CSE332: Data Abstractions

Lecture 3: Asymptotic Analysis

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Announcements

- **Room changes** – section and lecture, (see course web page) maybe still one change for section
- **Project 1** – phase A due next Wed Jan 12th
- **Homework 1** – due Friday Jan 14th at ***beginning*** of class

- Info sheets?
- Catalyst Survey

Today

- How to compare two algorithms?
- Analyzing code
- Big-Oh

Comparing Two Algorithms…

Gauging performance

- Uh, why not just run the program and time it?
  - Too much variability; not reliable:
    - Hardware: processor(s), memory, etc.
    - OS, version of Java, libraries, drivers
    - Programs running in the background
    - Implementation dependent
    - Choice of input
  - Timing doesn't really evaluate the *algorithm*; it evaluates an *implementation* in one very specific scenario

Comparing algorithms

When is one *algorithm* (not *implementation*) better than another?
- Various possible answers (clarity, security, …)
- But a big one is *performance*: for sufficiently large inputs, runs in less time (our focus) or less space

Large inputs (n) because probably any algorithm is "plenty good" for small inputs (if *n* is 10, probably anything is fast enough)

Answer will be *independent* of CPU speed, programming language, coding tricks, etc.

Answer is general and rigorous, complementary to "coding it up and timing it on some test cases"
- Can do analysis before coding!

Analyzing code ("worst case")

Basic operations take "some amount of" constant time

- Arithmetic (fixed-width)
- Assignment
- Access one Java field **or array index**
- Etc.

(This is an *approximation of reality*: a very useful "lie".)

Consecutive statements	Sum of times
Conditionals	Time of test plus slower branch
Loops	Sum of iterations
Calls	Time of call's body
Recursion	Solve *recurrence equation*

Example

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   ???
}
```


Linear search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case:

Worst case:

Linear search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case: 6ish steps = *O*(1)
Worst case: 6ish*(arr.length)
= *O*(arr.length)

Binary search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

- Can also be done non-recursively but "doesn't matter" here

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2; //i.e., lo+(hi-lo)/2
   if(lo==hi)        return false;
   if(arr[mid]==k)   return true;
   if(arr[mid]< k)   return help(arr,k,mid+1,hi);
   else              return help(arr,k,lo,mid);
}
```


Binary search

Best case: 8ish steps = *O*(1)
Worst case: *T*(*n*) = 10ish + *T*(*n*/2) where *n* is **`hi-lo`**

- *O*(**`log`** *n*) where *n* is **`array.length`**
- Solve *recurrence equation* to know that…

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2;
   if(lo==hi)        return false;
   if(arr[mid]==k)   return true;
   if(arr[mid]< k)   return help(arr,k,mid+1,hi);
   else              return help(arr,k,lo,mid);
}
```

Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
   – $T(n) = 10 + T(n/2)$ $\quad T(1) = 8$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.
3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case

Solving Recurrence Relations

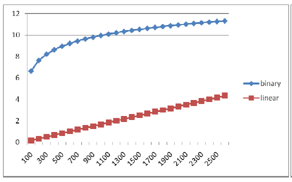
1. Determine the recurrence relation. What is the base case?
   – $T(n) = 10 + T(n/2)$ $\quad T(1) = 8$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.
   – $T(n) = 10 + 10 + T(n/4)$
     $= 10 + 10 + 10 + T(n/8)$
     $= \ldots$
     $= 10k + T(n/(2^k))$
3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case
   – $n/(2^k) = 1$ means $n = 2^k$ means $k = \log_2 n$
   – So $T(n) = 10 \log_2 n + 8$ (get to base case and do it)
   – So $T(n)$ is $O(\log n)$

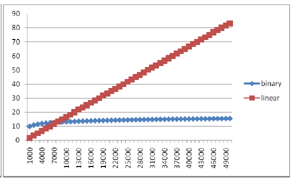
Ignoring constant factors

- So binary search is $O(\log n)$ and linear is $O(n)$
  – But which is faster
- Could depend on constant factors
  – How *many* assignments, additions, etc. for each *n*
  – And could depend on size of *n*
- But there exists some n_0 such that for all $n > n_0$ binary search wins
- Let's play with a couple plots to get some intuition…

Example

- Let's try to "help" linear search
  – Run it on a computer 100x as fast (say 2010 model vs. 1990)
  – Use a new compiler/language that is 3x as fast
  – Be a clever programmer to eliminate half the work
  – So doing each iteration is 600x as fast as in binary search
- Note: 600x still helpful for problems without logarithmic algorithms!





Another example: sum array

Two "obviously" linear algorithms: $T(n) = O(1) + T(n\text{-}1)$

Iterative:

```
int sum(int[] arr){
  int ans = 0;
  for(int i=0; i<arr.length; ++i)
     ans += arr[i];
  return ans;
}
```

Recursive:
– Recurrence is $k + k + \ldots + k$ for *n* times

```
int sum(int[] arr){
   return help(arr,0);
}
int help(int[]arr,int i) {
  if(i==arr.length)
     return 0;
  return arr[i] + help(arr,i+1);
}
```


What about a binary version?

```
int sum(int[] arr){
   return help(arr,0,arr.length);
}
int help(int[] arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

Recurrence is $T(n) = O(1) + 2T(n/2)$
– $1 + 2 + 4 + 8 + \ldots$ for $\log n$ times
– $2^{(\log n)} - 1$ which is proportional to *n* (definition of logarithm)

Easier explanation: it adds each number once while doing little else

"Obvious": You can't do better than *O(n)* – have to read whole array

Parallelism teaser

- But suppose we could do two recursive calls *at the same time*
  - *Like having a friend do half the work for you!*

```
int sum(int[]arr){
   return help(arr,0,arr.length);
}
int help(int[]arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

- If you have as many "friends of friends" as needed the recurrence is now $T(n) = O(1) + 1T(n/2)$
  - $O(\log n)$: same recurrence as for **find**

Really common recurrences

Should know how to solve recurrences but also recognize some really common ones:

$T(n) = O(1) + T(n\text{-}1)$	linear
$T(n) = O(1) + 2T(n/2)$	linear
$T(n) = O(1) + T(n/2)$	logarithmic
$T(n) = O(1) + 2T(n\text{-}1)$	exponential
$T(n) = O(n) + T(n\text{-}1)$	quadratic
$T(n) = O(n) + T(n/2)$	linear
$T(n) = O(n) + 2T(n/2)$	$O(n \log n)$

Note big-Oh can also use more than one variable

- Example: can sum all elements of an n-by-m matrix in $O(nm)$

Asymptotic notation

About to show formal definition, which amounts to saying:

1. Eliminate low-order terms
2. Eliminate coefficients

Examples:

- $4n + 5$
- $0.5n \log n + 2n + 7$
- $n^3 + 2^n + 3n$
- $n \log (10n^2)$

Examples

True or false?

1. 4+3n is O(n)	True
2. n+2logn is O(logn)	False
3. logn+2 is O(1)	False
4. n^{50} is $O(1.1^n)$	True

Notes:

- Do NOT ignore constants that are not multipliers:
  - n^3 is $O(n^2)$: FALSE
  - 3^n is $O(2^n)$: FALSE
- When in doubt, refer to the definition)

Big-Oh relates functions

We use O on a function f(n) (for example n^2) to mean *the set of functions with asymptotic behavior* ***less than or equal to*** f(n)

So $(3n^2+17)$ **is in** $O(n^2)$

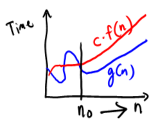
- $3n^2+17$ and n^2 have the same asymptotic behavior

Confusingly, we also say/write:

- $(3n^2+17)$ **is** $O(n^2)$
- $(3n^2+17)$ **=** $O(n^2)$

But we would never say $O(n^2) = (3n^2+17)$

Formally Big-Oh



> Definition: **g(n)** is in O(**f(n)**) iff there exist positive constants ***c*** and ***n_0*** such that
>
> **g(n)** ≤ ***c*** **f(n)** for all ***n*** ≥ ***n_0***

To show **g(n)** is in O(**f(n)**), pick a ***c*** large enough to "cover the constant factors" and ***n_0*** large enough to "cover the lower-order terms"

- Example: Let **g(n)** = $3n^2+17$ and **f(n)** = n^2
  - ***c*** = 5 and ***n_0*** = 10 is more than good enough

This is "less than or equal to"

- So $3n^2+17$ is also $O(n^5)$ and $O(2^n)$ etc.

Using the definition of Big-Oh (Example 1)

For g(n) = 4n & f(n) = n^2, prove g(n) is in O(f(n))
- A valid proof is to find valid c & n_0
- When n=4, g(n) =16 & f(n) =16; this is the crossing over point
- So we can choose n_0 = 4, and c = 1
- Note: There are many possible choices:
  ex: n_0 = 78, and c = 42 works fine

The Definition: g(*n*) is in O(f(*n*)) iff there exist *positive* constants *c* and n_0 such that
g(*n*) ≤ *c* f(*n*) for all $n \geq n_0$.

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Using the definition of Big-Oh (Example 2)

For g(n) = n^4 & f(n) = 2^n, prove g(n) is in O(f(n))
- A valid proof is to find valid c & n_0
- One possible answer: n_0 = 20, and c = 1

The Definition: g(*n*) is in O(f(*n*)) iff there exist *positive* constants *c* and n_0 such that
g(*n*) ≤ *c* f(*n*) for all $n \geq n_0$.

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What's with the c?

- To capture this notion of similar asymptotic behavior, we allow a constant multiplier (called c)
- Consider:
  - g(n) = 7n+5
  - f(n) = n
- These have the same asymptotic behavior (linear), so g(n) is in O(f(n)) even though g(n) is always larger
- There is no positive n_0 such that g(n) ≤ f(n) for all n ≥ n_0
- The 'c' in the definition allows for that:
  g(*n*) ≤ *c* f(*n*) for all $n \geq n_0$
- To prove g(n) is in O(f(n)), have c = 12, n_0 = 1

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Big Oh: Common Categories

From fastest to slowest

O(1)	constant (same as *O*(*k*) for constant *k*)
O(`log` *n*)	logarithmic
O(*n*)	linear
O(n `log` *n*)	"n `log` *n*"
O(n^2)	quadratic
O(n^3)	cubic
O(n^k)	polynomial (where is *k* is an constant)
O(k^n)	exponential (where *k* is any constant > 1)

Usage note: "exponential" does not mean "grows really fast", it means "grows at rate proportional to k^n for some *k*>1"
- A savings account accrues interest exponentially (*k*=1.01?)

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More Asymptotic Notation

- **Upper bound**: *O*(**f(n)**) is the set of all functions asymptotically less than or equal to **f(n)**
  - g(*n*) is in *O*(**f(n)**) if there exist constants *c* and n_0 such that
    g(*n*) ≤ *c* **f(n)** for all $n \geq n_0$
- **Lower bound**: Ω(**f(n)**) is the set of all functions asymptotically greater than or equal to **f(n)**
  - g(*n*) is in Ω(**f(n)**) if there exist constants *c* and n_0 such that
    g(*n*) ≥ *c* **f(n)** for all $n \geq n_0$
- **Tight bound**: θ(**f(n)**) is the set of all functions asymptotically equal to **f(n)**
  - Intersection of *O*(**f(n)**) and Ω(**f(n)**) (use *different c* values)

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Regarding use of terms

A common error is to say *O*(f(*n*)) when you mean θ(f(*n*))
- People often say O() to mean a tight bound
- Say we have f(n)=n; we could say f(n) is in O(n), which is true, but only conveys the upper-bound
- Since f(n)=n is *also* *O*(n^5), it's tempting to say "this algorithm is *exactly* *O*(*n*)"
- Somewhat incomplete; instead say it is θ(*n*)
- That means that it is not, for example *O*(`log` *n*)

Less common notation:
- "little-oh": like "big-Oh" but strictly less than
  - Example: sum is *o*(n^2) but not *o*(*n*)
- "little-omega": like "big-Omega" but strictly greater than
  - Example: sum is ω(`log` *n*) but not ω(*n*)

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What we are analyzing

- The most common thing to do is give an O or θ bound to the worst-case running time of an algorithm
- Example: True statements about binary-search algorithm
  - Common: θ(`log` *n*) running-time in the worst-case
  - Less common: θ(1) in the best-case (item is in the middle)
  - Less common: Algorithm is Ω(`log log` *n*) in the worst-case (it is not really, really, really fast asymptotically)
  - Less common (but very good to know): the find-in-sorted-array ***problem*** is Ω(`log` *n*) in the worst-case
    - *No* algorithm can do better (without parallelism)
    - A ***problem*** cannot be $O(f(n))$ since you can always find a slower algorithm, but can mean ***there exists*** an algorithm

Other things to analyze

- Space instead of time
  - Remember we can often use space to gain time
- Average case
  - Sometimes only if you assume something about the distribution of inputs
    - See CSE312 and STAT391
  - Sometimes uses randomization in the algorithm
    - Will see an example with sorting; also see CSE312
  - Sometimes an *amortized guarantee*
    - Will discuss in a later lecture

Summary

Analysis can be about:

- The problem or the algorithm (usually algorithm)
- Time or space (usually time)
  - Or power or dollars or …
- Best-, worst-, or average-case (usually worst)
- Upper-, lower-, or tight-bound (usually upper or tight)

Big-Oh Caveats

- Asymptotic complexity (Big-Oh) focuses on behavior for **large *n*** and is independent of any computer / coding trick
  - But you can "abuse" it to be misled about trade-offs
  - Example: $n^{1/10}$ vs. `log` *n*
    - Asymptotically $n^{1/10}$ grows more quickly
    - But the "cross-over" point is around $5 * 10^{17}$
    - So if you have input size less than 2^{58}, prefer $n^{1/10}$
- Comparing O() for ***small n*** values can be misleading
  - Quicksort: O(nlogn) (expected)
  - Insertion Sort: $O(n^2)$ (expected)
  - Yet in reality Insertion Sort is faster for small n's
  - We'll learn about these sorts later

Addendum: Timing vs. Big-Oh?

- At the core of CS is a backbone of theory & mathematics
  - Examine the algorithm itself, mathematically, not the implementation
  - Reason about performance as a function of n
  - Be able to mathematically prove things about performance
- Yet, timing has its place
  - In the real world, we do want to know whether implementation A runs faster than implementation B on data set C
  - Ex: Benchmarking graphics cards
  - We will do some timing in project 3 (and in 2, a bit)
- Evaluating an algorithm? Use asymptotic analysis
- Evaluating an implementation of hardware/software? Timing can be useful