CSE332: Data Abstractions

Lecture 11:More Hashing

Ruth Anderson

Winter 2011

Announcements

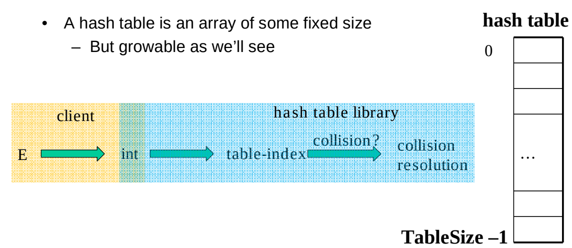
- **Homework 3**– due NOW!
- **Project 2** – Phase A due next Wed Feb 2nd at 11pm
- (No homework due next Friday)
- **Midterm – Monday Feb 7th during lecture**
- **Homework 4** – not due until Friday Feb 11th at the BEGINNING of lecture

Today

- Dictionaries
  - Hashing

Hash Tables: Review

- Aim for constant-time (i.e., *O*(1)) **`find`**, **`insert`**, and **`delete`**
  - "On average" under some reasonable assumptions
- A hash table is an array of some fixed size
  - But growable as we'll see



Hashing Choices

1. Choose a Hash function
2. Choose TableSize
3. Choose a Collision Resolution Strategy from these:
   - Separate Chaining
   - Open Addressing
     - Linear Probing
     - Quadratic Probing
     - Double Hashing

- Other issues to consider:
  - Deletion?
  - What to do when the hash table gets "too full"?

An Alternative to Separate Chaining: Open Addressing

- Why not use up the empty space in the table?
- Store directly in the array cell (no linked list)
- How to deal with collisions?
- If **`h(key)`** is already full,
  - try **`(h(key) + 1) % TableSize`**. If full,
  - try **`(h(key) + 2) % TableSize`**. If full,
  - try **`(h(key) + 3) % TableSize`**. If full…
- Example: insert 38, 19, 8, 109, 10

0	/
1	/
2	/
3	/
4	/
5	/
6	/
7	/
8	38
9	/

An Alternative to Separate Chaining: Open Addressing

- Another simple idea: If **`h(key)`** is already full,
  - try **`(h(key) + 1) % TableSize`**. If full,
  - try **`(h(key) + 2) % TableSize`**. If full,
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1	/
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5	/
6	/
7	/
8	38
9	19

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- Example: insert 38, 19, 8, 109, 10

0	8
1	/
2	/
3	/
4	/
5	/
6	/
7	/
8	38
9	19

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1	109
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An Alternative to Separate Chaining: Open Addressing

- Another simple idea: If **`h(key)`** is already full,
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- Example: insert 38, 19, 8, 109, 10

0	8
1	109
2	10
3	/
4	/
5	/
6	/
7	/
8	38
9	19

Open addressing

This is *one example* of open addressing
- More generally, we just need to describe *where to check next when one attempt fails* (cell already in use)
- Each version of open addressing involves specifying a sequence of indices to try

Trying the next spot is called probing
- Our **`i`**th probe was: **`(h(key) + i) % TableSize`**
  - This is called linear probing
- In general have some probe function **`f`** and use:

  `(h(key) + f(i)) % TableSize`

  `for the i`th **`probe (start at i=0)`**
  - **`For linear probing, f(i)=i`**

Open addressing does poorly with high load factor λ
- So want larger tables
- Too many probes means no more *O*(1)

Terminology

We and the book use the terms
- "chaining" or "separate chaining"
- "open addressing"

Very confusingly,
- "open hashing" is a synonym for "chaining"
- "closed hashing" is a synonym for "open addressing"

Open Addressing: Linear Probing

What about **`find`**? If value is in table? If not there? Worst case?

What about **`delete`**?

How does open addressing with linear probing compare to separate chaining?

Other operations

Okay, so **`insert`** finds an open table position using a probe function

What about **`find`**?

- Must use same probe function to "retrace the trail" and find the data
- Unsuccessful search when reach empty position

What about **`delete`**?

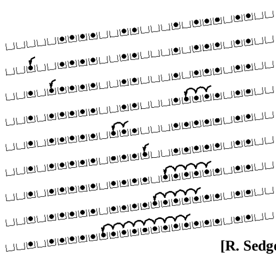
- ***Must*** use "lazy" deletion. Why?
- But here just means "no data here, but don't stop probing"
- Note: **`delete`** with chaining is plain-old list-remove

(Primary) Clustering

It turns out linear probing is a *bad idea*, even though the probe function is quick to compute (a good thing)

Tends to produce *clusters*, which lead to long probing sequences

- Called primary clustering
- Saw this starting in our example



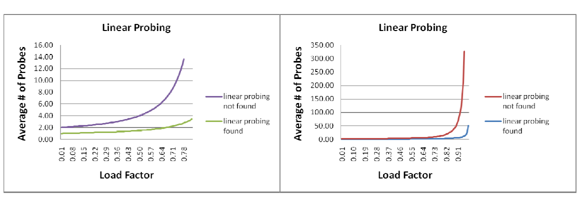
[R. Sedgewick]

Analysis of Linear Probing

- Trivial fact: For any $\lambda < 1$, linear probing will find an empty slot
  - It is "safe" in this sense: no infinite loop unless table is full
- Non-trivial facts we won't prove:
  Average # of probes given λ (in the limit as **`TableSize`** $\rightarrow \infty$)
  - Unsuccessful search: $\frac{1}{2}\left(1+\frac{1}{(1-\lambda)^2}\right)$
  - Successful search: $\frac{1}{2}\left(1+\frac{1}{(1-\lambda)}\right)$
- This is pretty bad: need to leave sufficient empty space in the table to get decent performance (see chart)

In a chart

- Linear-probing performance degrades rapidly as table gets full
  - (Formula assumes "large table" but point remains)
- By comparison, chaining performance is linear in λ and has no trouble with $\lambda > 1$



Open Addressing: linear probing

`(h(key) + f(i)) % TableSize`

- For linear probing:
  `f(i) = i`
- So probe sequence is:
  - 0^{th} probe: **`h(key) % TableSize`**
  - 1^{st} probe: **`(h(key) + 1) % TableSize`**
  - 2^{nd} probe: **`(h(key) + 2) % TableSize`**
  - 3^{rd} probe: **`(h(key) + 3) % TableSize`**
  - …
  - i^{th} probe: **`(h(key) + i) % TableSize`**

Open Addressing: Quadratic probing

- We can avoid primary clustering by changing the probe function…

  `(h(key) + f(i)) % TableSize`

  - For quadratic probing:

    `f(i) = i²`

  - So probe sequence is:
    - 0th probe: `h(key) % TableSize`
    - 1st probe: `(h(key) + 1) % TableSize`
    - 2nd probe: `(h(key) + 4) % TableSize`
    - 3rd probe: `(h(key) + 9) % TableSize`
    - …
    - ith probe: `(h(key) + i²) % TableSize`
- Intuition: Probes quickly "leave the neighborhood"

ith probe: `(h(key) + i²) % TableSize`

Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

TableSize=10
Insert:
89
18
49
58
79

Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	89

TableSize=10
Insert:
89
18
49
58
79

Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	
7	
8	18
9	89

TableSize=10
Insert:
89
18
49
58
79

Quadratic Probing Example

0	49
1	
2	
3	
4	
5	
6	
7	
8	18
9	89

TableSize=10
Insert:
89
18
49
58
79

Quadratic Probing Example

0	49
1	
2	58
3	
4	
5	
6	
7	
8	18
9	89

TableSize=10
Insert:
89
18
49
58
79

Quadratic Probing Example

0	49
1	
2	58
3	79
4	
5	
6	
7	
8	18
9	89

TableSize=10
Insert:
89
18
49
58
79

ith probe: (`h(key) + i²) % TableSize`

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	

TableSize = 7

Insert:
76 (76 % 7 = 6)
40 (40 % 7 = 5)
48 (48 % 7 = 6)
5 (5 % 7 = 5)
55 (55 % 7 = 6)
47 (47 % 7 = 5)

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	76

TableSize = 7

Insert:
76 (76 % 7 = 6)
40 (40 % 7 = 5)
48 (48 % 7 = 6)
5 (5 % 7 = 5)
55 (55 % 7 = 6)
47 (47 % 7 = 5)

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	40
6	76

TableSize = 7

Insert:
76 (76 % 7 = 6)
40 (40 % 7 = 5)
48 (48 % 7 = 6)
5 (5 % 7 = 5)
55 (55 % 7 = 6)
47 (47 % 7 = 5)

Another Quadratic Probing Example

0	48
1	
2	
3	
4	
5	40
6	76

TableSize = 7

Insert:
76 (76 % 7 = 6)
40 (40 % 7 = 5)
48 (48 % 7 = 6)
5 (5 % 7 = 5)
55 (55 % 7 = 6)
47 (47 % 7 = 5)

Another Quadratic Probing Example

0	48
1	
2	5
3	
4	
5	40
6	76

TableSize = 7

Insert:
76 (76 % 7 = 6)
40 (40 % 7 = 5)
48 (48 % 7 = 6)
5 (5 % 7 = 5)
55 (55 % 7 = 6)
47 (47 % 7 = 5)

ith probe: `(h(key) + i²) % TableSize`

Another Quadratic Probing Example

0	48
1	
2	5
3	55
4	
5	40
6	76

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

ith probe: `(h(key) + i²) % TableSize`

Another Quadratic Probing Example

0	48
1	
2	5
3	55
4	
5	40
6	76

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

Uh-oh: For all *n*, **`(5 +(n*n)) % 7 is 0, 2, 5, or 6`**

- Proof uses induction and **`(n²+5) % 7 = ((n-7)²+5) % 7`**
  - In fact, for all *c* and *k*, **`(n²+c) % k = ((n-k)²+c) % k`**

From bad news to good news

- The bad news is: After **`TableSize`** quadratic probes, we will just cycle through the same indices
- The good news:
  - Assertion #1: If **`T = TableSize`** is *prime* and λ < ½, then quadratic probing will find an empty slot in at most **`T/2`** probes
  - Assertion #2: For prime **`T`** and **`0 ≤ i,j ≤ T/2`** where **`i ≠ j`**,
    `(h(key) + i²) % T ≠ (h(key) + j²) % T`

  That is, if T is prime, the first T/2 quadratic probes map to different locations
  - Assertion #3: Assertion #2 is the "key fact" for proving Assertion #1
- So: If you keep λ < ½, no need to detect cycles

Quadratic Probing: Success guarantee for $\lambda < ½$

- If size is prime and λ < ½, then quadratic probing will find an empty slot in size/2 probes or fewer.
  - show for all **`0 ≤ i,j ≤ size/2`** and **`i ≠ j`**
    `(h(x) + i²) mod size ≠ (h(x) + j²) mod size`
  - by contradiction: suppose that for some i ≠ j:
    `(h(x) + i²) mod size = (h(x) + j²) mod size`
    ⇒ **`i² mod size = j² mod size`**
    ⇒ **`(i² - j²) mod size = 0`**
    ⇒ **`[(i + j)(i - j)] mod size = 0`**

BUT size does not divide **`(i-j)`** or **`(i+j)`**

How can **`i+j = 0`** or **`i+j = size`** when:
`i ≠ j` and **`0 ≤ i,j ≤ size/2`**?
Similarly how can **`i-j = 0`** or **`i-j = size ?`**

Clustering reconsidered

- Quadratic probing does not suffer from primary clustering: quadratic nature quickly escapes the neighborhood
- But it's no help if keys *initially hash to the same index*
  - Called secondary clustering
  - Any 2 keys that hash to the same value will have the same series of moves after that
- Can avoid secondary clustering with *a probe function that depends on the key*: double hashing…

Open Addressing: Double hashing

Idea: Given two good hash functions *h* and *g*, it is very unlikely that for some *key*, **`h(key) == g(key)`**

`(h(key) + f(i)) % TableSize`

- For double hashing:
  `f(i) = i*g(key)`
- So probe sequence is:
  - 0th probe: **`h(key) % TableSize`**
  - 1st probe: **`(h(key) + g(key)) % TableSize`**
  - 2nd probe: **`(h(key) + 2*g(key)) % TableSize`**
  - 3rd probe: **`(h(key) + 3*g(key)) % TableSize`**
  - …
  - ith probe: **`(h(key) + i*g(key)) % TableSize`**

- Detail: Make sure **`g(key)`** can't be 0

Resolving Collisions with Double Hashing

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

T = 10 (TableSize)
Hash Functions:
h(key) = key mod T
g(key) = 1 + ((key/T) mod (T-1))

Insert these values into the hash table in this order. Resolve any collisions with double hashing:

13
28
33
147
43

Double-hashing analysis

- Intuition: Since each probe is "jumping" by **`g(key)`** each time, we "leave the neighborhood" *and* "go different places from other initial collisions"
- But we could still have a problem like in quadratic probing where we are not "safe" (infinite loop despite room in table)
  - It is known that this cannot happen in at least one case:
    - **`h(key) = key % p`**
    - **`g(key) = q – (key % q)`**
    - **`2 < q < p`**
    - **`p`** and **`q`** are prime

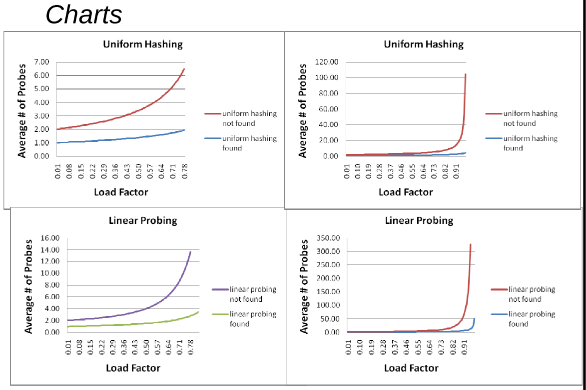
Yet another reason to use a prime TableSize

- So, for double hashing
  i^{th} probe: **`(h(key) + i*g(key))% TableSize`**
- Say g(key) divides Tablesize
  - That is, there is some integer x such that x*g(key)=Tablesize
  - After x probes, we'll be back to trying the same indices as before
- Ex:
  - Tablesize=50
  - g(key)=25
  - Probing sequence:
    - h(key)
    - h(key)+25
    - h(key)+50=h(key)
    - h(key)+75=h(key)+25
- Only 1 & itself divide a prime

More double-hashing facts

- Assume "uniform hashing"
  - Means probability of **`g(key1) % p == g(key2) % p`** is **`1/p`**
- Non-trivial facts we won't prove:
  Average # of probes given λ (in the limit as **`TableSize`** $\rightarrow \infty$)
  - Unsuccessful search (intuitive): $\dfrac{1}{1-\lambda}$
  - Successful search (less intuitive): $\dfrac{1}{\lambda}\log_e\left(\dfrac{1}{1-\lambda}\right)$
- Bottom line: unsuccessful bad (but not as bad as linear probing), but successful is not nearly as bad

Charts



Where are we?

- Separate Chaining is easy
  - **`insert`**, **`find`**, **`delete`** proportion to load factor on average (**`insert`** can be constant if just push on front of list)
- Open addressing uses probe functions, has clustering issues as table gets full
  - Why use it:
    - Less memory allocation?
    - Some run-time overhead for allocating linked list (or whatever) nodes; open addressing could be faster
    - Easier data representation?
- Now:
  - Growing the table when it gets too full (aka "rehashing")
  - Relation between hashing/comparing and connection to Java

Rehashing

- Like with array-based stacks/queues/lists, if table gets too full, create a bigger table and copy everything over
- Especially with chaining, we get to decide what "too full" means
  - Keep load factor reasonable (e.g., < 1)?
  - Consider average or max size of non-empty chains?
- For open addressing, half-full is a good rule of thumb
- New table size
  - Twice-as-big is a good idea, except, uhm, that won't be prime!
  - So go *about* twice-as-big
  - Can have a list of prime numbers in your code since you won't grow more than 20-30 times, and then calculate after that

More on rehashing

- What if we copy all data to the same indices in the new table?
  - Not going to work; calculated index based on TableSize – we may not be able to find it later
- Go through current table, do standard insert for each into new table; run-time?
  - O(n): Iterate through table
- But resize is an $O(n)$ operation, involving n calls to the hash function (1 for each insert in the new table)
  - Is there some way to avoid all those hash function calls again?
  - Space/time tradeoff: Could store **`h(key)`** with each data item, but since rehashing is rare, this is probably a poor use of space
    - And growing the table is still $O(n)$; only helps by a constant factor

Hashing and comparing

- For insert/find, as we go through the chain or keep probing, we have to *compare* each item we see to the key we're looking for
  - We need to have a comparator (or key's type needs to be comparable)
  - Don't actually need < & >; just =
- So a hash table needs a hash function and a comparator
  - In Project 2, you'll use two function objects
  - The Java standard library uses a more OO approach where each object has an **`equals`** method and a **`hashCode`** method:

```
class Object {
  boolean equals(Object o) {…}
  int hashCode() {…}
  …
}
```


Equal objects must hash the same

- The Java library (and your project hash table) make a very important assumption that clients must satisfy…
- OO way of saying it:

  If **`a.equals(b)`**, then we must require
  `a.hashCode()==b.hashCode()`
- Function object way of saying it:

  If **`c.compare(a,b) == 0`**, then we must require
  `h.hash(a) == h.hash(b)`
- Why is this essential?

Java bottom line

- Lots of Java libraries use hash tables, perhaps without your knowledge
- So: If you ever override **`equals`**, you need to override **`hashCode`** also in a consistent way
  - See CoreJava book, Chapter 5 for other "gotchas" with **`equals`**

Bad Example

- Think about using a hash table holding points

```
class PolarPoint {
  double r = 0.0;
  double theta = 0.0;
  void addToAngle(double theta2) { theta+=theta2; }
  …
  boolean equals(Object otherObject) {
     if(this==otherObject) return true;
     if(otherObject==null) return false;
     if(getClass()!=other.getClass()) return false;
     PolarPoint other = (PolarPoint)otherObject;
     double angleDiff =
         (theta – other.theta) % (2*Math.PI);
     double rDiff = r – other.r;
     return Math.abs(angleDiff) < 0.0001
            && Math.abs(rDiff) < 0.0001;
  }
  // wrong: must override hashCode!
}
```

Aside: Comparable/Comparator have rules too

We didn't emphasize some important "rules" about comparison functions for:

- all our dictionaries
- sorting (next major topic)

Comparison must impose a consistent, total ordering:

For all **a**, **b**, and **c**,

- If **`compare(a,b) < 0`**, then **`compare(b,a) > 0`**
- If **`compare(a,b) == 0`**, then **`compare(b,a) == 0`**
- If **`compare(a,b) < 0`** and **`compare(b,c) < 0`**, then **`compare(a,c) < 0`**

Some final arguments for a prime table size

If **`TableSize`** is 60 and…

- Lots of data items are multiples of 5, wasting 80% of table
- Lots of data items are multiples of 10, wasting 90% of table
- Lots of data items are multiples of 2, wasting 50% of table

If **`TableSize`** is 61…

- Collisions can still happen, but 5, 10, 15, 20, … will fill table
- Collisions can still happen but 10, 20, 30, 40, … will fill table
- Collisions can still happen but 2, 4, 6, 8, … will fill table

In general, if **`x`** and **`y`** are "co-prime" (means **`gcd(x,y)==1`**), then **`(a * x) % y == (b * x) % y`** if and only if **`a % y == b % y`**

- So, given table size y and keys as multiples of x, we'll get a decent distribution if x & y are co-prime
- Good to have a **`TableSize`** that has no common factors with any "likely pattern" **`x`**

Final word on hashing

- The hash table is one of the most important data structures
  - Supports only **`find, insert, and delete`** efficiently
  - **`FindMin, FindMax, predecessor`**, etc.: not so efficiently
  - Most likely data-structure to be asked about in interviews; many real-world applications
- Important to use a good hash function
  - Good distribution
  - Uses enough of key's values
- Important to keep hash table at a good size
  - Prime #
  - Preferable λ depends on type of table
- Side-comment: hash functions have uses beyond hash tables
  - Examples: Cryptography, check-sums