CSE 332 Data Abstractions:

A Heterozygous Forest of AVL, Splay, and B Trees

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From last time…

Binary search trees can give us great performance due to providing a structured binary search.

This only occurs if the tree is balanced.

Three Flavors of Balance

How to guarantee efficient search trees has been an active area of data structure research

We will explore three variations of "balancing":

- AVL Trees:
  Guaranteed balanced BST with only constant time additional overhead
- Splay Trees:
  Ignore balance, focus on recency
- B Trees:
  n-ary balanced search trees that work well with real world memory/disks

Arboreal masters of balance

AVL TREES

Achieving a Balanced BST (part 1)

For a BST with n nodes inserted in arbitrary order

- Average height is O(log n) – see text
- Worst case height is O(n)
- Simple cases, such as pre-sorted, lead to worst-case scenario
- Inserts and removes can and will destroy any current balance

Achieving a Balanced BST (part 2)

Shallower trees give better performance

- This happens when the tree's height is O(log n) ← like a perfect or complete tree

Solution: Require a **Balance Condition** that

1. ensures depth is always $O(\log n)$
2. is easy to maintain

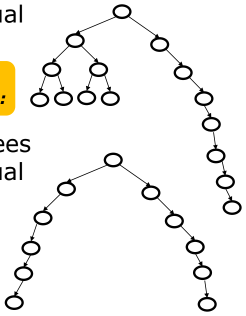
Potential Balance Conditions

1. Left and right subtrees of the *root* have equal number of nodes

Too weak!
Height mismatch example:

2. Left and right subtrees of the *root* have equal *height*

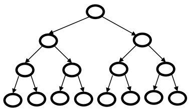
Too weak!
Double chain example:



Potential Balance Conditions

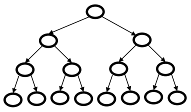
3. Left and right subtrees of every node have equal number of nodes

Too strong!
Only perfect trees
(2^n – 1 nodes)



4. Left and right subtrees of every node have equal *height*

Too strong!
Only perfect trees
(2^n – 1 nodes)



The AVL Balance Condition

Left and right subtrees of every node have heights differing by at most 1

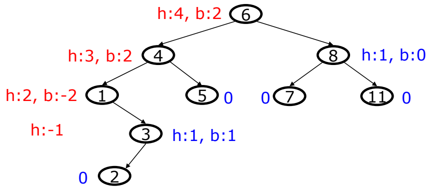
Mathematical Definition:

For every node x, –1 ≤ **balance**(x) ≤ 1 where

balance(node)
= height(node.left) – height(node.right)

An AVL Tree?

To check if this tree is an AVL, we calculate the heights and balances for each node



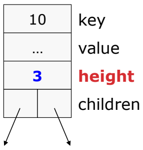
AVL Balance Condition

Ensures small depth

- Can prove by showing an AVL tree of height h must have nodes exponential in h

Efficient to maintain

- Requires adding a height parameter to the node class (Why?)
- Balance is maintained through constant time manipulations of the tree structure: *single* and *double rotations*



Calculating Height

What is the height of a tree with root r?

```
int treeHeight(Node root) {
  if(root == null)
    return -1;
  return 1 + max(treeHeight(root.left),
                 treeHeight(root.right));
}
```

Running time for tree with *n* nodes:
O(*n*) – single pass over tree

Very important detail of definition:
height of a null tree is -1, height of tree with a single node is 0

Height of an AVL Tree?

Using the AVL balance property, we can determine the minimum number of nodes in an AVL tree of height *h*

Recurrence relation:
Let $\mathbf{S}(h)$ be the minimum nodes in height *h*, then

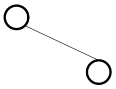
$$\mathbf{S}(h) = \mathbf{S}(h-1) + \mathbf{S}(h-2) + 1$$

where $\mathbf{S}(-1) = 0$ and $\mathbf{S}(0) = 1$

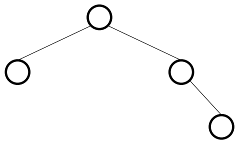
Solution of Recurrence: $\mathbf{S}(h) \approx 1.62^h$

Minimal AVL Tree (height = 0)

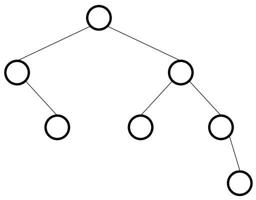
Minimal AVL Tree (height = 1)



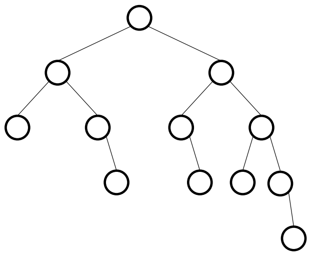
Minimal AVL Tree (height = 2)



Minimal AVL Tree (height = 3)



Minimal AVL Tree (height = 4)



AVL Tree Operations

AVL find:

- Same as BST find

AVL insert:

- Starts off the same as BST insert
- Then check balance of tree
- Potentially fix the AVL tree (4 imbalance cases)

AVL delete:

- Do the deletion
- Then handle imbalance (same as insert)

Insert / Detect Potential Imbalance

Insert the new node (at a leaf, as in a BST)

- For each node on the path from the new leaf to the root
- The insertion may, or may not, have changed the node's height

After recursive insertion in a subtree

- detect height imbalance
- perform a rotation to restore balance at that node

All the action is in defining the correct rotations to restore balance

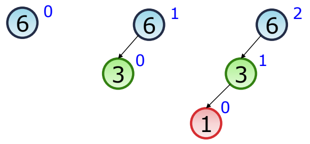
The Secret

If there is an imbalance, then there must be a deepest element that is imbalanced

- After rebalancing this deepest node, every node is then balanced
- Ergo, at most one node needs rebalancing

Example

Insert(6)
Insert(3)
Insert(1)



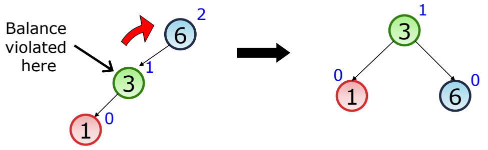
Third insertion violates balance

What is a way to fix this?

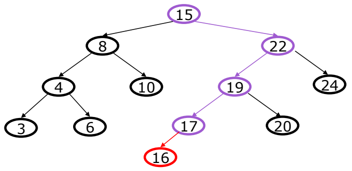
Single Rotation

The basic operation we use to rebalance

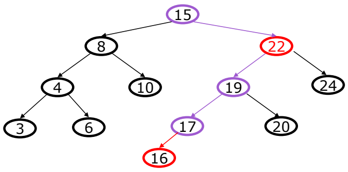
- Move child of unbalanced node into parent position
- Parent becomes a "other" child
- Other subtrees move as allowed by the BST


Balance violated here

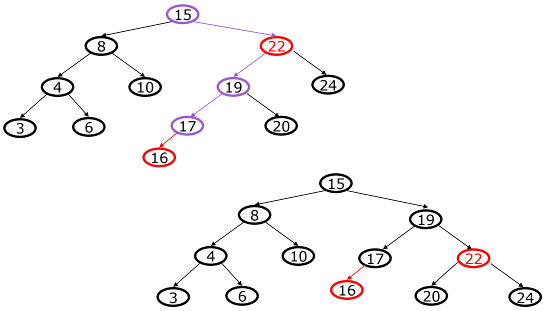
Single Rotation Example: Insert(16)



Single Rotation Example: Insert(16)



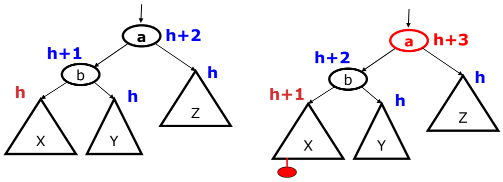
Single Rotation Example: Insert(16)



Left-Left Case

Node imbalanced due to insertion in left-left grandchild (1 of 4 imbalance cases)

First we did the insertion, which made a imbalanced

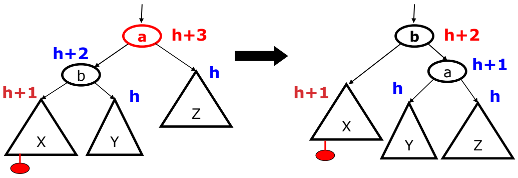


Left-Left Case

So we rotate at a, using BST facts:
X < b < Y < a < Z

A single rotation restores balance at the node

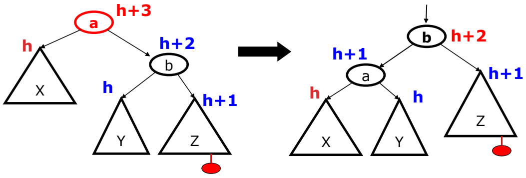
- Node is same height as before insertion, so ancestors now balanced



Right-Right Case

Mirror image to left-left case, so you rotate the other way

- Exact same concept, but different code

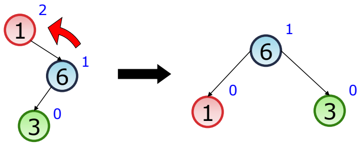


The Other Two Cases

Single rotations not enough for insertions left-right or right-left subtree

- Simple example: insert(1), insert(6), insert(3)

First wrong idea: single rotation as before

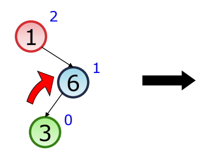


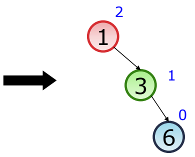
The Other Two Cases

Single rotations not enough for insertions left-right or right-left subtree

- Simple example: insert(1), insert(6), insert(3)

Second wrong idea: single rotation on child





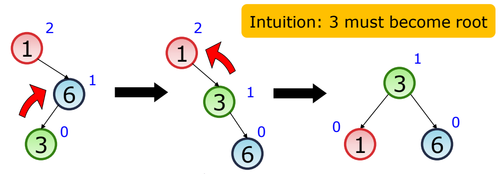
Double Rotation

First attempt at violated the BST property

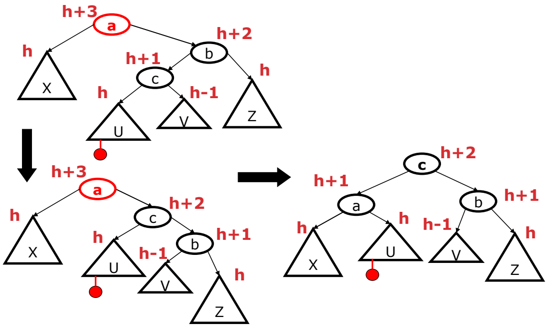
Second attempt did not fix balance

Double rotation: If we do both, it works!

- Rotate problematic child and grandchild
- Then rotate between self and new child


Intuition: 3 must become root

Right-Left Case

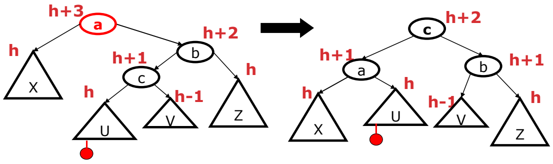


Right-Left Case

Height of the subtree after rebalancing is the same as before insert

- No ancestor in the tree will need rebalancing

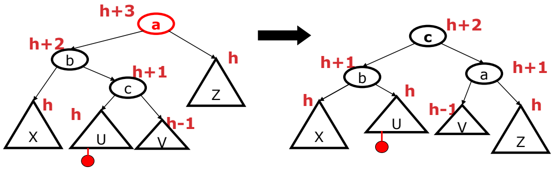
Does not have to be implemented as two rotations; can just do:



Left-Right Case

Mirror image of right-left

- No new concepts, just additional code to write

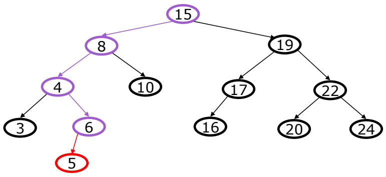


Memorizing Double Rotations

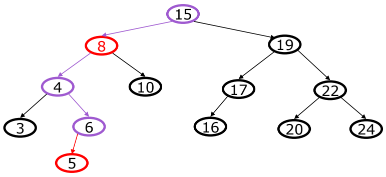
Easier to remember than you may think:

- Move grandchild c to grandparent's position
- Put grandparent a, parent b, and subtrees X, U, V, and Z in the only legal position

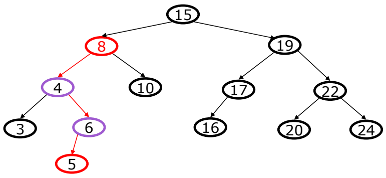
Double Rotation Example: Insert(5)



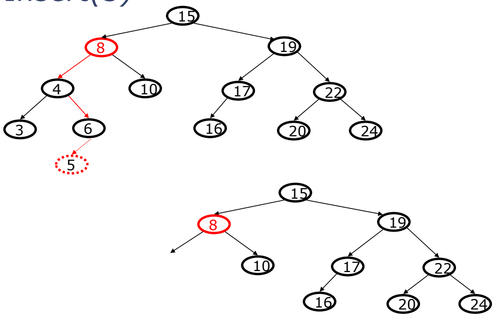
Double Rotation Example: Insert(5)



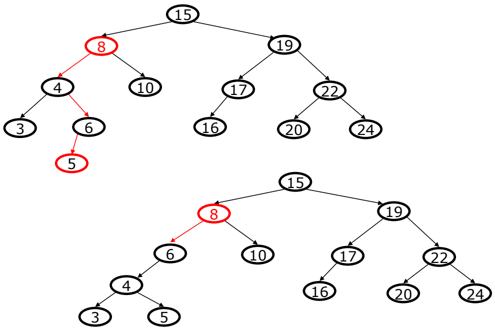
Double Rotation Example: Insert(5)



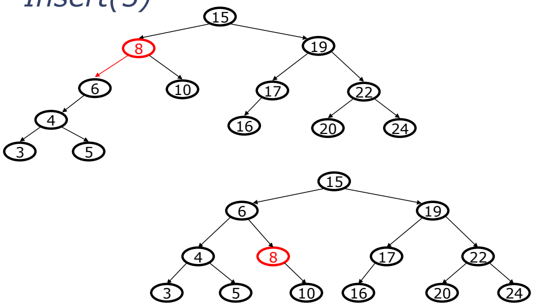
Double Rotation Example: Insert(5)



Double Rotation Example: Insert(5)



Double Rotation Example: Insert(5)



Summarizing Insert

Insert as in a BST

Check back up path for imbalance for 1 of 4 cases:
- node's left-left grandchild is too tall
- node's left-right grandchild is too tall
- node's right-left grandchild is too tall
- node's right-right grandchild is too tall

Only one case can occur, because tree was balanced before insert

After rotations, the smallest-unbalanced subtree now has the same height as before the insertion
- So all ancestors are now balanced

Efficiency

Worst-case complexity of **find**: *O*(log *n*)

Worst-case complexity of **insert**: *O*(log *n*)
- Rotation is *O*(1)
- There's an *O*(log *n*) path to root
- Even without "one-rotation-is-enough" fact this still means *O*(log n) time

Worst-case complexity of **buildTree**: *O*(*n* log *n*)

Delete

We will not cover delete in detail
- Read the textbook
- May cover in section

Basic idea:
- Do the delete as in a BST
- Where you start the balancing check depends on if a leaf or a node with children was removed
- In latter case, you will start from the predecessor/successor for the balancing check

delete is also *O*(log *n*)

If this were a medical class, we would be discussing urine thresholds and kidney function

SPLAY TREES

Balancing Takes a Lot of Work

To make AVL trees work, we needed:
- Extra info for each node
- Complex logic to detect imbalance
- Recursive bottom-up implementation

Can we do better with less work?

Splay Trees

Here's an insane idea:
- Let's take the rotating idea of AVL trees but do it without any care (ignore balance)
- Insert/Find always rotate node to the root

Seems crazy, right? But…
- Amortized time per operations is O(log n)
- Worst case time per operation is O(n) but is guaranteed to happen very rarely

Amortized Analysis

If a sequence of M operations takes O(M f(n)) time, we say the amortized runtime is O(f(n))

- Average time per operation for any sequence is O(f(n))
- Worst case time for any sequence of M operations is O(M f(n))
- Worst case time per operation can still be large, say O(n)

Amortized complexity is a worst-case guarantee for a sequences of operations

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Interpreting Amortized Analyses

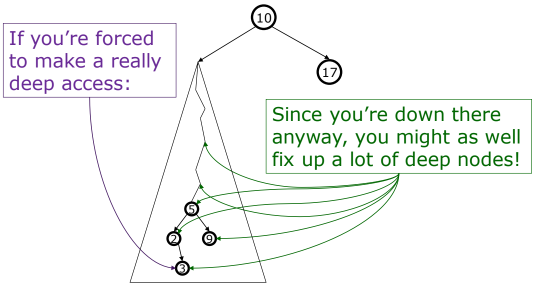
Is amortized guarantee any weaker than worst-case?
Yes, it is only for sequences of operations

Is amortized guarantee stronger than average-case?
Yes, it guarantees no bad sequences

Is average-case guarantee good enough in practice?
No, adversarial input can always happen

Is amortized guarantee good enough in practice?
Yes, due to promise of no bad sequences

The Splay Tree Idea



If you're forced to make a really deep access:

Since you're down there anyway, you might as well fix up a lot of deep nodes!

Find/Insert in Splay Trees

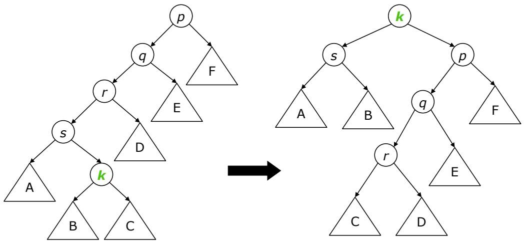
1. Find or insert a node k
2. **Splay k to the root using**: zig-zag, zig-zig, or plain old zig rotation

Splaying moves multiple nodes higher up in the tree (pushing some down too)

How do we do this?

Naïve Approach

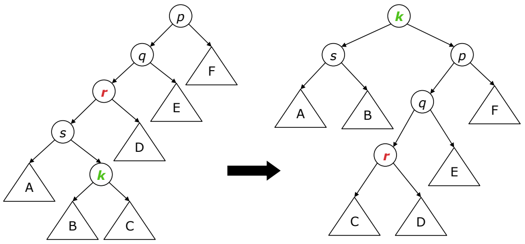
One option is to repeatedly use AVL single rotation until node **k** becomes the root:



Naïve Approach

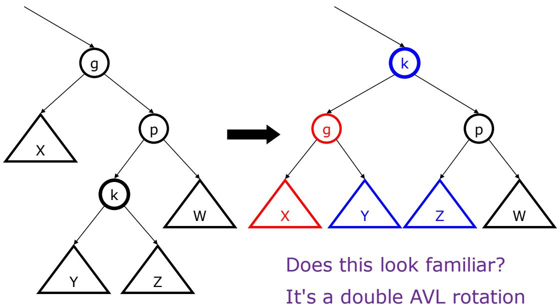
Why this is bad:

- ***r*** gets pushed almost as low as ***k*** was
- Bad sequence: find(k), find(r), find(k), etc.



Splay: Zig-Zag

Blue nodes are Helped
Red nodes are Hurt



Does this look familiar?

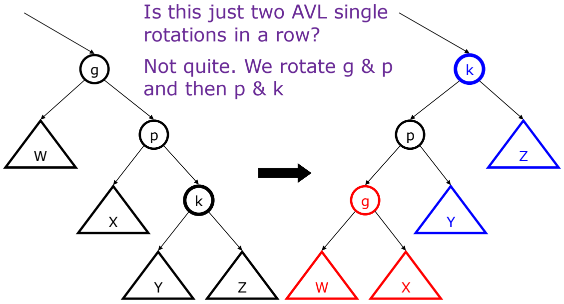
It's a double AVL rotation

Splay: Zig-Zig

Blue nodes are Helped
Red nodes are Hurt

Is this just two AVL single rotations in a row?

Not quite. We rotate g & p and then p & k

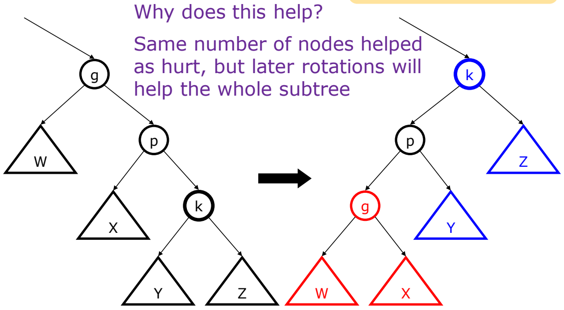


Splay: Zig-Zig

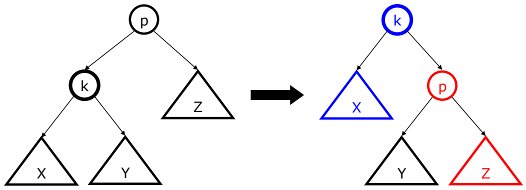
Blue nodes are Helped
Red nodes are Hurt

Why does this help?

Same number of nodes helped as hurt, but later rotations will help the whole subtree



Special Case for Root: Zig



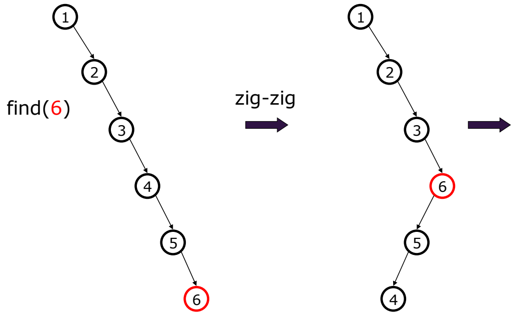
Relative depth of p, Y, and Z?
Down one level

Relative depth of everyone else?
Much better!

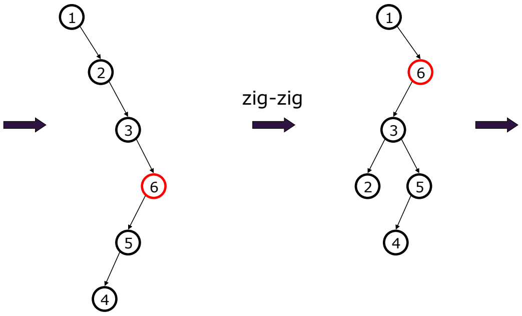
Why not drop zig-zig and just zig all the way?

No! Zig helps **one** child subtree. Zig-zig helps **two**!

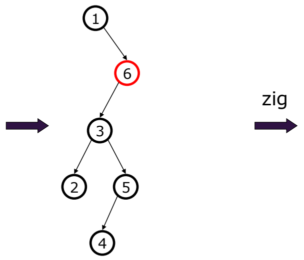
Splaying Example: find(6)



Still Splaying 6

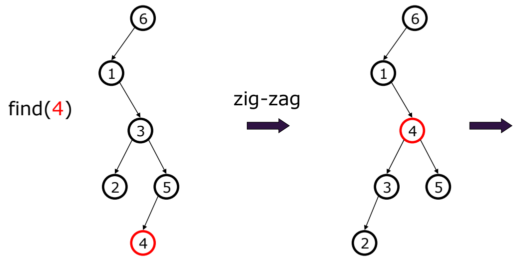


Stay on target…



zig

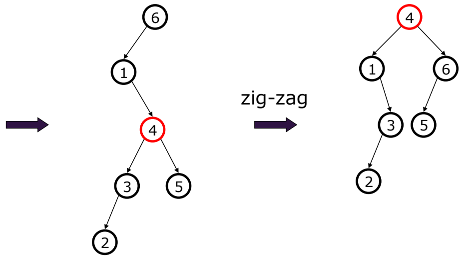
Splay Again: find(4)



find(4)

zig-zag

Almost there…



zig-zag

Wait a sec…

What happened here?

- Didn't the two find operations take linear time instead of logarithmic?
- What about the amortized O(log n) guarantee?

The guarantee still holds

- We must take into account the previous steps used to create this tree.
- The analysis says that some operations may be linear, but they average out in the long run

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Why Splaying Helps

If a node *k* on the access path is at depth *d* before the splay

It's at about depth *d/2* after the splay

Overall, nodes which are low on the access path tend to move closer to the root

Importantly, we fix up/balance the tree every time we do an expensive (deep) access

- This gives splaying its amortized O(log n) performance (Maybe not now, but soon, and for the rest of the operations)

Further Practical Benefits of Splaying

No heights to maintain/No imbalances to check

- Less storage per node
- Easier to code (seriously!)

Data accessed once is often soon accessed again

- Splaying does implicit *caching* to the root
- This important idea is known as *locality*

Splay Operations: find

1. Find the node in normal BST manner
2. Splay the node to the root
   - if node not found, splay what would have been the node's parent

What if we didn’t splay?

- The amortized guarantee would fail!
- Consider this sequence with k not in tree: find(k), find(k), find(k), …
- Splaying would make the second find(k) a constant time operation

Splay Operations: Insert

To insert, could do an ordinary BST insert

- That would not fix up tree
- A BST insert followed by a find and splay?

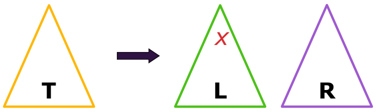
Better idea: Splay before the insert!

- How? A combination of find and split
- What's split?

Splitting in Binary Search Trees

split(T, *x*) creates from T two BSTs L and R:

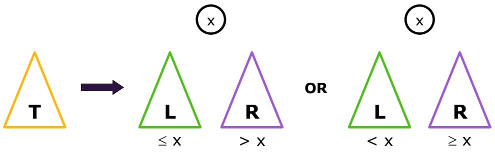
- All elements of T are in either subtree L or R (T = L ∪ R)
- All elements in L are ≤ *x*
- All elements in R are ≥ *x*
- L and R share no elements (L ∩ R = ∅)



Splay Operations: Split

To split, do a find on x:

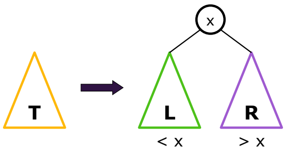
- If x is in T, then splay x to the root
- Otherwise splay the last node found to the root
- After splaying split the tree at the root



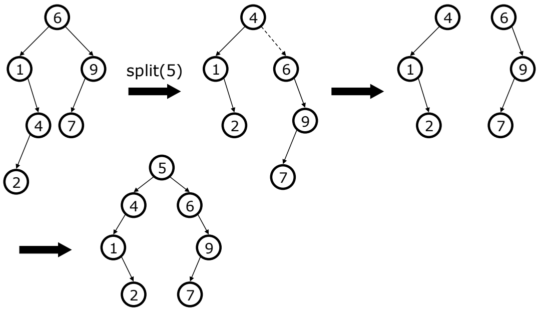
Back to Insert

insert(x):

- Split on x
- Join subtrees using x as root



Insert Example: insert(5)

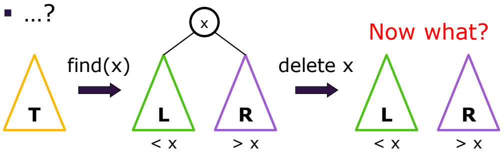


Splay Operations: Delete

The other operations splayed, so we'd better do that for delete as well

delete(x):

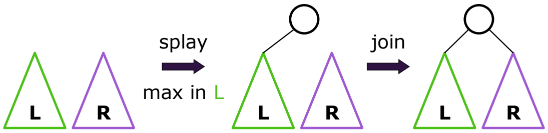
- find x and splay to root
- if x is there, remove it
- …?



Join Operation

Join(L, R) merges two trees L < R

- Splay on the maximum element in L then attach R



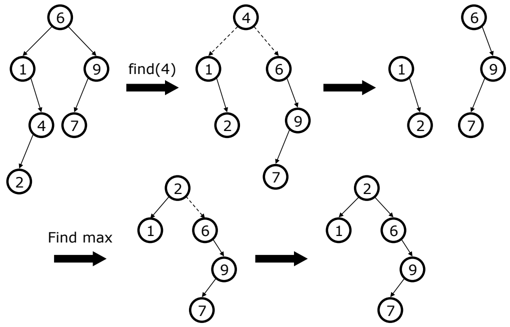
Similar to BST delete:
find max = find element with no right child

Splay Operations: Delete

delete(x):

- find x and splay to root
- if x is there, remove it
- join the resulting subtrees

Delete Example: delete(4)



Technically, they are called B+ trees but their name was lowered due to concerns of grade inflation

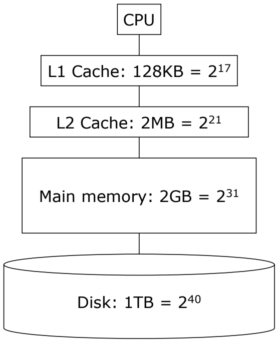
B TREES

Reality Bites

Despite our best efforts, AVL trees and splay trees can perform poorly on very large inputs

Why? It's the fault of hardware!

A Typical Memory Hierarchy



CPU

instructions (e.g., addition): 2^{30}/sec

L1 Cache: 128KB = 2^{17}

get data in L1: 2^{29}/sec = 2 insns

L2 Cache: 2MB = 2^{21}

get data in L2: 2^{25}/sec = 30 insns

Main memory: 2GB = 2^{31}

get data in main memory: 2^{22}/sec = 250 insns

Disk: 1TB = 2^{40}

get data from "new place" on disk: 2^{7}/sec =8,000,000 insns

"streamed": 2^{18}/sec

Moral of The Story

It is much faster to do:	Than:
5 million arithmetic ops	1 disk access
2500 L2 cache accesses	1 disk access
400 main memory accesses	1 disk access

Accessing the disk is EXPENSIVE!!!

Why are computers built this way?

- Physical realities of speed of light and relative closeness to CPU
- Cost (price per byte of different technologies)
- Disks get much bigger not much faster
  - 7200 RPM spin is slow compared to RAM
  - Disks unlikely to spin faster in the future
- Solid-state drives are faster than disks but still slower due to distance, write performance, etc.
- Speedups at higher levels generally make lower levels relatively slower

Dealing with Latency

Moving data up the memory hierarchy is slow because of latency

We can do better by grabbing surrounding memory with each request

- It is easy to do since we are there anyways
- Likely to be asked for soon (locality of reference)

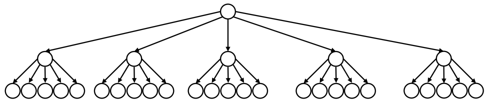
As defined by the operating system:

- Amount moved from disk to memory is called *block* or *page* size
- Amount moved from memory to cache is called the *line* size

M-ary Search Tree

Build a search tree with branching factor M:

- Have an array of sorted children (Node[])
- Choose M to fit snugly into a disk block (1 access for array)



Perfect tree of height h has $(M^{h+1}-1)/(M-1)$ nodes

hops for find: Use $\log_M n$ to calculate

- If M=256, that's an 8x improvement
- If n = 2^{40}, only 5 levels instead of 40 (5 disk accesses)

Runtime of find if balanced: $O(\log_2 M \log_M n)$

Problems with M-ary Search Trees

- What should the order property be?
- How would you rebalance (ideally without more disk accesses)?
- Any "useful" data at the internal nodes takes up disk-block space without being used by finds moving past it
- Use the branching-factor idea, but for a different kind of balanced tree
  - Not a binary search tree
  - But still logarithmic height for any M > 2

B+ Trees (will just say "B Trees")

Two types of nodes:

- Internal nodes and leaf nodes

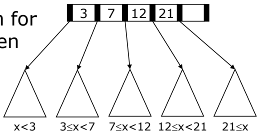
Each internal node has room for up to M-1 *keys* and M children

- All data are at the leaves!

Order property:

- Subtree between x and y
  Data that is ≥ x and < y
- Notice the ≥

Leaf has up to L sorted *data* items



As usual, we will focus only on the keys in our examples

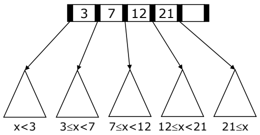
B Tree Find

We are used to data at internal nodes

But find is still an easy root-to-leaf algorithm

- At an internal node, binary search on the M-1 keys
- At the leaf do binary search on the ≤ L data items

To ensure logarithmic running time, we need to guarantee balance!



What should the balance condition be?

Structure Properties

Root (special case)

- If tree has ≤ L items, root is a leaf (occurs when starting up, otherwise very unusual)
- Otherwise, root has between 2 and M children

Internal Node

- Has between ⌈M/2⌉ and M children (at least half full)

Leaf Node

- All leaves at the same depth
- Has between ⌈L/2⌉ and L items (at least half full)

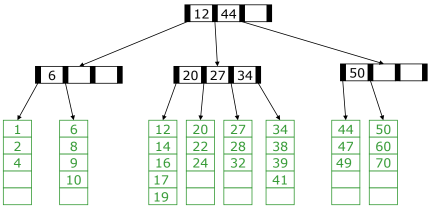
Any M > 2 and L will work

- Picked based on disk-block size

Example

Suppose: M=4 (max # children in internal node)
L=5 (max # data items at leaf)

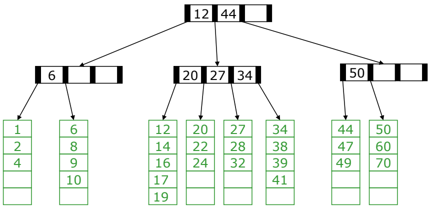
- All internal nodes have at least 2 children
- All leaves at same depth with at least 3 data items



Example

Note on notation:

- Inner nodes drawn horizontally
- Leaves drawn vertically to distinguish
- Includes all empty cells



Balanced enough

Not hard to show height *h* is logarithmic in number of data items *n*

Let M > 2 (if M = 2, then a list tree is legal → BAD!)

Because all nodes are at least half full (except root may have only 2 children) and all leaves are at the same level, the minimum number of data items n for a height h>0 tree is…

$$n \geq \underbrace{2 \lceil M/2 \rceil^{h-1}}_{\text{minimum number of leaves}} \cdot \underbrace{\lceil L/2 \rceil}_{\text{minimum data per leaf}}$$

Exponential in height because ⌈M/2⌉ > 1

What makes B trees so disk friendly?

Many keys stored in one **internal node**

- All brought into memory in one disk access
- But only if we pick *M* wisely
- Makes the binary search over *M*-1 keys worth it (insignificant compared to disk access times)

Internal nodes contain only keys

- Any **`find`** wants only one data item; wasteful to load unnecessary items with internal nodes
- Only bring one **leaf** of data items into memory
- Data-item size does not affect what *M* is

Maintaining Balance

So this seems like a great data structure

It is

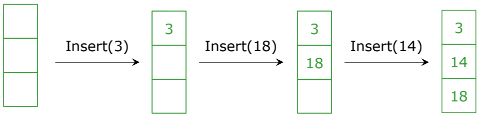
But we haven't implemented the other dictionary operations yet

- insert
- delete

As with AVL trees, the hard part is maintaining structure properties

Building a B-Tree

M = 3 *L* = 3

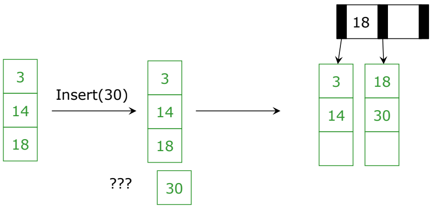


The empty B-Tree (the root will be a leaf at the beginning)

Simply need to keep data sorted

Building a B-Tree

M = 3 *L* = 3



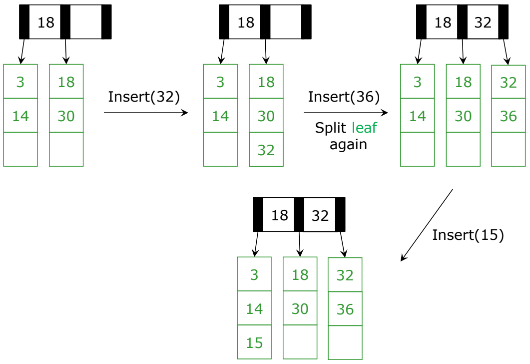
When we 'overflow' a leaf, we split it into 2 leaves

- Parent gains another child
- If there is no parent, we create one

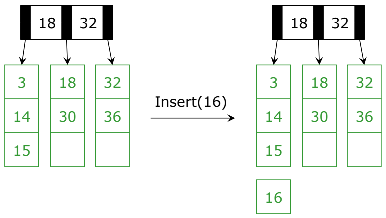
How do we pick the new key?

- Smallest element in right subtree

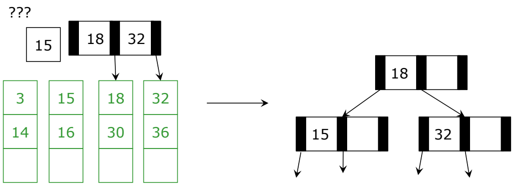
M = 3 *L* = 3



M = 3 *L* = 3

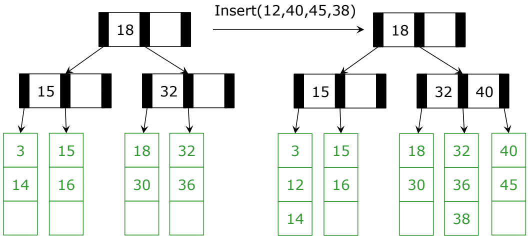


$M = 3$ $L = 3$



Split the internal node
(in this case, the root)

$M = 3$ $L = 3$



Given the leaves and the structure of the tree, we can always fill in internal node keys using the rule:
What is the smallest value in my right branch?

Insertion Algorithm

1. Insert the data in its leaf in sorted order
2. If the leaf now has L+1 items, overflow!
   a. Split the leaf into two nodes:
      - Original leaf with $\lceil (L+1)/2 \rceil$ smaller items
      - New leaf with $\lfloor (L+1)/2 \rfloor = \lceil L/2 \rceil$ larger items
   b. Attach the new child to the parent
      - Adding new key to parent in sorted order
3. If Step 2 caused the parent to have M+1 children, overflow the parent!

Insertion Algorithm (cont)

4. If an internal node (parent) has M+1 kids
   a. Split the node into two nodes
      - Original node with $\lceil (M+1)/2 \rceil$ smaller items
      - New node with $\lfloor (M+1)/2 \rfloor = \lceil M/2 \rceil$ larger items
   b. Attach the new child to the parent
      - Adding new key to parent in sorted order

Step 4 could make the parent overflow too

- Repeat up the tree until a node does not overflow
- If the root overflows, make a new root with two children. This is the only the tree height inceases

Worst-Case Efficiency of Insert

Find correct leaf:	$O(\log_2 M \log_M n)$
Insert in leaf:	$O(L)$
Split leaf:	$O(L)$
Split parents all the way to root:	$O(M \log_M n)$
Total	$O(L + M \log_M n)$

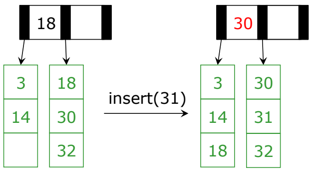
But it's not that bad:

- Splits are rare (only if a node is FULL)
- M and L are likely to be large
- After a split, nodes will be half empty
- Splitting the **root** is thus extremely rare
- Reducing disk accesses is name of the game: inserts are thus $O(\log_M n)$ on average

Adoption for Insert

We can sometimes avoid splitting via a process called adoption

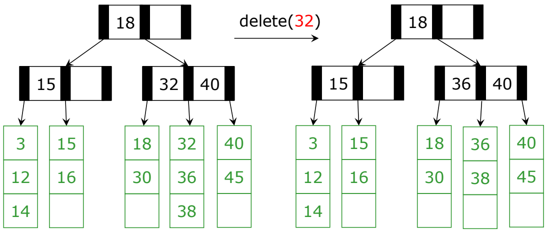
Example:



- Notice correction by changing parent keys
- Implementation not necessary for efficiency

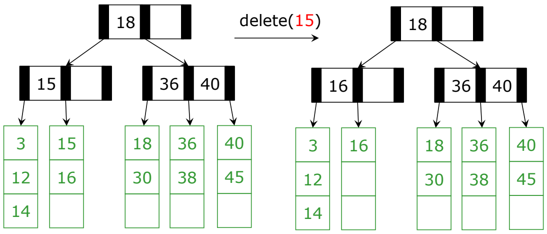
Deletion

M = 3 *L* = 3



delete(32)

M = 3 *L* = 3



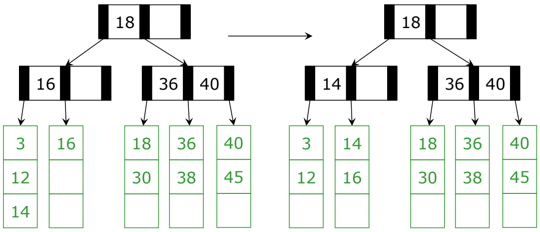
delete(15)

Are we okay?

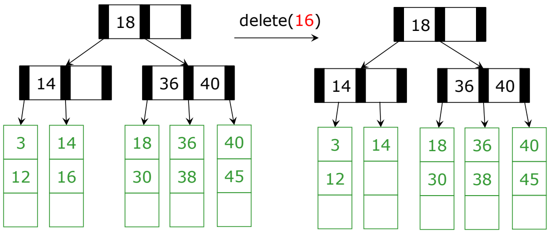
Dang, not half full

Are you using that 14?
Can I borrow it?

M = 3 *L* = 3



M = 3 *L* = 3

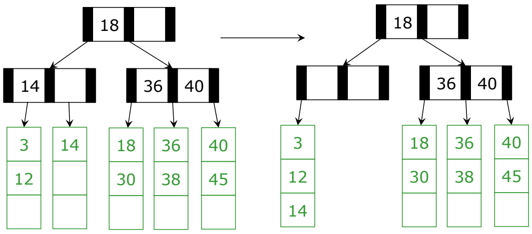


delete(16)

Are you using that 12? Yes

Are you using that 18? Yes

M = 3 *L* = 3

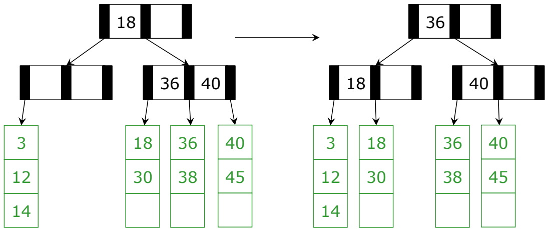


Well, let's just consolidate our leaves since we have the room

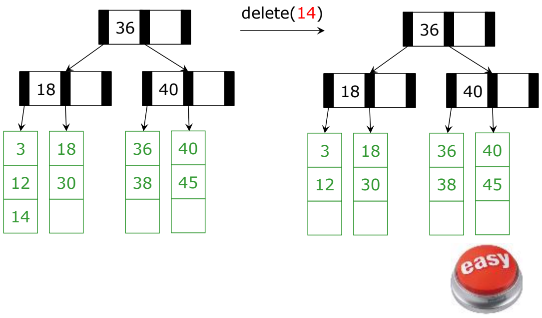
Oops. Not enough leaves

Are you using that 18/30?

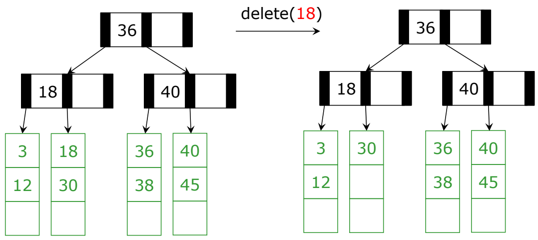
M = 3 *L* = 3



M = 3 *L* = 3

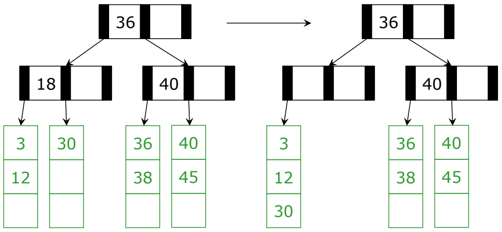


M = 3 *L* = 3



Oops. Not enough leaves

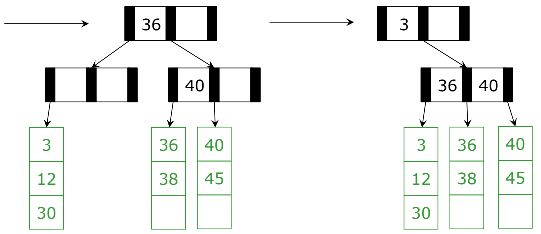
M = 3 *L* = 3



We will borrow as before

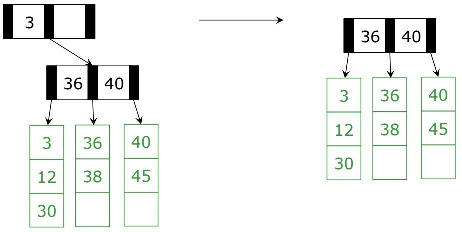
Oh no. Not enough leaves and we cannot borrow!

M = 3 *L* = 3



We have to move up a node and collapse into a new root.

M = 3 *L* = 3



Huh, the root is pretty small. Let's reduce the tree's height.

Deletion Algorithm

1. Remove the data from its leaf
2. If the leaf now has $\lceil L/2 \rceil - 1$, underflow!
   - If a neighbor has $> \lceil L/2 \rceil$ items, adopt and update parent
   - Else merge node with neighbor
     - Guaranteed to have a legal number of items $\lfloor L/2 \rfloor + \lceil L/2 \rceil = L$
     - Parent now has one less node
1. If Step 2 caused parent to have $\lceil M/2 \rceil - 1$ children, underflow!

Deletion Algorithm

4. If an internal node has ⌈M/2⌉ - 1 children
   - If a neighbor has >⌈M/2⌉ items, adopt and update parent
   - Else merge node with neighbor
     - Guaranteed to have a legal number of items
     - Parent now has one less node, may need to continue underflowing up the tree

Fine if we merge all the way up to the root

- If the root went from 2 children to 1, delete the root and make child the root
- This is the only case that decreases tree height

Worst-Case Efficiency of Delete

Find correct leaf:	$O(\log_2 M \log_M n)$
Insert in leaf:	$O(L)$
Split leaf:	$O(L)$
Split parents all the way to root:	$O(M \log_M n)$
Total	$O(L + M \log_M n)$

But it's not that bad:

- Merges are not that common
- After a merge, a node will be over half full
- Reducing disk accesses is name of the game: deletions are thus $O(\log_M n)$ on average

Implementing B Trees in Java?

Assuming our goal is efficient number of disk accesses, Java was not designed for this

This is not a programming languages course

Still, it is worthwhile to know enough about "how Java works" and why this is probably a bad idea for B trees

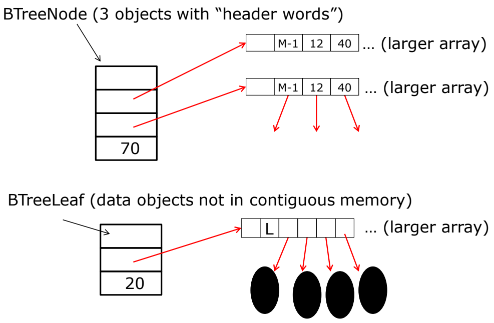
The key issue is extra levels of indirection…

Naïve Approach

Even if we assume data items have int keys, you cannot get the data representation you want for "really big data"

```
interface Keyed<E> {
  int key(E);
}
class BTreeNode<E implements Keyed<E>> {
  static final int M = 128;
  int[] keys = new int[M-1];
  BTreeNode<E>[] children = new BTreeNode[M];
  int numChildren = 0;
  …
}
class BTreeLeaf<E> {
  static final int L = 32;
  E[] data = (E[])new Object[L];
  int numItems = 0;
  …
}
```

What that looks like



The moral

The point of B trees is to keep related data in contiguous memory

All the red references on the previous slide are inappropriate

- As minor point, beware the extra "header words"

But that is "the best you can do" in Java

- Again, the advantage is generic, reusable code
- But for your performance-critical web-index, not the way to implement your B-Tree for terabytes of data

Other languages better support "flattening objects into arrays"

Did we actually get here in one lecture?

FINAL THOUGHTS

Conclusion: Balanced Trees

Balanced trees make good dictionaries because they guarantee logarithmic-time find, insert, and delete

- Essential and beautiful computer science
- But only if you can maintain balance within the time bound and the underlying computer architecture

Another great balanced tree which we sadly will not cover (but easy to read about)

- Red-black trees: all leaves have depth within a factor of 2