# CSE332: Data Abstractions

# Lecture 3: Asymptotic Analysis

Dan Grossman

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# *Gauging performance*

- Uh, why not just run the program and time it
  - Too much *variability*, not reliable or *portable*:
    - Hardware: processor(s), memory, etc.
    - OS, Java version, libraries, drivers
    - Other programs running
    - Implementation dependent
  - Choice of input
    - Testing (inexhaustive) may *miss* worst-case input
    - Timing does not *explain* relative timing among inputs (what happens when *n* doubles in size)

- Often want to evaluate an *algorithm*, not an implementation
  - Even *before* creating the implementation ("coding it up")

# *Comparing algorithms*

When is one *algorithm* (not *implementation*) better than another?

- Various possible answers (clarity, security, …)
- But a big one is *performance*: for sufficiently large inputs, runs in less time (our focus) or less space

*Large inputs* because probably any algorithm is “plenty good” for small inputs (if *n* is 10, probably anything is fast)

Answer will be *independent* of CPU speed, programming language, coding tricks, etc.

Answer is general and rigorous, complementary to “coding it up and timing it on some test cases”

# *Analyzing code ("worst case")*

Basic operations take "some amount of" constant time

- Arithmetic (fixed-width)
- Assignment
- Access one Java field **or array index**
- Etc.

(This is an *approximation of reality*: a very useful "lie".)

| | |
|---|---|
| Consecutive statements | Sum of times |
| Conditionals | Time of test plus slower branch |
| Loops | Sum of iterations |
| Calls | Time of call's body |
| Recursion | Solve *recurrence equation* |

# *Example*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 16 | 37 | 50 | 73 | 75 | 126 |

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
    ???
}
```

# *Linear search*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 16 | 37 | 50 | 73 | 75 | 126 |

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
         return true;
   return false;
}
```

Best case: 6ish steps = $O(1)$

Worst case: 6ish*(arr.length)
= $O(\text{arr.length})$

# *Binary search*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 16 | 37 | 50 | 73 | 75 | 126 |

Find an integer in a *sorted* array

- Can also be done non-recursively but “doesn’t matter” here

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2; // i.e., lo+(hi-lo)/2
   if(lo==hi)        return false;
   if(arr[mid]==k)   return true;
   if(arr[mid]< k)   return help(arr,k,mid+1,hi);
   else              return help(arr,k,lo,mid);
}
```

# *Binary search*

Best case: 8ish steps = $O(1)$

Worst case: $T(n)$ = 10ish + $T(n/2)$ where $n$ is **`hi-lo`**

- $O($**`log`** $n)$ where $n$ is **`array.length`**
- Solve *recurrence equation* to know that…

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2;
   if(lo==hi)       return false;
   if(arr[mid]==k)  return true;
   if(arr[mid]< k)  return help(arr,k,mid+1,hi);
   else             return help(arr,k,lo,mid);
}
```

# *Solving Recurrence Relations*

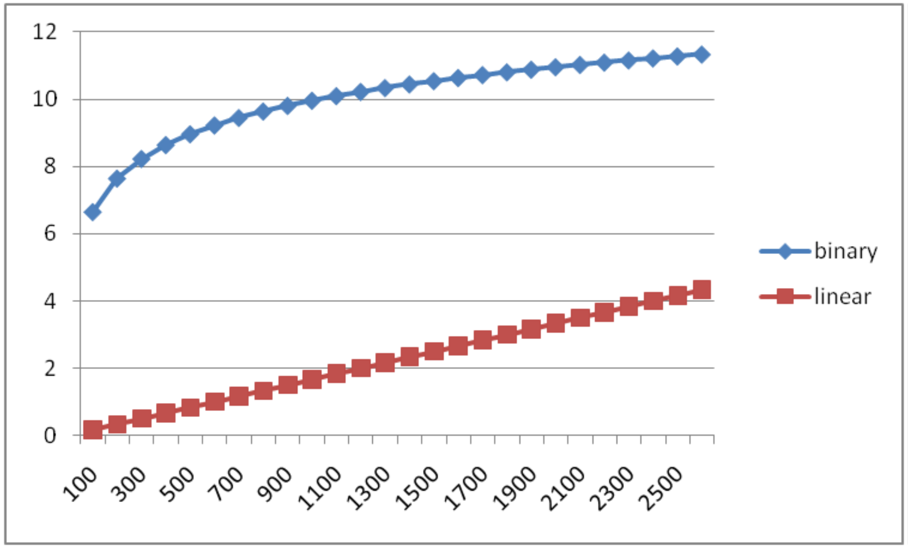
1. Determine the recurrence relation. What is the base case?
   - $T(n) = 10 + T(n/2)$ $\qquad$ $T(1) = 8$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.
   - $T(n) = 10 + 10 + T(n/4)$
     $= 10 + 10 + 10 + T(n/8)$
     $= \ldots$
     $= 10k + T(n/(2^k))$
3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case
   - $n/(2^k) = 1$ means $n = 2^k$ means $k = \log_2 n$
   - So $T(n) = 10 \log_2 n + 8$ (get to base case and do it)
   - So $T(n)$ is $O(\log n)$

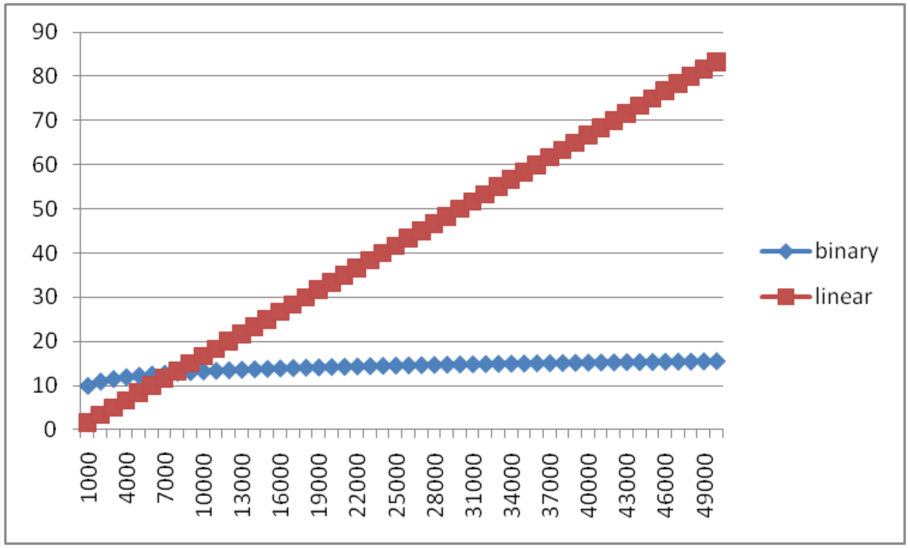
# *Ignoring constant factors*

- So binary search is $O(\texttt{log}\ n)$ and linear is $O(n)$
  - But which is faster
- Could depend on constant factors
  - How *many* assignments, additions, etc. for each $n$
  - And could depend on size of $n$
- But there exists some $n_0$ such that for all $n > n_0$ binary search wins
- Let's play with a couple plots to get some intuition…

# Example

- Let's try to "help" linear search
  - Run it on a computer 100x as fast (say 2010 model vs. 1990)
  - Use a new compiler/language that is 3x as fast
  - Be a clever programmer to eliminate half the work
  - So doing each iteration is 600x as fast as in binary search
- Note: 600x still helpful for problems without logarithmic algorithms!



# *Another example: sum array*

Two "obviously" linear algorithms: $T(n) = O(1) + T(n\text{-}1)$

Iterative:

```
int sum(int[] arr){
  int ans = 0;
  for(int i=0; i<arr.length; ++i)
     ans += arr[i];
  return ans;
}
```

Recursive:

- Recurrence is $k + k + \ldots + k$ for $n$ times

```
int sum(int[] arr){
  return help(arr,0);
}
int help(int[]arr,int i) {
  if(i==arr.length)
    return 0;
  return arr[i] + help(arr,i+1);
}
```

# *What about a binary version?*

```
int sum(int[] arr){
   return help(arr,0,arr.length);
}
int help(int[] arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

Recurrence is $T(n) = O(1) + 2T(n/2)$

- $1 + 2 + 4 + 8 + \ldots$ for **`log`** *n* times
- $2^{(\log n)} - 1$ which is proportional to *n* (definition of logarithm)

Easier explanation: it adds each number once while doing little else

"Obvious": You can't do better than *O(n)* – have to read whole array

# Parallelism teaser

- But suppose we could do two recursive calls *at the same time*
  - *Like having a friend do half the work for you!*

```
int sum(int[]arr){
   return help(arr,0,arr.length);
}
int help(int[]arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

- If you have as many "friends of friends" as needed the recurrence is now $T(n) = O(1) + 1T(n/2)$
  - $O($**log** $n)$ : same recurrence as for **find**

# *Really common recurrences*

Should know how to solve recurrences but also recognize some really common ones:

| | |
|---|---|
| $T(n) = O(1) + T(n\text{-}1)$ | linear |
| $T(n) = O(1) + 2T(n/2)$ | linear |
| $T(n) = O(1) + T(n/2)$ | logarithmic |
| $T(n) = O(1) + 2T(n\text{-}1)$ | exponential |
| $T(n) = O(n) + T(n\text{-}1)$ | quadratic (see previous lecture) |
| $T(n) = O(n) + T(n/2)$ | linear |
| $T(n) = O(n) + 2T(n/2)$ | $O$(n **`log`** n) |

Note big-Oh can also use more than one variable

- Example: can sum all elements of an $n$-by-$m$ matrix in $O(nm)$

# *Asymptotic notation*

About to show formal definition, which amounts to saying:

1. Eliminate low-order terms
2. Eliminate coefficients

Examples:

- $4n + 5$
- $0.5n$ **`log`** $n + 2n + 7$
- $n^3 + 2^n + 3n$
- $n$ **`log`** $(10n^2)$

# *Big-Oh relates functions*

We use $O$ on a function f($n$) (for example $n^2$) to mean *the set of functions with asymptotic behavior* ***less than or equal to*** f($n$)

So $(3n^2+17)$ **is in** $O(n^2)$

- $3n^2+17$ and $n^2$ have the same asymptotic behavior

Confusingly, we also say/write:

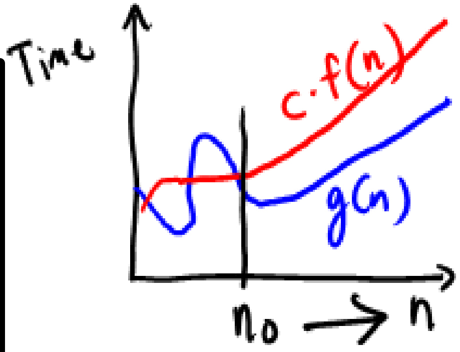
- $(3n^2+17)$ **is** $O(n^2)$
- $(3n^2+17) = O(n^2)$

But we would never say $O(n^2) = (3n^2+17)$

# *Formally Big-Oh (Dr? Ms? Mr? ☺)*

Definition:

g($n$) is in O( f($n$) ) if there exist constants $c$ and $n_0$ such that g($n$) $\leq$ $c$ f($n$) for all $n \geq n_0$




- To show g($n$) is in O( f($n$) ), pick a $c$ large enough to "cover the constant factors" and $n_0$ large enough to "cover the lower-order terms"
  - Example: Let g($n$) = $3n^2$+17 and f($n$) = $n^2$

    $c$=5 and $n_0$=10 is more than good enough

- This is "less than or equal to"
  - So $3n^2$+17 is also $O(n^5)$ and $O(2^n)$ etc.

# *More examples, using formal definition*

- Let $g(n) = 4n$ and $f(n) = n^2$
  - A valid proof is to find valid $c$ and $n_0$
  - The "cross-over point" is $n$=4
  - So we can choose $n_0$=4 and $c$=1
    - Many other possible choices, e.g., larger $n_0$ and/or $c$

Definition:

$g(n)$ is in $O(f(n))$ if there exist constants $c$ and $n_0$ such that $g(n) \leq c\, f(n)$ for all $n \geq n_0$

# *More examples, using formal definition*

- Let $g(n) = n^4$ and $f(n) = 2^n$
  - A valid proof is to find valid $c$ and $n_0$
  - We can choose $n_0=20$ and $c=1$

Definition:

g($n$) is in O( f($n$) ) if there exist constants $c$ and $n_0$ such that $g(n) \leq c\, f(n)$ for all $n \geq n_0$

# *What's with the c*

- The constant multiplier $c$ is what allows functions that differ only in their largest coefficient to have the same asymptotic complexity

- Example: $g(n) = 7n+5$ and $f(n) = n$
  - For any choice of $n_0$, need a $c > 7$ (or more) to show $g(n)$ is in $O(f(n))$

Definition:

$g(n)$ is in $O(f(n))$ if there exist constants $c$ and $n_0$ such that $g(n) \leq c\, f(n)$ for all $n \geq n_0$

# *What you can drop*

- Eliminate coefficients because we don't have units anyway
  - $3n^2$ versus $5n^2$ doesn't mean anything when we have not specified the cost of constant-time operations (can re-scale)

- Eliminate low-order terms because they have vanishingly small impact as $n$ grows

- Do NOT ignore constants that are not multipliers
  - $n^3$ is not $O(n^2)$
  - $3^n$ is not $O(2^n)$

(This all follows from the formal definition)

# *More Asymptotic Notation*

- Upper bound: $O(\, f(n)\, )$ is the set of all functions asymptotically less than or equal to $f(n)$
  - $g(n)$ is in $O(\, f(n)\, )$ if there exist constants $c$ and $n_0$ such that

    $g(n) \leq c\, f(n)$ for all $n \geq n_0$

- Lower bound: $\Omega(\, f(n)\, )$ is the set of all functions asymptotically greater than or equal to $f(n)$
  - $g(n)$ is in $\Omega(\, f(n)\, )$ if there exist constants $c$ and $n_0$ such that

    $g(n) \geq c\, f(n)$ for all $n \geq n_0$

- Tight bound: $\theta(\, f(n)\, )$ is the set of all functions asymptotically equal to $f(n)$
  - Intersection of $O(\, f(n)\, )$ and $\Omega(\, f(n)\, )$ (use *different* $c$ values)

# *Correct terms, in theory*

A common error is to say $O( f(n) )$ when you mean $\theta( f(n) )$

- Since a linear algorithm is also $O(n^5)$, it's tempting to say "this algorithm is exactly $O(n)$"
- That doesn't mean anything, say it is $\theta(n)$
- That means that it is not, for example $O(\mathtt{log}\ n)$

Less common notation:

- "little-oh": intersection of "big-Oh" and *not* "big-Theta"
  - *For all c*, there exists an $n_0$ such that… $\leq$
  - Example: array sum is $o(n^2)$ but not $o(n)$
- "little-omega": intersection of "big-Omega" and *not* "big-Theta"
  - *For all c*, there exists an $n_0$ such that… $\geq$
  - Example: array sum is $\omega(\mathtt{log}\ n)$ but not $\omega(n)$

# *What we are analyzing*

- The most common thing to do is give an $O$ or $\theta$ bound to the worst-case running time of an algorithm

- Example: binary-search algorithm
  - Common: $\theta(\texttt{log}\ n)$ running-time in the worst-case
  - Less common: $\theta(1)$ in the best-case (item is in the middle)
  - Less common: Algorithm is $\Omega(\texttt{log log}\ n)$ in the worst-case (it is not really, really, really fast asymptotically)
  - Less common (but very good to know): the find-in-sorted-array ***problem*** is $\Omega(\texttt{log}\ n)$ in the worst-case
    - *No* algorithm can do better (without parallelism)
    - A ***problem*** cannot be $O(f(n))$ since you can always find a slower algorithm, but can mean ***there exists*** an algorithm

# *Other things to analyze*

- Space instead of time
  - Remember we can often use space to gain time

- Average case
  - Sometimes only if you assume something about the distribution of inputs
    - See CSE312 and STAT391
  - Sometimes uses randomization in the algorithm
    - Will see an example with sorting; also see CSE312
  - Sometimes an *amortized guarantee*
    - Will discuss in a later lecture

# *Summary*

Analysis can be about:

- The problem or the algorithm (usually algorithm)
- Time or space (usually time)
  - Or power or dollars or …
- Best-, worst-, or average-case (usually worst)
- Upper-, lower-, or tight-bound  (usually upper or tight)

# *Usually asymptotic is valuable*

- Asymptotic complexity focuses on behavior for large *n* and is independent of any computer / coding trick
- But you can "abuse" it to be misled about trade-offs
- Example: $n^{1/10}$ vs. **`log`** *n*
  - Asymptotically $n^{1/10}$ grows more quickly
  - But the "cross-over" point is around $5 * 10^{17}$
  - So if you have input size less than $2^{58}$, prefer $n^{1/10}$
- For *small n*, an algorithm with worse asymptotic complexity might be faster
  - Here the constant factors can matter, if you care about performance for small *n*

# *Timing vs. Big-Oh Summary*

- Big-oh is an essential part of computer science's mathematical foundation
  - Examine the algorithm itself, not the implementation
  - Reason about (even prove) performance as a function of *n*

- Timing also has its place
  - Compare implementations
  - Focus on data sets you care about (versus worst case)
  - Determine what the constant factors "really are"
  - Will do some timing on the projects too