Announcements

- **Project 2** – let us know by tonight if you plan on working with a partner. Phase A due next Wednesday
- **Homework 3** – due Friday Feb 1st at the BEGINNING of lecture
Today

• Dictionaries
  – Hashing
### Motivating Hash Tables

For a dictionary with $n$ key/value pairs:

<table>
<thead>
<tr>
<th>Data Structure</th>
<th>Insert</th>
<th>Find</th>
<th>Delete</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unsorted linked-list</td>
<td>$O(1)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>Unsorted array</td>
<td>$O(1)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>Sorted linked list</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>Sorted array</td>
<td>$O(n)$</td>
<td>$O(\log n)$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>Balanced tree</td>
<td>$O(\log n)$</td>
<td>$O(\log n)$</td>
<td>$O(\log n)$</td>
</tr>
</tbody>
</table>
Hash Tables

- Aim for constant-time (i.e., $O(1)$) find, insert, and delete
  - “On average” under some reasonable assumptions
- A hash table is an array of some fixed size
- Basic idea:

hash function: \[
\text{index} = h(\text{key})
\]

key space (e.g., integers, strings)
Aside: Hash Tables vs. Balanced Trees

• In terms of a Dictionary ADT for just insert, find, delete, hash tables and balanced trees are just different data structures
  – Hash tables $O(1)$ on average (assuming few collisions)
  – Balanced trees $O(\log n)$ worst-case

• Constant-time is better, right?
  – Yes, but you need “hashing to behave” (must avoid collisions)
  – Yes, but findMin, findMax, predecessor, and successor go from $O(\log n)$ to $O(n)$, printSorted from $O(n)$ to $O(n \log n)$
    • Why your textbook considers this to be a different ADT
    • Not so important to argue over the definitions
Hash Tables

- There are $m$ possible keys ($m$ typically large, even infinite)
- We expect our table to have only $n$ items
- $n$ is much less than $m$ (often written $n << m$)

Many dictionaries have this property

- Compiler: All possible identifiers allowed by the language vs. those used in some file of one program
- Database: All possible student names vs. students enrolled
- AI: All possible chess-board configurations vs. those considered by the current player
- ...
Hash functions

An ideal hash function:
- Is fast to compute
- “Rarely” hashes two “used” keys to the same index
  - Often impossible in theory; easy in practice
  - Will handle collisions a bit later
Who hashes what?

• Hash tables can be generic
  – To store elements of type $E$, we just need $E$ to be:
    1. Comparable: order any two $E$ (like with all dictionaries)
    2. Hashable: convert any $E$ to an int

• When hash tables are a reusable library, the division of responsibility generally breaks down into two roles:

  ![Diagram of hash table interaction]

• We will learn both roles, but most programmers “in the real world” spend more time as clients while understanding the library
More on roles

Some ambiguity in terminology on which parts are “hashing”

Two roles must both contribute to minimizing collisions (heuristically)

- Client should aim for different ints for expected items
  - Avoid “wasting” any part of \( E \) or the 32 bits of the int
- Library should aim for putting “similar” ints in different indices
  - Conversion to index is almost always “mod table-size”
  - Using prime numbers for table-size is common
What to hash?

• We will focus on two most common things to hash: ints and strings

• If you have objects with several fields, it is usually best to have most of the “identifying fields” contribute to the hash to avoid collisions

• Example:
  
  ```java
  class Person {
      String first; String middle; String last;
      Date birthdate;
  }
  ```

• An inherent trade-off: hashing-time vs. collision-avoidance
  – Bad idea(?): Only use first name
  – Good idea(?): Only use middle initial
  – Admittedly, what-to-hash is often an unprincipled guess 😞
# Hashing integers

key space = integers

Simple hash function:

\[ h(key) = key \mod TableSize \]

- Client: \( f(x) = x \)
- Library \( g(x) = f(x) \mod TableSize \)
- Fairly fast and natural

Example:

- TableSize = 10
- Insert 7, 18, 41, 34, 10
- (As usual, ignoring corresponding data)
Hashing integers (Soln)

key space = integers

Simple hash function:

\[ h(key) = key \mod TableSize \]

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Collision-avoidance

- With \( x \% \) TableSize the number of collisions depends on
  - the ints inserted (obviously)
  - TableSize

- Larger table-size tends to help, but not always
  - Example: 70, 24, 56, 43, 10
    with TableSize = 10 and TableSize = 60

- Technique: Pick table size to be prime. Why?
  - Real-life data tends to have a pattern
  - “Multiples of 61” are probably less likely than “multiples of 60”
  - We’ll see some collision strategies do better with prime size
More arguments for a prime table size

If `TableSize` is 60 and…
- Lots of data items are multiples of 5, wasting 80% of table
- Lots of data items are multiples of 10, wasting 90% of table
- Lots of data items are multiples of 2, wasting 50% of table

If `TableSize` is 61…
- Collisions can still happen, but 5, 10, 15, 20, … will fill table
- Collisions can still happen but 10, 20, 30, 40, … will fill table
- Collisions can still happen but 2, 4, 6, 8, … will fill table

In general, if \( x \) and \( y \) are “co-prime” (means \( \gcd(x,y) == 1 \)), then
- \((a \times x) \mod y == (b \times x) \mod y \) if and only if \( a \mod y == b \mod y \)
- Given table size \( y \) and keys as multiples of \( x \), we’ll get a decent distribution if \( x \) & \( y \) are co-prime
- So good to have a `TableSize` that has no common factors with any “likely pattern” \( x \)
What if the key is not an int?

- If keys aren’t ints, the client must convert to an int
  - Trade-off: speed and distinct keys hashing to distinct ints

- Common and important example: Strings
  - Key space $K = s_0s_1s_2…s_{m-1}$
    - where $s_i$ are chars: $s_i \in [0,256]$
  - Some choices: Which avoid collisions best?

1. $h(K) = s_0 \% \text{TableSize}$

2. $h(K) = \left( \sum_{i=0}^{m-1} s_i \right) \% \text{TableSize}$

3. $h(K) = \left( \sum_{i=0}^{m-1} s_i \cdot 37^i \right) \% \text{TableSize}$
Specializing hash functions

How might you hash differently if all your strings were web addresses (URLs)?
Aside: Combining hash functions

A few rules of thumb / tricks:

1. Use all 32 bits (careful, that includes negative numbers)
2. Use different overlapping bits for different parts of the hash
   - This is why a factor of $37^i$ works better than $256^i$
   - Example: “abcde” and “ebcda”
3. When smashing two hashes into one hash, use bitwise-xor
   - bitwise-and produces too many 0 bits
   - bitwise-or produces too many 1 bits
4. Rely on expertise of others; consult books and other resources
5. If keys are known ahead of time, choose a perfect hash
Collision resolution

Collision:
When two keys map to the same location in the hash table

We try to avoid it, but number-of-keys exceeds table size

So hash tables should support collision resolution
    – Ideas?
Flavors of Collision Resolution

Separate Chaining

Open Addressing
- Linear Probing
- Quadratic Probing
- Double Hashing
Separate Chaining

Chaining: All keys that map to the same table location are kept in a list (a.k.a. a "chain" or "bucket")

As easy as it sounds

Example: insert 10, 22, 107, 12, 42 with mod hashing and TableSize = 10
Separate Chaining

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Worst case time for find?
Thoughts on separate chaining

• Worst-case time for find?
  – Linear
  – But only with really bad luck or bad hash function
  – So not worth avoiding (e.g., with balanced trees at each bucket)
    • Keep # of items in each bucket small
    • Overhead of AVL tree, etc. not worth it for small n

• Beyond asymptotic complexity, some “data-structure engineering” can improve constant factors
  – Linked list vs. array or a hybrid of the two
  – Move-to-front (part of Project 2)
  – Leave room for 1 element (or 2?) in the table itself, to optimize constant factors for the common case
    • A time-space trade-off…
Time vs. space (constant factors only here)

0  | 10 /
1  | /
2  | 42  → 12 → 22 /
3  | /
4  | /
5  | /
6  | /
7  | 107 /
8  | /
9  | /

0  | 10 /
1  | / X
2  | 42
3  | / X
4  | / X
5  | / X
6  | / X
7  | 107 /
8  | / X
9  | / X
More rigorous separate chaining analysis

Definition: The load factor, $\lambda$, of a hash table is

$$\lambda = \frac{N}{\text{TableSize}} \quad \leftarrow \text{number of elements}$$

Under chaining, the average number of elements per bucket is ___

So if some inserts are followed by random finds, then on average:
- Each unsuccessful find compares against ____ items
- Each successful find compares against _____ items

- How big should TableSize be??
More rigorous separate chaining analysis

Definition: The **load factor**, $\lambda$, of a hash table is

$$\lambda = \frac{N}{\text{TableSize}} \quad \leftarrow \text{number of elements}$$

Under chaining, the average number of elements per bucket is $\lambda$

So if some inserts are followed by *random* finds, then on average:
- Each unsuccessful `find` compares against $\lambda$ items
- Each successful `find` compares against $\lambda/2$ items
- If $\lambda$ is low, find & insert likely to be O(1)
- We like to keep $\lambda$ around 1 for separate chaining
Load Factor?

\[ \lambda = \frac{n}{TableSize} = \frac{5}{10} = 0.5 \]
**Load Factor?**

![Load Factor Diagram]

\[
\lambda = \frac{n}{TableSize} = \frac{21}{10} = 2.1
\]
Separate Chaining Deletion?
Separate Chaining Deletion

• Not too bad
  – Find in table
  – Delete from bucket
• Say, delete 12
• Similar run-time as insert