSE 332:
Sorting Bound, and Radix Sort

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How fast can we sort?

Heapsort, Mergesort, Heapsort, AVL sort all have O(N log N) **worst** case running time.

These algorithms, along with Quicksort, also have O(N log N) **average** case running time.

Can we do any better?

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Permutations

- Suppose you are given *N* elements
  – Assume no duplicates
- How many possible orderings can you get?
  – Example: a, b, c (N = 3)

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Permutations

- How many possible orderings can you get?
  – Example: a, b, c (N = 3)
  – (a b c), (a c b), (b a c), (b c a), (c a b), (c b a)
  – 6 orderings = 3·2·1 = 3! (i.e., "3 factorial")
- For *N* elements
  – *N* choices for the first position, (*N*-1) choices for the second position, …, (2) choices, 1 choice
  – $N(N-1)(N-2)\cdots(2)(1)$= <u>*N*! possible orderings</u>

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Sorting Model

Recall our basic sorting assumption:

We can only compare two elements at a time.

These comparisons prune the space of possible orderings.

We can represent these concepts in a…

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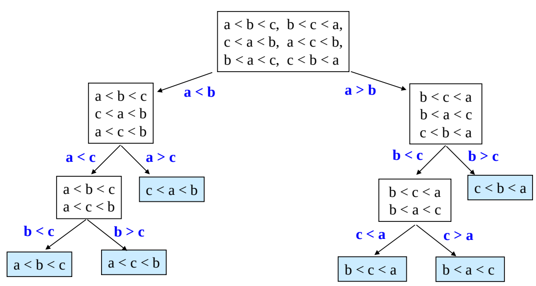
Decision Tree



The leaves contain all the possible orderings of a, b, c.

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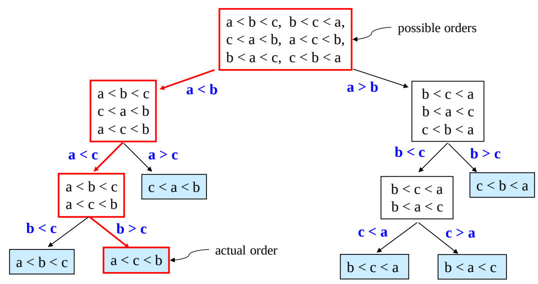
Decision Trees

- A Decision Tree is a Binary Tree such that:
  - Each node = a set of orderings
    - i.e., the remaining solution space
  - Each edge = 1 comparison
  - Each leaf = 1 unique ordering
  - How many leaves for *N* distinct elements?

- Only 1 leaf has the ordering that is the desired correctly sorted arrangement

Decision Tree Example

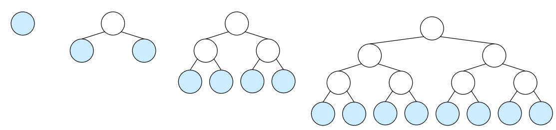


Decision Trees and Sorting

- Every sorting algorithm corresponds to a decision tree
  - Finds correct leaf by choosing edges to follow
    - i.e., by making comparisons
- We will focus on worst case run time
- Observations:
  - Worst case run time ≥ max number of comparisons
  - Max number of comparisons
    = length of the longest path in the decision tree
    = tree height

How many leaves on a tree?

Suppose you have a binary tree of height *h*. How many leaves in a perfect tree?



We can prune a perfect tree to make any binary tree of same height. Can # of leaves increase?

Lower bound on Height

- A binary tree of height h has at most 2^h leaves
  - Can prove by induction
- A decision tree has *N*! leaves. What is its minimum height?

An Alternative Explanation

At each decision point, one branch has ≤ ½ of the options remaining, the other has ≥ ½ remaining.

Worst case: we always end up with ≥ ½ remaining.

Best algorithm, in the worst case: we always end up with *exactly* ½ remaining.

Thus, in the worst case, the best we can hope for is halving the space *d* times (with *d* comparisons), until we have an answer, i.e., until the space is reduced to size = 1.

The space starts at *N*! in size, and halving *d* times means multiplying by $1/2^d$, giving us a lower bound on the worst case:

$$\frac{N!}{2^d}=1 \Rightarrow N!=2^d \Rightarrow d=\log_2(N!)$$

Lower Bound on log(N!)

Ω(N log N)

Worst case run time of any comparison-based sorting algorithm is **Ω(*N* log *N*)** .

Can also show that **average case** run time is also **Ω(*N* log *N*)** .

Can we do better if we don't use comparisons? (Huh?)

Can we sort in O(n)?

- Suppose keys are integers between 0 and 1000

BucketSort (aka BinSort)

If all values to be sorted are integers between 1 and *B*, create an array `count` of size *B*, **increment** counts while traversing the input, and finally output the result.

Example *B*=5. Input = (5,1,3,4,3,2,1,1,5,4,5)

`count` array	
1	
2	
3	
4	
5	

Running time to sort n items?

What about our Ω(n log n) bound?

Dependence on *B*

What if *B* is very large (e.g., 2^{64})?

Fixing impracticality: RadixSort

- RadixSort: generalization of BucketSort for large integer keys
- Origins go back to the 1890 census.
- Radix = “The base of a number system”
  – We’ll use 10 for convenience, but could be anything
- Idea:
  – BucketSort on one digit at a time
  – After k^{th} sort, the last k digits are sorted
  – Set number of buckets: *B* = radix.

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Radix Sort Example

Input: 478, 537, 9, 721, 3, 38, 123, 67

BucketSort on 1’s

0	1	2	3	4	5	6	7	8	9

BucketSort on 10’s

0	1	2	3	4	5	6	7	8	9

BucketSort on 100’s

0	1	2	3	4	5	6	7	8	9

Output:

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Radix Sort Example (1^{st} pass)

Input data

478
537
9
721
3
38
123
67

Bucket sort by 1’s digit

0	1	2	3	4	5	6	7	8	9
	721		3 123				537 67	478 38	9

After 1^{st} pass

721
3
123
537
67
478
38
9

This example uses B=10 and base 10 digits for simplicity of demonstration. Larger bucket counts should be used in an actual implementation.

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Radix Sort Example (2^{nd} pass)

After 1^{st} pass

721
3
123
537
67
478
38
9

Bucket sort by 10’s digit

0	1	2	3	4	5	6	7	8	9
03 09		721 123	537 38			67	478		

After 2^{nd} pass

3
9
721
123
537
38
67
478

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Radix Sort Example (3^{rd} pass)

After 2^{nd} pass

3
9
721
123
537
38
67
478

Bucket sort by 100’s digit

0	1	2	3	4	5	6	7	8	9
003 009 038 067	123			478	537		721		

After 3^{rd} pass

3
9
38
67
123
478
537
721

Invariant: after k passes the low order k digits are sorted.

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Radixsort: Complexity

In our examples, we had:
– Input size, N
– Number of buckets, B = 10
– Maximum value, $M < 10^3$
– Number of passes, P =

How much work per pass?

Total time?

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Choosing the Radix

Run time is roughly proportional to:

$$P(B+N) = \log_B M(B+N)$$

Can show that this is minimized when:

$$B \log_e B \approx N$$

In theory, then, the best base (radix) depends only on N.

For fast computation, prefer $B = 2^b$. Then best b is:

$$b + \log_2 b \approx \log_2 N$$

Example:

- N = 1 million (i.e., ~2^{20}) 64 bit numbers, $M = 2^{64}$
- $\log_2 N \approx 20 \rightarrow b = 16$
- $B = 2^{16} = 65{,}536$ and $P = \log_{(2^{16})} 2^{64} = 4$.

In practice, memory word sizes, space, other architectural considerations, are important in choosing the radix.

Big Data: External Sorting

Goal: **<u>minimize disk/tape access</u>** time:

- Quicksort and Heapsort both jump all over the array, leading to expensive random disk accesses
- Mergesort scans linearly through arrays, leading to (relatively) efficient sequential disk access

Basic Idea:

- Load chunk of data into Memory, sort, store this "run" on disk/tape
- Use the Merge routine from Mergesort to merge runs
- Repeat until you have only one run (one sorted chunk)
- Mergesort can leverage multiple disks
- Weiss gives some examples

Sorting Summary

$O(N^2)$ average, worst case:

- **Selection Sort**, **Bubblesort**, **Insertion Sort**

$O(N \log N)$ average case:

- **Heapsort**: In-place, not stable.
- **BST Sort**: $O(N)$ extra space (including tree pointers, possibly poor memory locality), stable.
- **Mergesort**: $O(N)$ extra space, stable.
- **Quicksort**: claimed fastest in practice, but $O(N^2)$ worst case. Recursion/stack requirement. Not stable.

$\Omega(N \log N)$ worst and average case:

- **Any comparison-based sorting algorithm**

$O(N)$

- **Radix Sort**: fast and stable. Not comparison based. Not in-place. Poor memory locality can undercut performance.