CSE332: Data Abstractions

Lecture 11: Hash Tables

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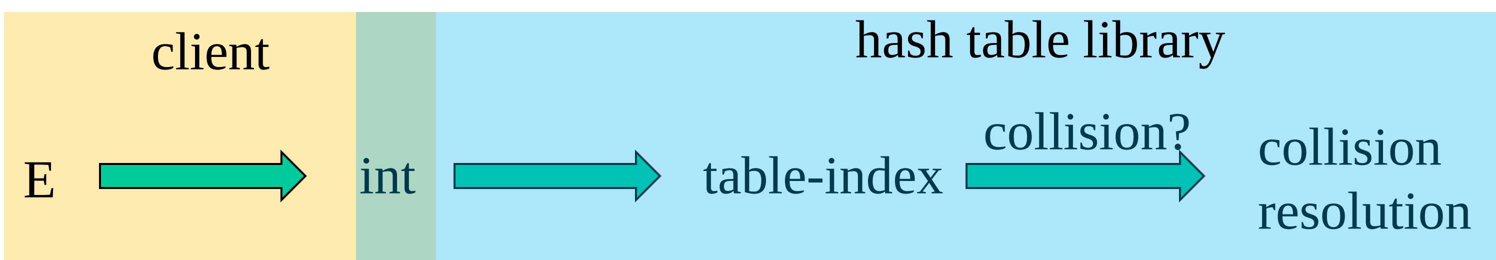
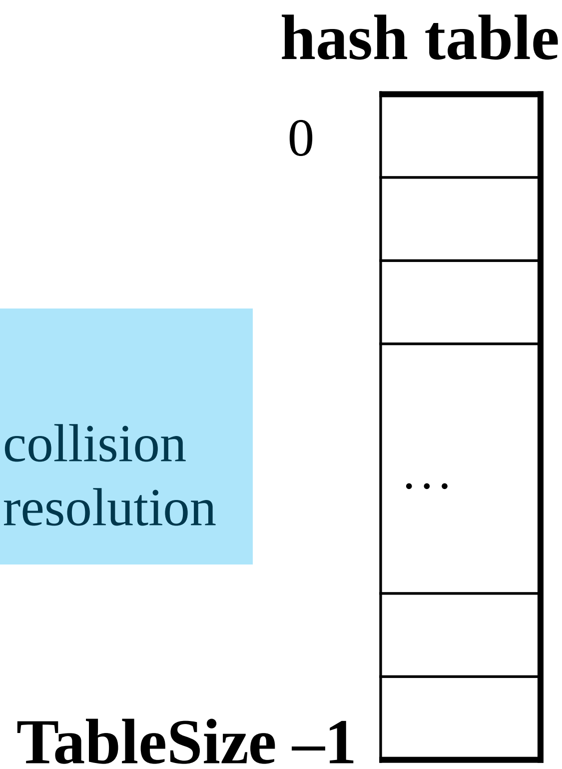
Hash Tables: Review

- Aim for constant-time (i.e., *O*(1)) **`find`**, **`insert`**, and **`delete`**
  - "On average" under some reasonable assumptions
- A hash table is an array of some fixed size
  - But growable as we'll see





Collision resolution

Collision:

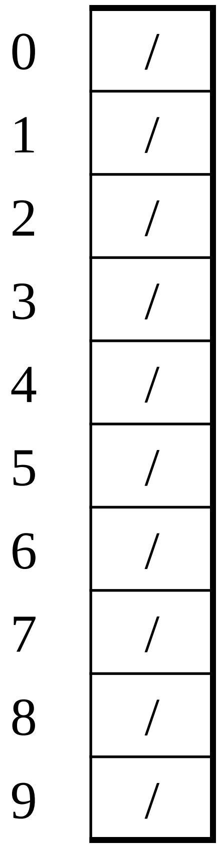
When two keys map to the same location in the hash table

We try to avoid it, but number-of-keys exceeds table size

So hash tables should support collision resolution

– Ideas?

Separate Chaining



Chaining:

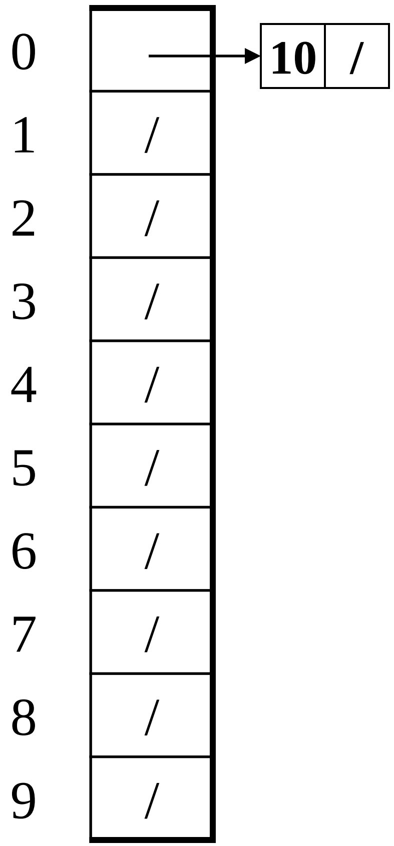
All keys that map to the same table location are kept in a list (a.k.a. a “chain” or “bucket”)

As easy as it sounds

Example:

insert 10, 22, 107, 12, 42 with mod hashing and **`TableSize`** = 10

Separate Chaining



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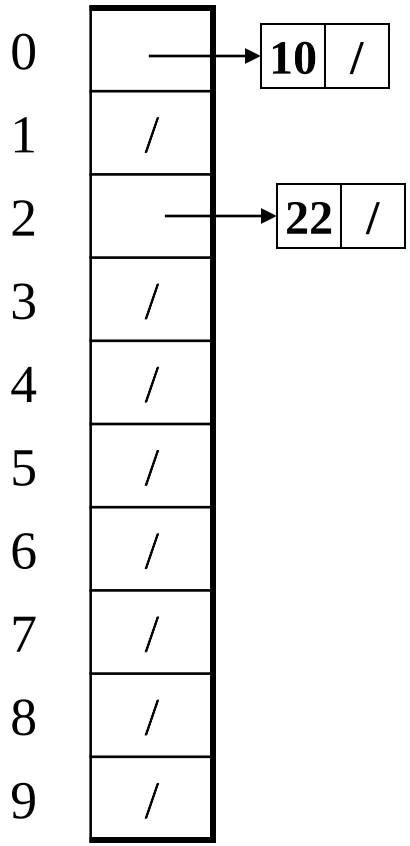

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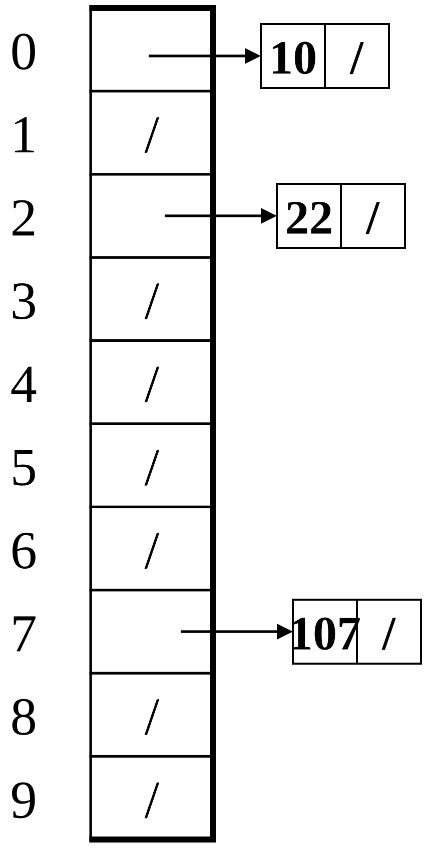

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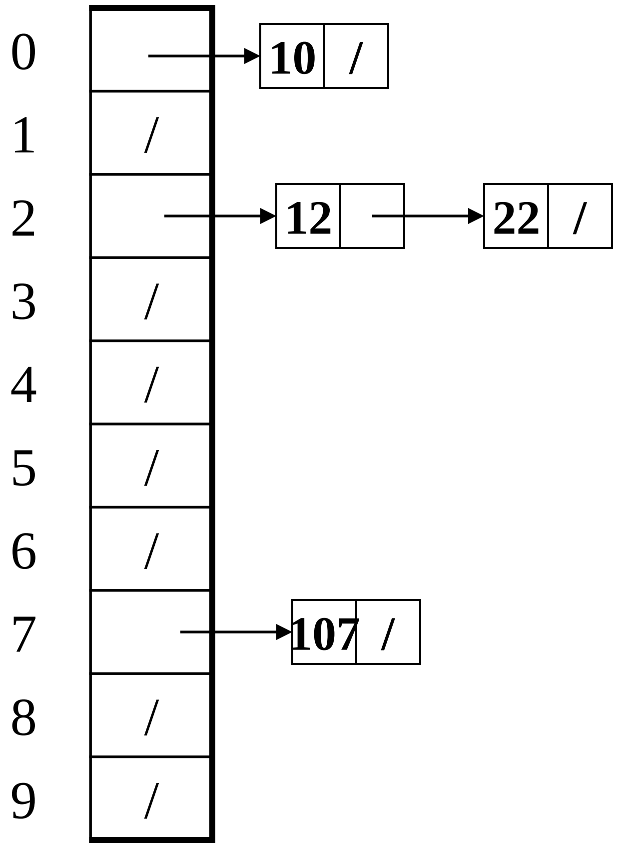

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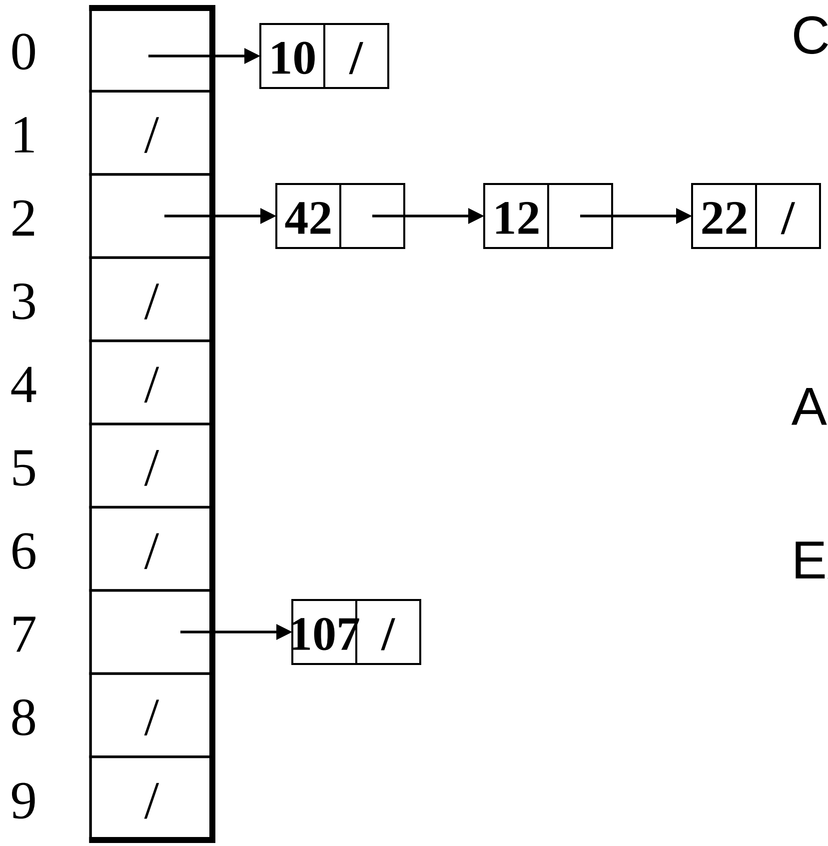

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As easy as it sounds

Example:

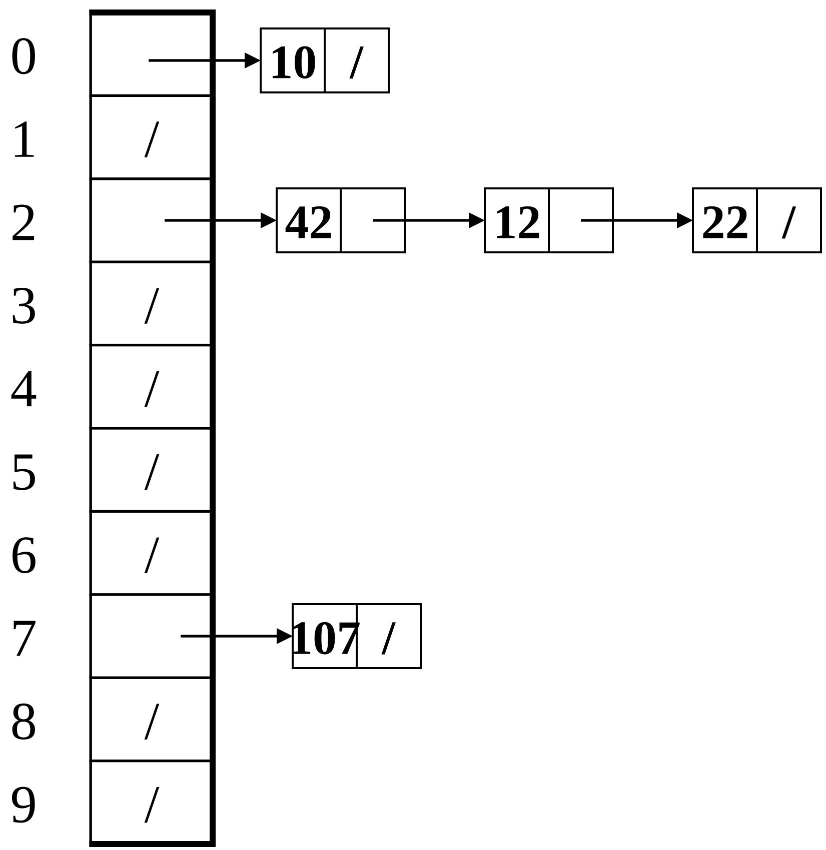
insert 10, 22, 107, 12, 42 with mod hashing and **`TableSize`** = 10

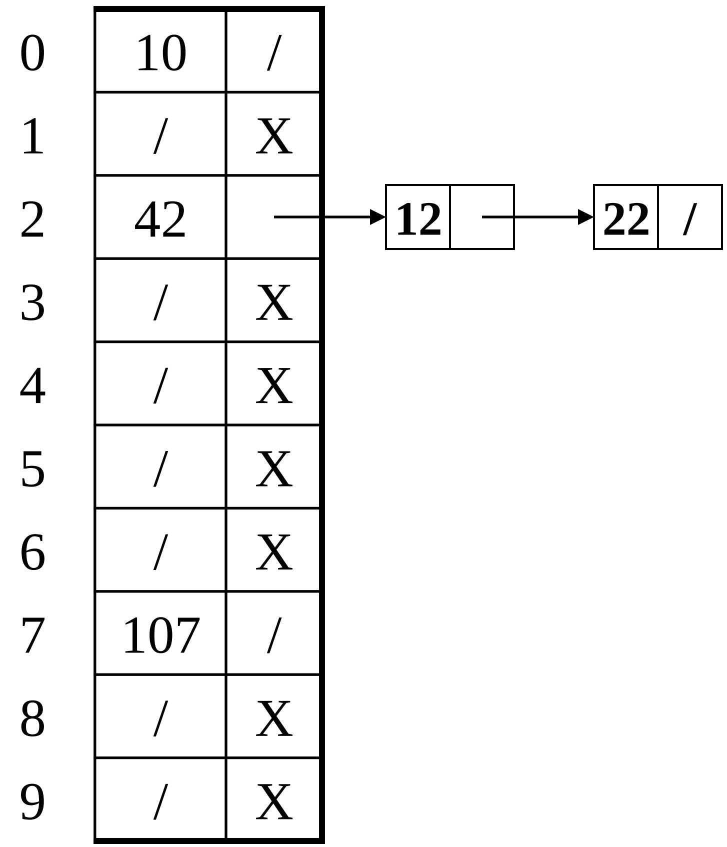
Thoughts on chaining

- Worst-case time for **`find`**?
  - Linear
  - But only with really bad luck or bad hash function
  - So not worth avoiding (e.g., with balanced trees at each bucket)

- Beyond asymptotic complexity, some “data-structure engineering” may be warranted
  - Linked list vs. array vs. chunked list (lists should be short!)
  - Move-to-front (cf. Project 2)
  - Better idea: Leave room for 1 element (or 2?) in the table itself, to optimize constant factors for the common case
    - A time-space trade-off…

Time vs. space (constant factors only here)





More Rigorous Chaining Analysis

Definition: The load factor, λ, of a hash table is

$$\lambda = \frac{\text{N}}{\text{TableSize}} \qquad \leftarrow \textbf{number of elements}$$

Under chaining, the average number of elements per bucket is ____

So if some inserts are followed by *random* finds, then on average:
- Each unsuccessful **`find`** compares against ____ items
- Each successful **`find`** compares against _____ items

More rigorous chaining analysis

Definition: The load factor, λ, of a hash table is

$$\lambda = \frac{\text{N}}{\text{TableSize}} \quad \leftarrow \textbf{number of elements}$$

Under chaining, the average number of elements per bucket is λ

So if some inserts are followed by *random* finds, then on average:

- Each unsuccessful **`find`** compares against λ items
- Each successful **`find`** compares against $\lambda / 2$ items

So we like to keep λ fairly low (e.g., 1 or 1.5 or 2) for chaining

Alternative: Use empty space in the table

- Another simple idea: If **`h(key)`** is already full,
  - try **`(h(key) + 1) % TableSize`**. If full,
  - try **`(h(key) + 2) % TableSize`**. If full,
  - try **`(h(key) + 3) % TableSize`**. If full…

- Example: insert 38, 19, 8, 109, 10

0	/
1	/
2	/
3	/
4	/
5	/
6	/
7	/
8	38
9	/

Alternative: Use empty space in the table

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- Example: insert 38, 19, 8, 109, 10

0	/
1	/
2	/
3	/
4	/
5	/
6	/
7	/
8	38
9	19

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0	8
1	/
2	/
3	/
4	/
5	/
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- Example: insert 38, 19, 8, 109, 10

0	8
1	109
2	10
3	/
4	/
5	/
6	/
7	/
8	38
9	19

Open addressing

This is *one example* of open addressing

In general, open addressing means resolving collisions by trying a sequence of other positions in the table.

Trying the next spot is called probing

- We just did linear probing
  - i^{th} probe was **`(h(key) + i) % TableSize`**
- In general have some probe function **`f`** and use **`h(key) + f(i) % TableSize`**

Open addressing does poorly with high load factor λ

- So want larger tables
- Too many probes means no more $O(1)$

Terminology

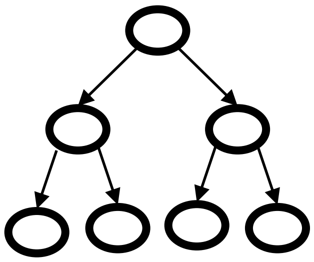
We and the book use the terms

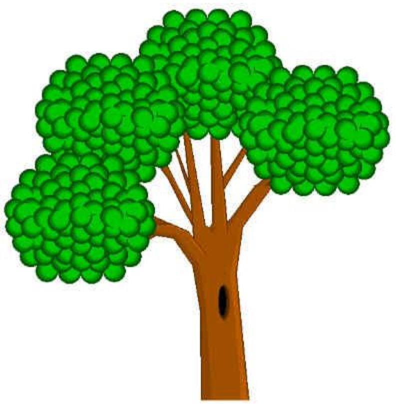
- “chaining” or “separate chaining”
- “open addressing”

Very confusingly,

- “open hashing” is a synonym for “chaining”
- “closed hashing” is a synonym for “open addressing”

(If it makes you feel any better,
most trees in CS grow upside-down ☺)





Other operations

`insert` finds an open table position using a probe function

What about **`find`**?

- Must use same probe function to “retrace the trail” for the data
- Unsuccessful search when reach empty position

What about **`delete`**?

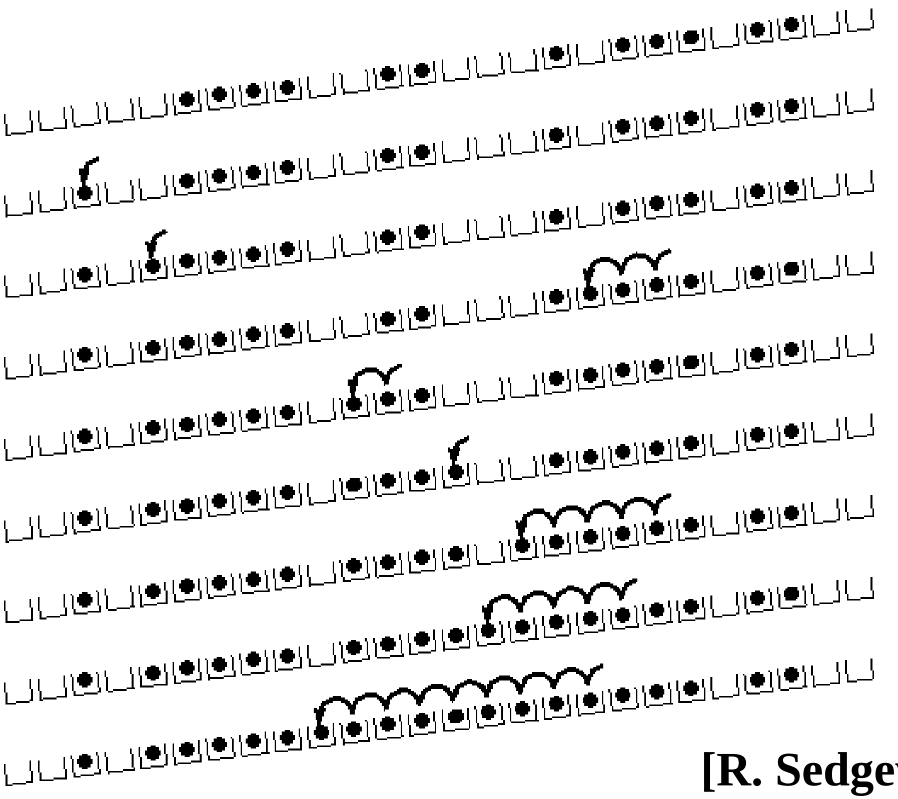
- ***Must*** use “lazy” deletion. Why?
  - Marker indicates “no data here, but don’t stop probing”
- Note: **`delete`** with chaining is plain-old list-remove

(Primary) Clustering

It turns out linear probing is a *bad idea*, even though the probe function is quick to compute (which is a good thing)

Tends to produce *clusters*, which lead to long probing sequences

- Called primary clustering
- Saw this starting in our example



[R. Sedgewick]

Analysis of Linear Probing

- Trivial fact: For any $\lambda < 1$, linear probing will find an empty slot
  - It is "safe" in this sense: no infinite loop unless table is full

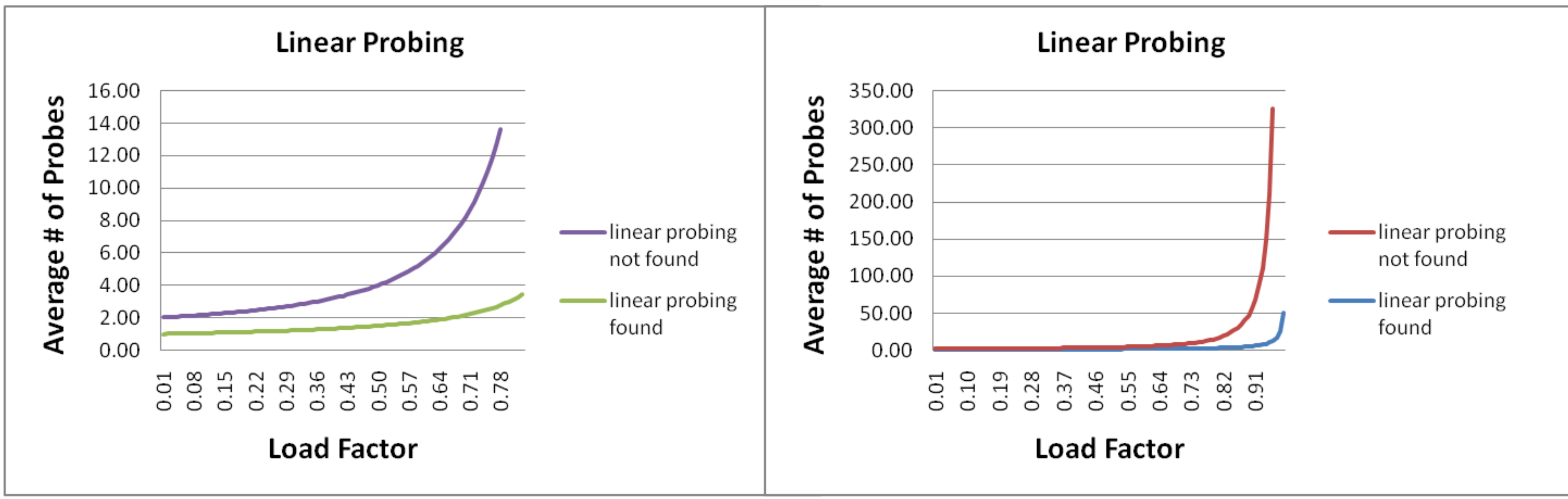
- Non-trivial facts we won't prove:

  Average # of probes given λ (in the limit as **`TableSize`** $\rightarrow \infty$)
  - Unsuccessful search: $$\frac{1}{2}\left(1+\frac{1}{(1-\lambda)^2}\right)$$
  - Successful search: $$\frac{1}{2}\left(1+\frac{1}{(1-\lambda)}\right)$$

- This is pretty bad: need to leave sufficient empty space in the table to get decent performance (see chart)

In a chart

- Linear-probing performance degrades rapidly as table gets full
  - (Formula assumes "large table" but point remains)



- By comparison, chaining performance is linear in λ and has no trouble with $\boldsymbol{\lambda>1}$

Quadratic probing

- We can avoid primary clustering by changing the probe function

  `(h(key) + f(i)) % TableSize`

- A common technique is quadratic probing:

  `f(i) =` i^2

  - So probe sequence is:
    - 0^{th} probe: **`h(key) % TableSize`**
    - 1^{st} probe: **`(h(key) + 1) % TableSize`**
    - 2^{nd} probe: **`(h(key) + 4) % TableSize`**
    - 3^{rd} probe: **`(h(key) + 9) % TableSize`**
    - …
    - i^{th} probe: **`(h(key) + i`**^{2}**`) % TableSize`**

- Intuition: Probes quickly "leave the neighborhood"

Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

TableSize=10
Insert:
89
18
49
58
79

Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	89

TableSize=10
Insert:
89
18
49
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79

Quadratic Probing Example

0	
1	
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3	
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4	
5	
6	
7	
8	18
9	89

TableSize=10
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Quadratic Probing Example

0	49
1	
2	58
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4	
5	
6	
7	
8	18
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Quadratic Probing Example

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3	79
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5	
6	
7	
8	18
9	89

TableSize=10
Insert:
89
18
49
58
79

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	76

TableSize = 7

Insert:

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48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
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Another Quadratic Probing Example

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4	
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TableSize = 7

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Another Quadratic Probing Example

0	48
1	
2	
3	
4	
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6	76

TableSize = 7

Insert:

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48	**(48 % 7 = 6)**
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47	**(47 % 7 = 5)**

Another Quadratic Probing Example

0	48
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2	5
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TableSize = 7

Insert:

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Doh!: For all *n*, `((n*n) +5) % 7 is 0, 2, 5, or 6`

- Excel shows takes “at least” 50 probes and a pattern
- Proof uses induction and $(n^2+5) \% 7 = ((n-7)^2+5) \% 7$
  - In fact, for all *c* and *k*, $(n^2+c) \% k = ((n-k)^2+c) \% k$

From Bad News to Good News

- Bad news:
  - Quadratic probing can cycle through the same full indices, never terminating despite table not being full
- Good news:
  - If **`TableSize`** is *prime* and $\lambda < ½$, then quadratic probing will find an empty slot in at most **`TableSize/2`** probes
  - So: If you keep $\lambda < ½$ and **`TableSize`** is *prime*, no need to detect cycles
  - Proof is posted in **`lecture11.txt`**
    - Also, slightly less detailed proof in textbook
    - Key fact: For prime **`T`** and **`0 < i,j < T/2`** where $i \neq j$, $(k + i^2) \% T \neq (k + j^2) \% T$ (i.e., no index repeat)

Clustering reconsidered

- Quadratic probing does not suffer from primary clustering: no problem with keys initially hashing to the same neighborhood

- But it's no help if keys initially hash to the same index
  - Called secondary clustering

- Can avoid secondary clustering with a probe function that depends on the key: double hashing…

Double hashing

Idea:

- Given two good hash functions *h* and *g*, it is very unlikely that for some *key*, **`h(key) == g(key)`**
- So make the probe function **`f(i) = i*g(key)`**

Probe sequence:

- 0^{th} probe: **`h(key) % TableSize`**
- 1^{st} probe: **`(h(key) + g(key))   % TableSize`**
- 2^{nd} probe: **`(h(key) + 2*g(key)) % TableSize`**
- 3^{rd} probe: **`(h(key) + 3*g(key)) % TableSize`**
- …
- i^{th} probe: **`(h(key) + i*g(key)) % TableSize`**

Detail: Make sure **`g(key)`** cannot be **`0`**

Double-hashing analysis

- Intuition: Because each probe is "jumping" by **`g(key)`** each time, we "leave the neighborhood" *and* "go different places from other initial collisions"

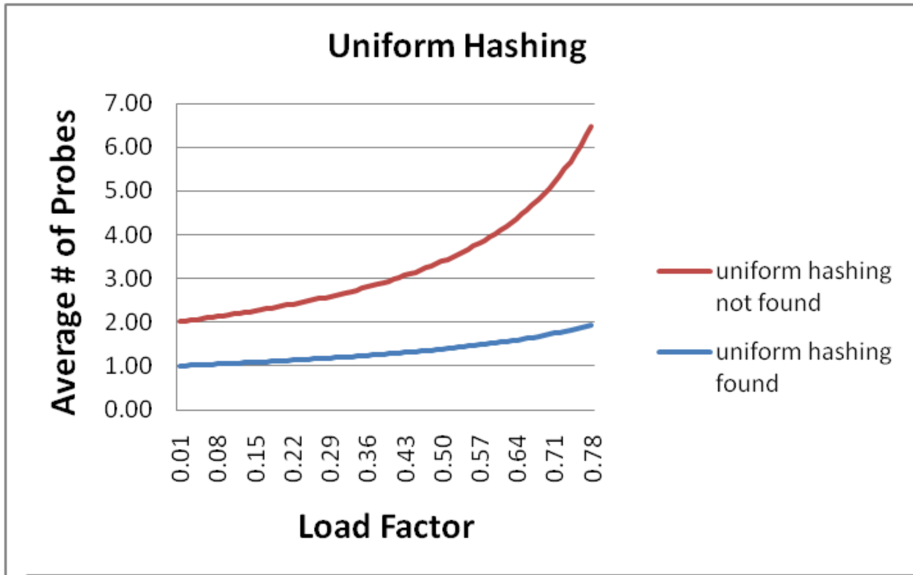
- But we could still have a problem like in quadratic probing where we are not "safe" (infinite loop despite room in table)
  - It is known that this cannot happen in at least one case:
    - **`h(key) = key % p`**
    - **`g(key) = q - (key % q)`**
    - **`2 < q < p`**
    - **`p`** and **`q`** are prime

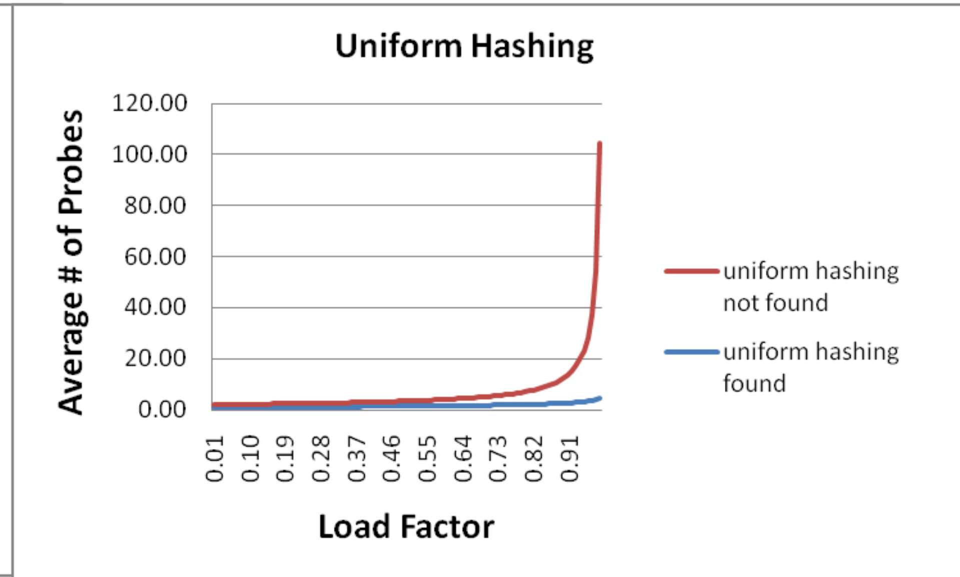
More double-hashing facts

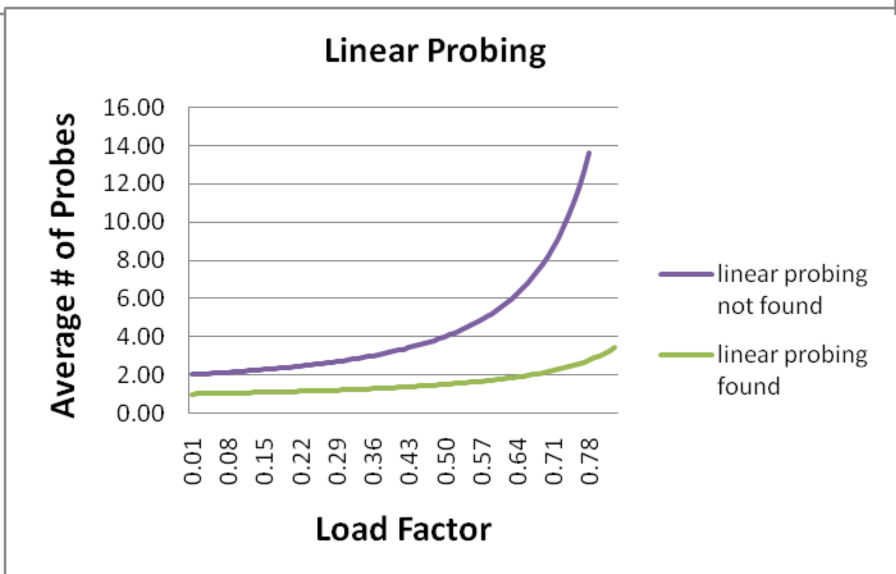
- Assume "uniform hashing"
  - Means probability of **`g(key1) % p == g(key2) % p`** is **`1/p`**
- Non-trivial facts we won't prove:

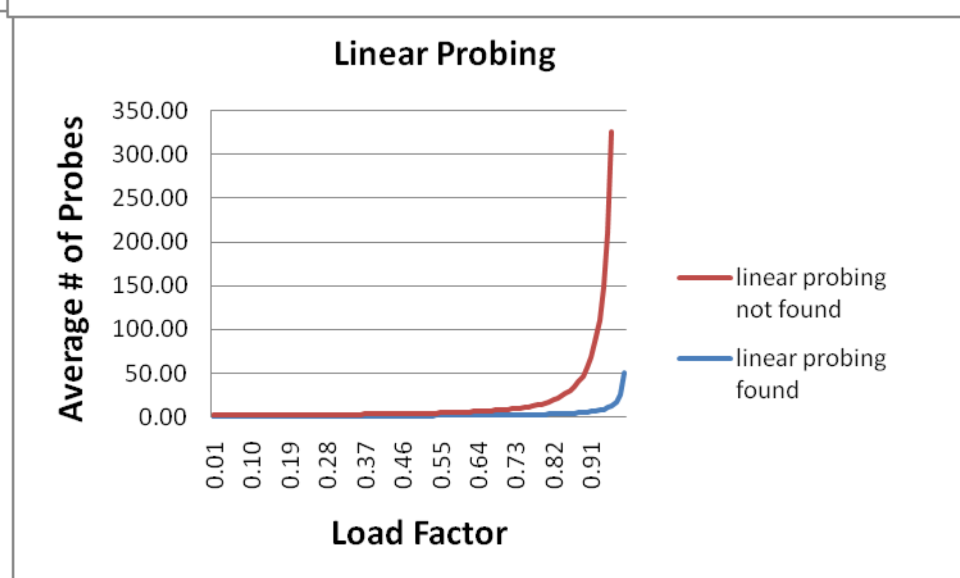
  Average # of probes given λ (in the limit as **`TableSize`** $\rightarrow\infty$)
  - Unsuccessful search (intuitive): $$\frac{1}{1-\lambda}$$
  - Successful search (less intuitive): $$\frac{1}{\lambda}\log_e\left(\frac{1}{1-\lambda}\right)$$
- Bottom line: unsuccessful bad (but not as bad as linear probing), but successful is not nearly as bad

Charts


Uniform Hashing
7.00
6.00
5.00
4.00
3.00
2.00
1.00
0.00
Average # of Probes
0.01 0.08 0.15 0.22 0.29 0.36 0.43 0.50 0.57 0.64 0.71 0.78
Load Factor
uniform hashing not found
uniform hashing found


Uniform Hashing
120.00
100.00
80.00
60.00
40.00
20.00
0.00
Average # of Probes
0.01 0.10 0.19 0.28 0.37 0.46 0.55 0.64 0.73 0.82 0.91
Load Factor
uniform hashing not found
uniform hashing found


Linear Probing
16.00
14.00
12.00
10.00
8.00
6.00
4.00
2.00
0.00
Average # of Probes
0.01 0.08 0.15 0.22 0.29 0.36 0.43 0.50 0.57 0.64 0.71 0.78
Load Factor
linear probing not found
linear probing found


Linear Probing
350.00
300.00
250.00
200.00
150.00
100.00
50.00
0.00
Average # of Probes
0.01 0.10 0.19 0.28 0.37 0.46 0.55 0.64 0.73 0.82 0.91
Load Factor
linear probing not found
linear probing found

Where are we?

- *Chaining* is easy
  - **`find`**, **`delete`** proportional to load factor on average
  - **`insert`** can be constant if just push on front of list

- *Open addressing* uses probing, has clustering issues as table fills
  - Why use it:
    - Less memory allocation?
    - Easier data representation?

- Now:
  - Growing the table when it gets too full (“rehashing”)
  - Relation between hashing/comparing and connection to Java

Rehashing

- As with array-based stacks/queues/lists, if table gets too full, create a bigger table and copy everything

- With chaining, we get to decide what “too full” means
  - Keep load factor reasonable (e.g., < 1)?
  - Consider average or max size of non-empty chains?

- For open addressing, half-full is a good rule of thumb

- New table size
  - Twice-as-big is a good idea, except, uhm, that won’t be prime!
  - So go *about* twice-as-big
  - Can have a list of prime numbers in your code since you won’t grow more than 20-30 times

More on rehashing

- What if we copy all data to the same indices in the new table?
  - Will not work; we calculated the index based on **`TableSize`**

- Go through table, do standard insert for each into new table
  - Run-time?
  - *O*(n): Iterate through old table

- Resize is an *O*(*n*) operation, involving *n* calls to the hash function
  - Is there some way to avoid all those hash function calls?
  - Space/time tradeoff: Could store **`h(key)`** with each data item
  - Growing the table is still *O*(*n*); only helps by a constant factor

Hashing and comparing

- Need to emphasize a critical detail:
  - We initially *hash* **`E`** to get a table index
  - While chaining or probing we *compare* to **`E`**
    - Just need equality testing (i.e., "is it what I want")
- So a hash table needs a hash function and a comparator
  - In Project 2, you will use two function objects
  - The Java library uses a more object-oriented approach: each object has an **`equals`** method and a **`hashCode`** method

```
class Object {
  boolean equals(Object o) {…}
  int hashCode() {…}
  …
}
```

Equal Objects Must Hash the Same

- The Java library (and your project hash table) make a very important assumption that clients must satisfy…
- Object-oriented way of saying it:

  If **`a.equals(b)`**, then we must require
  `a.hashCode()==b.hashCode()`
- Function-object way of saying it:

  If **`c.compare(a,b) == 0`**, then we must require
  `h.hash(a) == h.hash(b)`
- Why is this essential?

Java bottom line

- Lots of Java libraries use hash tables, perhaps without your knowledge

- So: If you ever override **`equals`**, you need to override **`hashCode`** also in a consistent way
  - See CoreJava book, Chapter 5 for other “gotchas” with **`equals`**

Bad Example

- Think about using a hash table holding points

```
class PolarPoint {
  double r = 0.0;
  double theta = 0.0;
  void addToAngle(double theta2) { theta+=theta2; }
  …
  boolean equals(Object otherObject) {
     if(this==otherObject) return true;
     if(otherObject==null) return false;
     if(getClass()!=other.getClass()) return false;
     PolarPoint other = (PolarPoint)otherObject;
     double angleDiff =
         (theta - other.theta) % (2*Math.PI);
     double rDiff = r - other.r;
     return Math.abs(angleDiff) < 0.0001
            && Math.abs(rDiff) < 0.0001;
  }
  // wrong: must override hashCode!
}
```

By the way: comparison has rules too

We have not empahsized important “rules” about comparison for:

- All our dictionaries
- Sorting (next major topic)

Comparison must impose a consistent, total ordering:

For all **`a`**, **`b`**, and **`c`**,

- If **`compare(a,b) < 0`**, then **`compare(b,a) > 0`**
- If **`compare(a,b) == 0`**, then **`compare(b,a) == 0`**
- If **`compare(a,b) < 0`** and **`compare(b,c) < 0`**, then **`compare(a,c) < 0`**

Final word on hashing

- The hash table is one of the most important data structures
  - Supports only **`find`**, **`insert`**, and **`delete`** efficiently

- Important to use a good hash function

- Important to keep hash table at a good size

- What we skipped: Perfect hashing, universal hash functions, hopscotch hashing, cuckoo hashing

- Side-comment: hash functions have uses beyond hash tables
  - Examples: Cryptography, check-sums