CSE 332: Data Abstractions
Lecture 14: Introduction to Graphs

Ruth Anderson
Winter 2013
Announcements

• **Midterm** – **Monday Feb 11\textsuperscript{th} during lecture**, info about midterm has been posted
  – Review session Sat noon, EEB 037
  – Ruth has extra office hours Mon Feb 11\textsuperscript{th}, 12:30pm-2pm

• **Homework 4** – due Friday Feb 15\textsuperscript{th} at the BEGINNING of lecture

• **Project 2** – Phase B due Tues Feb 19\textsuperscript{th} at 11pm
Today

• Sorting
  – Beyond comparison sorting
• Graphs
  – Intro & Definitions
Where We Are

We have learned about the essential ADTs and data structures:
• Regular and Circular Arrays (dynamic sizing)
• Linked Lists
• Stacks, Queues
• Priority Queues, Heaps
• Unbalanced and Balanced Search Trees, B-Trees
• Hash Tables

We have also learned important algorithms
• Tree traversals
• Floyd's buildheap Method
• Sorting algorithms
Where We Are Going

More on algorithms and related problems that require constructing data structures to make the solutions efficient

Topics will include:

• Graphs
• Parallelism
• Concurrency
Graphs

- A graph is a formalism for representing relationships among items
  - Very general definition because very general concept

- A graph is a pair
  \[ G = (V, E) \]
  - A set of vertices, also known as nodes
    \[ V = \{v_1, v_2, \ldots, v_n\} \]
  - A set of edges
    \[ E = \{e_1, e_2, \ldots, e_m\} \]
    - Each edge \( e_i \) is a pair of vertices \((v_j, v_k)\)
    - An edge “connects” the vertices

- Graphs can be directed or undirected

\[ V = \{Han, Leia, Luke\} \]
\[ E = \{(Luke, Leia), (Han, Leia), (Leia, Han)\} \]
An ADT?

• Can think of graphs as an ADT with operations like \( \text{isEdge}((v_j, v_k)) \)

• But it is unclear what the “standard operations” are

• Instead we tend to develop algorithms over graphs and then use data structures that are efficient for those algorithms

• Many important problems can be solved by:
  1. Formulating them in terms of graphs
  2. Applying a standard graph algorithm

• To make the formulation easy and standard, we have a lot of standard terminology about graphs
Some graphs

For each, what are the vertices and what are the edges?

• Web pages with links
• Facebook friends
• “Input data” for the Kevin Bacon game
• Methods in a program that call each other
• Road maps (e.g., Google maps)
• Airline routes
• Family trees
• Course pre-requisites
• …

Wow: Using the same algorithms for problems across so many domains sounds like “core computer science and engineering”
Undirected Graphs

- In undirected graphs, edges have no specific direction
  - Edges are always “two-way”

- Thus, \((u, v) \in E\) implies \((v, u) \in E\).
  - Only one of these edges needs to be in the set; the other is implicit

- Degree of a vertex: number of edges containing that vertex
  - Put another way: the number of adjacent vertices
Directed Graphs

- In directed graphs (sometimes called digraphs), edges have a direction.

Thus, \((u, v) \in E\) does not imply \((v, u) \in E\).
- Let \((u, v) \in E\) mean \(u \rightarrow v\)
- Call \(u\) the source and \(v\) the destination

In-Degree of a vertex: number of in-bound edges, i.e., edges where the vertex is the destination
Out-Degree of a vertex: number of out-bound edges i.e., edges where the vertex is the source
**Self-edges, connectedness**

- A **self-edge** a.k.a. a **loop** is an edge of the form \((u, u)\)
  - Depending on the use/algorithm, a graph may have:
    - No self edges
    - Some self edges
    - All self edges (often therefore implicit, but we will be explicit)

- A node can have a degree / in-degree / out-degree of **zero**

- A graph does not have to be **connected** (In an undirected graph, this means we can follow edges from any node to every other node), even if every node has non-zero degree
More notation

For a graph $G = (V, E)$:

- $|V|$ is the number of vertices
- $|E|$ is the number of edges
  - Minimum?
  - Maximum for undirected?
  - Maximum for directed?

- If $(u, v) \in E$
  - Then $v$ is a neighbor of $u$, i.e., $v$ is adjacent to $u$
  - Order matters for directed edges
    - $u$ is not adjacent to $v$ unless $(v, u) \in E$
More notation

For a graph $G = (V, E)$:

- $|V|$ is the number of vertices
- $|E|$ is the number of edges
  - Minimum? $0$
  - Maximum for undirected? $|V| |V+1| / 2 \in O(|V|^2)$
  - Maximum for directed? $|V|^2 \in O(|V|^2)$
    (assuming self-edges allowed, else subtract $|V|$)

- If $(u, v) \in E$
  - Then $v$ is a neighbor of $u$, i.e., $v$ is adjacent to $u$
  - Order matters for directed edges
    - $u$ is not adjacent to $v$ unless $(v, u) \in E$
Examples again

Which would use directed edges? Which would have self-edges? Which could have 0-degree nodes?

• Web pages with links
• Facebook friends
• “Input data” for the Kevin Bacon game
• Methods in a program that call each other
• Road maps (e.g., Google maps)
• Airline routes
• Family trees
• Course pre-requisites
• ...

2/08/2013
Weighted graphs

- In a weighed graph, each edge has a weight a.k.a. cost
  - Typically numeric (most examples will use ints)
  - Orthogonal to whether graph is directed
  - Some graphs allow negative weights; many don’t

![Graph Diagram]

- Clinton
- Mukilteo
- Kingston
- Edmonds
- Bainbridge
- Seattle
- Bremerton
Examples

What, if anything, might weights represent for each of these? Do negative weights make sense?

- Web pages with links
- Facebook friends
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Road maps (e.g., Google maps)
- Airline routes
- Family trees
- Course pre-requisites
- …
Paths and Cycles

- A **path** is a list of vertices \([v_0, v_1, \ldots, v_n]\) such that 
  \((v_i, v_{i+1}) \in E\) for all \(0 \leq i < n\). Say “a path from \(v_0\) to \(v_n\)”

- A **cycle** is a path that begins and ends at the same node \((v_0 == v_n)\)

Example path (that also happens to be a cycle):
[Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle]
Path Length and Cost

- Path length: Number of edges in a path (also called “unweighted cost”)
- Path cost: Sum of the weights of each edge

Example where:

P = [Seattle, Salt Lake City, Chicago, Dallas, San Francisco]

\[
\text{length}(P) = 4 \\
\text{cost}(P) = 9.5
\]
Simple paths and cycles

- A simple path repeats no vertices, (except the first might be the last):
  - [Seattle, Salt Lake City, San Francisco, Dallas]
  - [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]

- Recall, a cycle is a path that ends where it begins:
  - [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]
  - [Seattle, Salt Lake City, Seattle, Dallas, Seattle]

- A simple cycle is a cycle and a simple path:
  - [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]
Paths/cycles in directed graphs

Example:

Is there a path from A to D?

Does the graph contain any cycles?
Paths/cycles in directed graphs

Example:

Is there a path from A to D?  No

Does the graph contain any cycles?  No
**Undirected graph connectivity**

- An undirected graph is **connected** if for all pairs of vertices $u, v$, there exists a *path* from $u$ to $v$

  ![Connected graph](image)

  ![Disconnected graph](image)

- An undirected graph is **complete**, a.k.a. **fully connected** if for all pairs of vertices $u, v$, there exists an *edge* from $u$ to $v$

  ![Complete graph](image)

  *(plus self edges)*
Directed graph connectivity

- A directed graph is **strongly connected** if there is a path from every vertex to every other vertex.

- A directed graph is **weakly connected** if there is a path from every vertex to every other vertex *ignoring direction of edges*.

- A **complete** a.k.a. **fully connected** directed graph has an edge from every vertex to every other vertex *(plus self edges)*.
Examples

For undirected graphs: connected?
For directed graphs: strongly connected? weakly connected?

- Web pages with links
- Facebook friends
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Road maps (e.g., Google maps)
- Airline routes
- Family trees
- Course pre-requisites
- …
Trees as graphs

When talking about graphs, we say a tree is a graph that is:
- undirected
- acyclic
- connected

So all trees are graphs, but not all graphs are trees

How does this relate to the trees we know and love?...

Example:
Rooted Trees

- We are more accustomed to rooted trees where:
  - We identify a unique ("special") root
  - We think of edges as directed: parent to children

- Given a tree, once you pick a root, you have a unique rooted tree (just drawn differently and with undirected edges)
Rooted Trees (Another example)

• We are more accustomed to rooted trees where:
  – We identify a unique (“special”) root
  – We think of edges as directed: parent to children

• Given a tree, once you pick a root, you have a unique rooted tree (just drawn differently and with undirected edges)
Directed acyclic graphs (DAGs)

- A DAG is a directed graph with no (directed) cycles
  - Every rooted directed tree is a DAG
    - But not every DAG is a rooted directed tree:
      - Not a rooted directed tree, Has a cycle (in the undirected sense)

- Every DAG is a directed graph
  - But not every directed graph is a DAG:
Examples

Which of our directed-graph examples do you expect to be a DAG?

- Web pages with links
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Airline routes
- Family trees
- Course pre-requisites
- …
Density / sparsity

- Recall: In an undirected graph, $0 \leq |E| < |V|^2$
- Recall: In a directed graph: $0 \leq |E| \leq |V|^2$
- So for any graph, $|E|$ is $O(|V|^2)$
- One more fact: If an undirected graph is connected, then $|E| \geq |V|-1$
- Because $|E|$ is often much smaller than its maximum size, we do not always approximate as $|E|$ as $O(|V|^2)$
  - This is a correct bound, it just is often not tight
  - If it is tight, i.e., $|E|$ is $\Theta(|V|^2)$ we say the graph is dense
    - More sloppily, dense means “lots of edges”
  - If $|E|$ is $O(|V|)$ we say the graph is sparse
    - More sloppily, sparse means “most (possible) edges missing”
What is the Data Structure?

- So graphs are really useful for lots of data and questions
  - For example, “what’s the lowest-cost path from x to y”

- But we need a data structure that represents graphs

- The “best one” can depend on:
  - Properties of the graph (e.g., dense versus sparse)
  - The common queries (e.g., “is \((u,v)\) an edge?” versus “what are the neighbors of node \(u\)?”)

- So we’ll discuss the two standard graph representations
  - Adjacency Matrix and Adjacency List
  - Different trade-offs, particularly time versus space
**Adjacency matrix**

- Assign each node a number from 0 to \(|V| - 1\)
- A \(|V| \times |V|\) matrix (i.e., 2-D array) of Booleans (or 1 vs. 0)
  - If \(M\) is the matrix, then \(M[u][v] == \text{true}\)
    means there is an edge from \(u\) to \(v\)
Adjacency Matrix Properties

• Running time to:
  – Get a vertex’s out-edges:
  – Get a vertex’s in-edges:
  – Decide if some edge exists:
  – Insert an edge:
  – Delete an edge:

• Space requirements:

• Best for sparse or dense graphs?
## Adjacency Matrix Properties

- **Running time to:**
  - Get a vertex’s out-edges: $O(|V|)$
  - Get a vertex’s in-edges: $O(|V|)$
  - Decide if some edge exists: $O(1)$
  - Insert an edge: $O(1)$
  - Delete an edge: $O(1)$

- **Space requirements:**
  - $|V|^2$ bits

- **Best for sparse or dense graphs?**
  - Best for dense graphs

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>B</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>C</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>D</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
Adjacency Matrix Properties

- How will the adjacency matrix vary for an undirected graph?

- How can we adapt the representation for weighted graphs?

\[
\begin{array}{cccc}
A & B & C & D \\
\hline
A & F & T & F & F \\
B & T & F & F & F \\
C & F & T & F & T \\
D & F & F & F & F \\
\end{array}
\]
Adjacency Matrix Properties

• How will the adjacency matrix vary for an undirected graph?
  – Undirected will be symmetric about diagonal axis

• How can we adapt the representation for weighted graphs?
  – Instead of a Boolean, store a number in each cell
  – Need some value to represent ‘not an edge’
    • In some situations, 0 or -1 works

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>B</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>C</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>D</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
**Adjacency List**

- Assign each node a number from 0 to $|V| - 1$
- An array of length $|V|$ in which each entry stores a list of all adjacent vertices (e.g., linked list)
Adjacency List Properties

• Running time to:
  – Get all of a vertex’s out-edges:
  – Get all of a vertex’s in-edges:
  – Decide if some edge exists:
    – Insert an edge:
    – Delete an edge:

• Space requirements:

• Best for dense or sparse graphs?
Adjacency List Properties

- Running time to:
  - Get all of a vertex’s out-edges: \( O(d) \) where \( d \) is out-degree of vertex
  - Get all of a vertex’s in-edges: \( O(|E|) \) (but could keep a second adjacency list for this!)
  - Decide if some edge exists: \( O(d) \) where \( d \) is out-degree of source
  - Insert an edge: \( O(1) \) (unless you need to check if it’s there)
  - Delete an edge: \( O(d) \) where \( d \) is out-degree of source

- Space requirements:
  - \( O(|V|+|E|) \)

- Best for dense or sparse graphs?
  - Best for sparse graphs, so usually just stick with linked lists
**Undirected Graphs**

Adjacency matrices & adjacency lists both do fine for undirected graphs

- **Matrix:** Can save roughly $\frac{1}{2}$ the space
  - But may slow down operations in languages with “proper” 2D arrays (not Java, which has only arrays of arrays)
  - How would you “get all neighbors”?

- **Lists:** Each edge in two lists to support efficient “get all neighbors”

Example:

```
A   B   C   D
A   F   T   F   F
B   T   F   T   F
C   F   T   F   T
D   F   F   T   F
```

```plaintext
Example:
A   B   C   D
A   B / 
B   A   C / 
C   D   B / 
D   C / 
```
Which is better?

Graphs are often sparse:

- Streets form grids
  - every corner is not connected to every other corner
- Airlines rarely fly to all possible cities
  - or if they do it is to/from a hub rather than directly to/from all small cities to other small cities

Adjacency lists should generally be your default choice

- Slower performance compensated by greater space savings
Okay, we can represent graphs

Now let’s implement some useful and non-trivial algorithms

- **Topological sort**: Given a DAG, order all the vertices so that every vertex comes before all of its neighbors

- **Shortest paths**: Find the shortest or lowest-cost path from $x$ to $y$
  - Related: Determine if there even is such a path