CSE 332: Data Structures

Asymptotic Analysis

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Announcements

- Homework requires you get the textbook (either E2 or E3)
- Go to Thursdays sections
- Homework #1 out on today (Wednesday)
  - Due at the **beginning of class next Wednesday**(Jan 17).

Algorithm Analysis

- Correctness:
  - Does the algorithm do what is intended.
- Performance:
  - Speed   time complexity
  - Memory   space complexity
- Why analyze?
  - To make good design decisions
  - Enable you to look at an algorithm (or code) and identify the bottlenecks, etc.

Correctness

Correctness of an algorithm is established by proof. Common approaches:

- (Dis)proof by counterexample
- Proof by contradiction
- Proof by induction
  - Especially useful in recursive algorithms

Proof by Induction

- **Base Case:** The algorithm is correct for a base case or two by inspection.
- **Inductive Hypothesis (n=k):** Assume that the algorithm works correctly for the first k cases.
- **Inductive Step (n=k+1):** Given the hypothesis above, show that the k+1 case will be calculated correctly.

Recursive algorithm for *sum*

- Write a *recursive* function to find the sum of the first **n** integers stored in array **v**.

```
sum(int array v, int n) returns int
  if n = 0 then
    sum = 0
  else
    sum = nth number + sum of first n-1 numbers
  return sum
```

Program Correctness by Induction

- Base Case:
- Inductive Hypothesis (n=k):
- Inductive Step (n=k+1):

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How to measure performance?

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Analyzing Performance

We will focus on analyzing time complexity. First, we have some "rules" to help measure how long it takes to do things:

Basic operations	Constant time
Consecutive statements	Sum of times
Conditionals	Test, plus larger branch cost
Loops	Sum of iterations
Function calls	Cost of function body
Recursive functions	Solve recurrence relation…

Second, we will be interested in **best** and **worst** case performance.

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Complexity cases

We'll start by focusing on two cases.

Problem size **N**

- **Worst-case complexity**: **max** # steps algorithm takes on "most challenging" input of size **N**
- **Best-case complexity: min** # steps algorithm takes on "easiest" input of size **N**

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Exercise - Searching

2	3	5	16	37	50	73	75

```
bool ArrayContains(int array[], int n, int key){
  // Insert your algorithm here



}
```

What algorithm would you choose to implement this code snippet?

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Linear Search Analysis

```
bool LinearArrayContains(int array[], int n, int key ) {
  for( int i = 0; i < n; i++ ) {
     if( array[i] == key )
        // Found it!
        return true;
  }
  return false;
}
```

Best Case:

Worst Case:

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Binary Search Analysis

2	3	5	16	37	50	73	75

```
bool BinArrayContains( int array[], int low, int high, int key ) {
    // The subarray is empty
    if( low > high ) return false;

    // Search this subarray recursively
    int mid = (high + low) / 2;
    if( key == array[mid] ) {
        return true;
    } else if( key < array[mid] ) {
        return BinArrayFind( array, low, mid-1, key );
    } else {
        return BinArrayFind( array, mid+1, high, key );
}
```

Best case:

Worst case:

Solving Recurrence Relations

1. Determine the recurrence relation and base case(s).

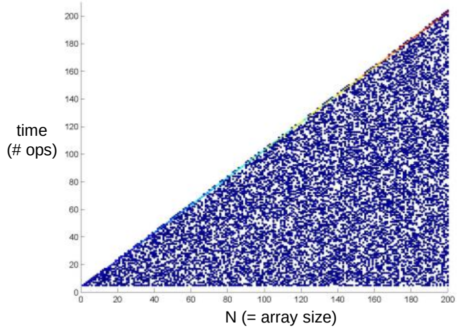
2. "Expand" the original relation to find an equivalent expression *in terms of the number of expansions (k)*.

3. Find a closed-form expression by setting *k* to a value which reduces the problem to a base case

Linear Search vs Binary Search

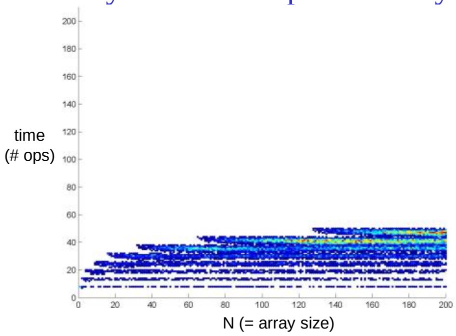
	Linear Search	Binary Search
Best Case	4	5 at [middle]
Worst Case	3n+3	$7 \lfloor \log n \rfloor + 9$

Linear search—empirical analysis



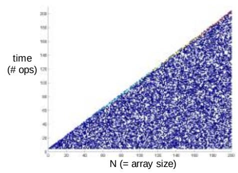
Each search produces a dot in above graph.
Blue = less frequently occurring, Red = more frequent

Binary search—empirical analysis

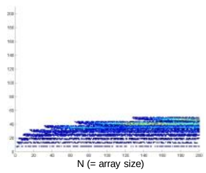


Each search produces a dot in above graph.
Blue = less frequently occurring, Red = more frequent

Empirical comparison



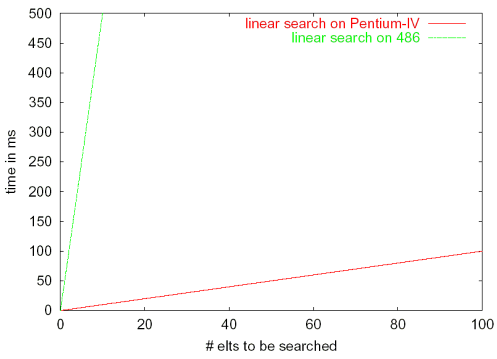
Linear search



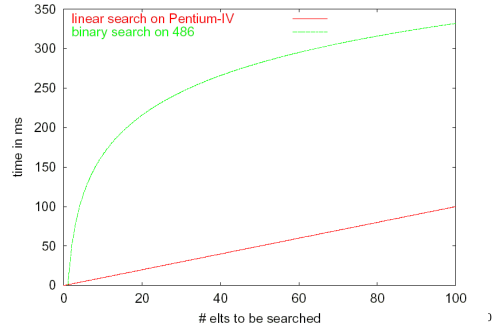
Binary search

Gives additional information

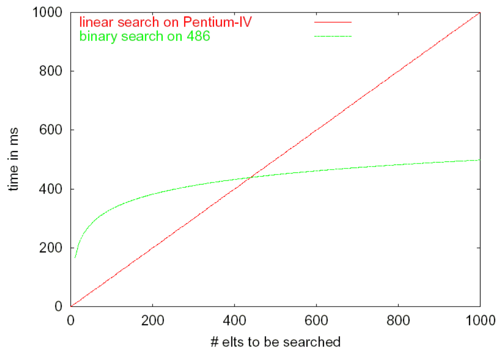
Fast Computer vs. Slow Computer



Fast Computer vs. Smart Programmer (small data)



Fast Computer vs. Smart Programmer (big data)



Asymptotic Analysis

- Consider only the *order* of the running time
  - A valuable tool when the input gets "large"
  - **Ignores** the effects of ***different machines*** or ***different implementations*** of same algorithm

Asymptotic Analysis

- To find the asymptotic runtime, throw away the constants and low-order terms
  - Linear search is $T^{LS}_{worst}(n) = 3n + 3 \in O(n)$
  - Binary search is $T^{BS}_{worst}(n) = 7\lfloor \log_2 n \rfloor + 9 \in O(\log n)$

Remember: the "fastest" algorithm has the slowest growing function for its runtime

Asymptotic Analysis

Eliminate low order terms
- $4n + 5 \Rightarrow$
- $0.5\ n \log n + 2n + 7 \Rightarrow$
- $n^3 + 3\ 2^n + 8n \Rightarrow$

Eliminate coefficients
- $4n \Rightarrow$
- $0.5\ n \log n \Rightarrow$
- $3\ 2^n$ =>

Properties of Logs

Basic:
- $A^{\log_A B} = B$
- $\log_A A =$

Independent of base:
- $\log(AB) =$
- $\log(A/B) =$
- $\log(A^B) =$
- $\log((A^B)^C) =$

Properties of Logs

Changing base → multiply by constant
- For example: $\log_2 x = 3.22 \log_{10} x$
- More generally

$$\log_A n = \left(\frac{1}{\log_B A}\right)\log_B n$$

- Means we can ignore the base for asymptotic analysis (since we're ignoring constant multipliers)

Another example

- Eliminate low-order terms
- Eliminate constant coefficients

$16n^3\log_8(10n^2) + 100n^2$

Comparing functions

- f(n) is an **upper bound** for h(n) if h(n) ≤ f(n) for all n

This is too strict – we mostly care about *large* n

Still too strict if we want to ignore *scale factors*

Definition of Order Notation

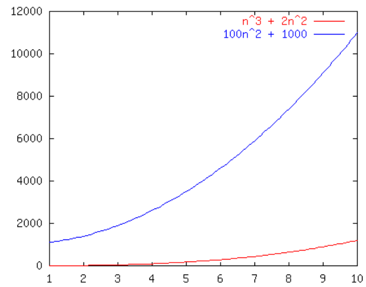
- h(n) ∊ O(f(n)) Big-O "Order"
  if there exist positive constants c and n_0 such that h(n) ≤ c f(n) for all n ≥ n_0

O(f(n)) defines a class (set) of functions

Order Notation: Intuition

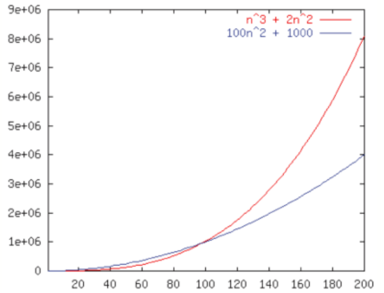
$a(n) = n^3 + 2n^2$

$b(n) = 100n^2 + 1000$



Although not yet apparent, as *n* gets "sufficiently large", $a(n)$ will be "greater than or equal to" $b(n)$

Order Notation: Example



$100n^2 + 1000 \le (n^3 + 2n^2)$ for all $n \ge 100$

So $100n^2 + 1000 \in O(n^3 + 2n^2)$

Example

$h(n) \in O(f(n))$ iff there exist positive constants c and n_0 such that: $h(n) \le c\, f(n)$ for all $n \ge n_0$

Example:

$100n^2 + 1000 \le 1\,(n^3 + 2n^2)$ for all $n \ge 100$

So $100n^2 + 1000 \in O(n^3 + 2n^2)$

Constants are not unique

$h(n) \in O(f(n))$ iff there exist positive constants c and n_0 such that: $h(n) \le c\, f(n)$ for all $n \ge n_0$

Example:

$100n^2 + 1000 \le 1\,(n^3 + 2n^2)$ for all $n \ge 100$

$100n^2 + 1000 \le 1/2\,(n^3 + 2n^2)$ for all $n \ge 198$

Another Example: Binary Search

$h(n) \in O(f(n))$ iff there exist positive constants c and n_0 such that: $h(n) \le c\, f(n)$ for all $n \ge n_0$

Is $7\log_2 n + 9 \in O(\log_2 n)$?

Order Notation: Worst Case Binary Search

Some Notes on Notation

Sometimes you'll see (e.g., in Weiss)

$h(n) = O(f(n))$

or

$h(n)$ is $O(f(n))$

These are equivalent to

$h(n) \in O(f(n))$

Big-O: Common Names

- constant: O(1)
- logarithmic: O(log n) ($\log_k n$, $\log n^2 \in$ O(log n))
- linear: O(n)
- log-linear: O(n log n)
- quadratic: $O(n^2)$
- cubic: $O(n^3)$
- polynomial: $O(n^k)$ (k is a constant)
- exponential: $O(c^n)$ (c is a constant > 1)

Asymptotic Lower Bounds

- $\Omega(\ g(n)\ )$ is the set of all functions asymptotically greater than or equal to $g(n)$
- $h(n) \in \Omega(\ g(n)\ )$ iff
  There exist $c>0$ and $n_0>0$ such that $h(n) \geq c\, g(n)$ for all $n \geq n_0$

Asymptotic Tight Bound

- $\theta(\ f(n)\ )$ is the set of all functions asymptotically equal to $f(n)$
- $h(n) \in \theta(\ f(n)\ )$ iff
  $h(n) \in O(\ f(n)\ )$ and $h(n) \in \Omega(f(n)\ )$
  - This is equivalent to:
    $\lim_{n\to\infty} h(n)/f(n) = c \neq 0$

Full Set of Asymptotic Bounds

- $O(\ f(n)\ )$ is the set of all functions asymptotically less than or equal to $f(n)$
  - $o(f(n)\ )$ is the set of all functions asymptotically strictly less than $f(n)$
- $\Omega(\ g(n)\ )$ is the set of all functions asymptotically greater than or equal to $g(n)$
  - $\omega(\ g(n)\ )$ is the set of all functions asymptotically strictly greater than $g(n)$
- $\theta(\ f(n)\ )$ is the set of all functions asymptotically equal to $f(n)$

Formal Definitions

- $h(n) \in O(\ f(n)\ )$ iff
  There exist $c>0$ and $n_0>0$ such that $h(n) \leq c\, f(n)$ for all $n \geq n_0$
- $h(n) \in o(f(n))$ iff
  There exists an $n_0>0$ such that $h(n) < c\, f(n)$ for all $c>0$ and $n \geq n_0$
  - This is equivalent to: $\lim_{n\to\infty} h(n)/f(n) = 0$
- $h(n) \in \Omega(\ g(n)\ )$ iff
  There exist $c>0$ and $n_0>0$ such that $h(n) \geq c\, g(n)$ for all $n \geq n_0$
- $h(n) \in \omega(\ g(n)\ )$ iff
  There exists an $n_0>0$ such that $h(n) > c\, g(n)$ for all $c>0$ and $n \geq n_0$
  - This is equivalent to: $\lim_{n\to\infty} h(n)/g(n) = \infty$
- $h(n) \in \theta(\ f(n)\ )$ iff
  $h(n) \in O(\ f(n)\ )$ and $h(n) \in \Omega(f(n)\ )$
  - This is equivalent to: $\lim_{n\to\infty} h(n)/f(n) = c \neq 0$

Big-Omega et al. Intuitively

Asymptotic Notation	Mathematics Relation
O	$\leq$
Ω	$\geq$
θ	=
o	<
ω	>

Complexity cases (revisited)

Problem size **N**

- **Worst-case complexity**: **max** # steps algorithm takes on "most challenging" input of size **N**
- **Best-case complexity: min** # steps algorithm takes on "easiest" input of size **N**
- **Average-case complexity**: **avg** # steps algorithm takes on *random* inputs of size **N**
- **Amortized complexity**: **max** total # steps algorithm takes on **M** "most challenging" *consecutive* inputs of size **N**, divided by **M** (i.e., divide the max total by **M**).

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Bounds vs. Cases

Two orthogonal axes:

- Bound Flavor
  - Upper bound (O, o)
  - Lower bound (Ω, ω)
  - Asymptotically tight (θ)
- Analysis Case
  - Worst Case (Adversary), $T_{worst}(n)$
  - Average Case, $T_{avg}(n)$
  - Best Case, $T_{best}(n)$
  - Amortized, $T_{amort}(n)$

One can estimate the bounds for any given case.

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Bounds vs. Cases

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Pros and Cons of Asymptotic Analysis

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Big-Oh Caveats

- Asymptotic complexity (Big-Oh) considers only ***large n***
  - You can "abuse" it to be misled about trade-offs
  - Example: $n^{1/10}$ vs. `log` n
    - Asymptotically $n^{1/10}$ grows more quickly
    - But the "cross-over" point is around 5 * 10^{17}
    - So $n^{1/10}$ better for almost any real problem
- Comparing O() for ***small n*** values can be misleading
  - Quicksort: O(nlogn)
  - Insertion Sort: O(n^2)
  - Yet in reality Insertion Sort is faster for small n
  - We'll learn about these sorts later

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