CSE332: Data Abstractions

Lecture 3: Asymptotic Analysis

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Announcements

- **Room changes** – section and lecture, (see course web page) maybe still one change for section
- **Project 1** – phase A due next Wed Jan 12th
- **Homework 1** – due Friday Jan 14th at ***beginning*** of class

- Info sheets?
- Catalyst Survey

Today

- How to compare two algorithms?
- Analyzing code
- Big-Oh

Comparing Two Algorithms…

Jessica

Jenny
~~4 mins~~
~~Lmins~~
2 ses

Mark
~~2mins~~
30 secs
0 sec

Gauging performance

- Uh, why not just run the program and time it?
  - Too much variability; not reliable:
    - Hardware: processor(s), memory, etc.
    - OS, version of Java, libraries, drivers
    - Programs running in the background
    - Implementation dependent
    - Choice of input
  - Timing doesn't really evaluate the *algorithm*; it evaluates an *implementation* in one very specific scenario

Comparing algorithms

When is one *algorithm* (not *implementation*) better than another?
- Various possible answers (clarity, security, …)
- But a big one is *performance*: for sufficiently large inputs, runs in less time (our focus) or less space

Large inputs (n) because probably any algorithm is "plenty good" for small inputs (if *n* is 10, probably anything is fast enough)

Answer will be *independent* of CPU speed, programming language, coding tricks, etc.

Answer is general and rigorous, complementary to "coding it up and timing it on some test cases"
- Can do analysis before coding!

Analyzing code ("worst case")

if (cond)
A
else
B

Basic operations take "some amount of" constant time
- Arithmetic (fixed-width)
- Assignment
- Access one Java field **or array index**
- Etc.

(This is an *approximation of reality*: a very useful "lie".)

Consecutive statements	Sum of times
Conditionals	Time of test plus slower branch
Loops	Sum of iterations
Calls	Time of call's body
Recursion	Solve *recurrence equation*

Example

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   ???
}
```


Linear search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
         return true;
   return false;
}
```

Best case: 6ish (k = first elem in array)
Worst case: 6ish * size array (k is not in the array)

O(1) O(n)

Linear search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
         return true;
   return false;
}
```

Best case: 6ish steps = $O(1)$
Worst case: 6ish*(arr.length)
= O(arr.length)

Binary search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array
- Can also be done non-recursively but "doesn't matter" here

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2; //i.e., lo+(hi-lo)/2
   if(lo==hi)        return false;
   if(arr[mid]==k)   return true;
   if(arr[mid]< k)   return help(arr,k,mid+1,hi);
   else              return help(arr,k,lo,mid);
}
```


Binary search

Best case: 8ish steps = $O(1)$
Worst case: $T(n)$ = 10ish + $T(n/2)$ where n is `hi-lo`
- $O(\log n)$ where n is `array.length`
- Solve *recurrence equation* to know that…

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2;
   if(lo==hi)        return false;
   if(arr[mid]==k)   return true;
   if(arr[mid]< k)   return help(arr,k,mid+1,hi);
   else              return help(arr,k,lo,mid);
}
```

Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
   - $\boxed{T(n) = 10 + T(n/2)}$ $T(1) = 8$ ish
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.

$$T(n) = 10 + (10 + T(\tfrac{n}{4}))$$
$$= 10 + (10 + (10 + T(\tfrac{n}{8})))$$
$$= 10k + T(\tfrac{n}{2^k})$$

3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case

$$\frac{n}{2^k} = 1 \qquad n = 2^k \qquad K = \log_2 n$$

$$T(n) = 10k + T(\tfrac{n}{2^k})$$
$$= 10 \log_2 n + T(1)$$
$$O(\log n) \leftarrow T(n) = 10 \log_2 n + 8$$

Solving Recurrence Relations

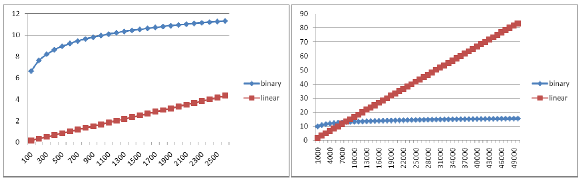
1. Determine the recurrence relation. What is the base case?
   - $T(n) = 10 + T(n/2)$ $T(1) = 8$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.
   - $T(n) = 10 + 10 + T(n/4)$
     $= 10 + 10 + 10 + T(n/8)$
     $= \ldots$
     $= 10k + T(n/(2^k))$
3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case
   - $n/(2^k) = 1$ means $n = 2^k$ means $k = \log_2 n$
   - So $T(n) = 10 \log_2 n + 8$ (get to base case and do it)
   - So $T(n)$ is $O(\log n)$

Ignoring constant factors

- So binary search is $O(\log n)$ and linear is $O(n)$
  - But which is faster
- Could depend on constant factors
  - How *many* assignments, additions, etc. for each n
  - And could depend on size of n
- But there exists some n_0 such that for all $n > n_0$ binary search wins
- Let's play with a couple plots to get some intuition…

Example

- Let's try to "help" linear search
  - Run it on a computer 100x as fast (say 2010 model vs. 1990)
  - Use a new compiler/language that is 3x as fast
  - Be a clever programmer to eliminate half the work
  - So doing each iteration is 600x as fast as in binary search
- Note: 600x still helpful for problems without logarithmic algorithms!



Another example: sum array

Two "obviously" linear algorithms: $T(n) = O(1) + T(n\text{-}1)$

Iterative:

```
int sum(int[] arr){
  int ans = 0;
  for(int i=0; i<arr.length; ++i)
     ans += arr[i];
  return ans;
}
```

Recursive:
- Recurrence is $k + k + \ldots + k$ for n times

```
int sum(int[] arr){
  return help(arr,0);
}
int help(int[]arr,int i) {
  if(i==arr.length)
     return 0;
  return arr[i] + help(arr,i+1);
}
```


What about a binary version?

```
int sum(int[] arr){
   return help(arr,0,arr.length);
}
int help(int[] arr, int lo, int hi) {
   if(lo==hi)    return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

Recurrence is $T(n) = O(1) + 2T(n/2)$
- $1 + 2 + 4 + 8 + \ldots$ for $\log n$ times
- $2^{(\log n)} - 1$ which is proportional to n (definition of logarithm)

Easier explanation: it adds each number once while doing little else

"Obvious": You can't do better than $O(n)$ – have to read whole array

Parallelism teaser

- But suppose we could do two recursive calls *at the same time*
  - *Like having a friend do half the work for you!*

```
int sum(int[]arr){
   return help(arr,0,arr.length);
}
int help(int[]arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

- If you have as many "friends of friends" as needed the recurrence is now $T(n) = O(1) + 1T(n/2)$
  - $O(\log n)$: same recurrence as for **find**

Really common recurrences

Should know how to solve recurrences but also recognize some really common ones:

$T(n) = O(1) + T(n\text{-}1)$	linear
$T(n) = O(1) + 2T(n/2)$	linear
$T(n) = O(1) + T(n/2)$	logarithmic
$T(n) = O(1) + 2T(n\text{-}1)$	exponential
$T(n) = O(n) + T(n\text{-}1)$	quadratic
$T(n) = O(n) + T(n/2)$	linear
$T(n) = O(n) + 2T(n/2)$	$O(n \log n)$

Note big-Oh can also use more than one variable

- Example: can sum all elements of an n-by-m matrix in $O(nm)$

Asymptotic notation

About to show formal definition, which amounts to saying:

1. Eliminate low-order terms
2. Eliminate coefficients

Examples:

- $4n + 5$ → $O(n)$
- $0.5n \log n + 2n + \cancel{7}$ → $O(n \log n)$
- $n^3 + 2^n + 3n$ → $O(2^n)$
- $n \log (10n^2)$ → $O(n \log n)$

Examples

True or false?

1. 4+3n is O(n) — True
2. n+2logn is O(logn) — False
3. logn+2 is O(1) — False
4. n^{50} is $O(1.1^n)$ — True

Notes:

- Do NOT ignore constants that are not multipliers:
  - n^3 is $O(n^2)$: FALSE
  - 3^n is $O(2^n)$: FALSE
- When in doubt, refer to the definition)

Big-Oh relates functions

We use O on a function $f(n)$ (for example n^2) to mean *the set of functions with asymptotic behavior* ***less than or equal to*** $f(n)$

So $(3n^2+17)$ **is in** $O(n^2)$

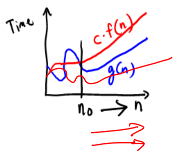
- $3n^2+17$ and n^2 have the same asymptotic behavior

Confusingly, we also say/write:

- $(3n^2+17)$ **is** $O(n^2)$
- $(3n^2+17)$ **=** $O(n^2)$

But we would never say $O(n^2) = (3n^2+17)$

Formally Big-Oh



> Definition: **g(n)** is in O(**f(n)**) iff there exist positive constants ***c*** and n_0 such that
>
> **g(n) ≤ c·f(n)** for all $n \geq n_0$

To show **g(n)** is in O(**f(n)**), pick a ***c*** large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms" → $O(n^2)$

- Example: Let $g(n) = 3n^2+17$ and $f(n) = n^2$
  - $c = 5$ and $n_0 = 10$ is more than good enough

This is "less than or equal to"

- So $3n^2+17$ is also $O(n^5)$ and $O(2^n)$ etc.

$3n^2 + 17 \leq c \cdot n^2$
$3n^2 + 17 \leq 5 \cdot n^2$
$3 \cdot 10^2 + 17 \leq 5 \cdot 10^2$
$3 \cdot 11^2 + 17 \leq 5 \cdot 11^2$
$3 \cdot 12^2 + 17 \leq 5 \cdot 12^2$

Using the definition of Big-Oh (Example 1)

For **g(n)** = 4n & **f(n)** = n^2, prove **g(n)** is in O(**f(n)**)

- A valid proof is to find valid **c** & $\mathbf{n_0}$
- When n=4, **g(n)** =16 & **f(n)** =16; this is the crossing over point
- So we can choose $\mathbf{n_0}$ = 4, and **c** = 1
- Note: There are many possible choices: ex: $\mathbf{n_0}$ = 78, and **c** = 42 works fine

The Definition: g(n) is in O(f(n)) iff there exist *positive* constants c and n_0 such that
g(n) $\leq$ c f(n) for all $n \geq n_0$.

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Using the definition of Big-Oh (Example 2)

For **g(n)** = n^4 & **f(n)** = 2^n, prove **g(n)** is in O(**f(n)**)

- A valid proof is to find valid **c** & $\mathbf{n_0}$
- One possible answer: $\mathbf{n_0}$ = 20, and **c** = 1

The Definition: g(n) is in O(f(n)) iff there exist *positive* constants c and n_0 such that
g(n) $\leq$ c f(n) for all $n \geq n_0$.

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What's with the c?

- To capture this notion of similar asymptotic behavior, we allow a constant multiplier (called **c**)
- Consider:
  - **g(n)** = 7n+5
  - **f(n)** = n
- These have the same asymptotic behavior (linear), so **g(n)** is in O(**f(n)**) even though **g(n)** is always larger
- There is no positive $\mathbf{n_0}$ such that **g(n)** ≤ **f(n)** for all n ≥ $\mathbf{n_0}$
- The '**c**' in the definition allows for that:
  - **g(n)** ≤ ***c*** **f(n)** for all $n \geq n_0$
- To prove **g(n)** is in O(**f(n)**), have **c** = 12, $\mathbf{n_0}$ = 1

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Big Oh: Common Categories

From fastest to slowest

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\log n)$	logarithmic
$O(n)$	linear
O(n $\log$ n)	"n $\log$ n"
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where is k is an constant)
$O(k^n)$	exponential (where k is any constant > 1)

Usage note: "exponential" does not mean "grows really fast", it means "grows at rate proportional to k^n for some k>1"

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More Asymptotic Notation

- **Upper bound**: O(**f(n)**) is the set of all functions asymptotically less than or equal to **f(n)**
  - g(n) is in O(**f(n)**) if there exist constants ***c*** and n_0 such that g(n) ≤ ***c*** **f(n)** for all $n \geq n_0$
- **Lower bound**: Ω(**f(n)**) is the set of all functions asymptotically greater than or equal to **f(n)**
  - g(n) is in Ω(**f(n)**) if there exist constants ***c*** and n_0 such that g(n) ≥ ***c*** **f(n)** for all $n \geq n_0$
- **Tight bound**: θ(**f(n)**) is the set of all functions asymptotically equal to **f(n)**
  - Intersection of O(**f(n)**) and Ω(**f(n)**) (use *different* **c** values)

g(n) is θ(f(n)) iff Both g(n) = O(f(n)) AND g(n) = Ω(f(n))

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Regarding use of terms

A common error is to say O(f(n)) when you mean θ(f(n))

- People often say O() to mean a tight bound
- Say we have f(n)=n; we could say f(n) is in O(n), which is true, but only conveys the upper-bound
- Since f(n)=n is also $O(n^5)$, it's tempting to say "this algorithm is *exactly* $O(n)$"
- Somewhat incomplete; instead say it is θ(n)
- That means that it is not, for example $O(\log n)$

Less common notation:

- "little-oh": like "big-Oh" but strictly less than
  - Example: sum is $o(n^2)$ but not $o(n)$
- "little-omega": like "big-Omega" but strictly greater than
  - Example: sum is $\omega(\log n)$ but not $\omega(n)$

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What we are analyzing

- The most common thing to do is give an O or θ bound to the worst-case running time of an algorithm
- Example: True statements about binary-search algorithm
  - Common: $\theta($`log` $n)$ running-time in the worst-case
  - Less common: $\theta(1)$ in the best-case (item is in the middle)
  - Less common: Algorithm is $\Omega($`log log` $n)$ in the worst-case (it is not really, really, really fast asymptotically)
  - Less common (but very good to know): the find-in-sorted-array ***problem*** is $\Omega($`log` $n)$ in the worst-case
    - *No* algorithm can do better (without parallelism)
    - A ***problem*** cannot be $O(f(n))$ since you can always find a slower algorithm, but can mean ***there exists*** an algorithm

Other things to analyze

- Space instead of time
  - Remember we can often use space to gain time
- Average case
  - Sometimes only if you assume something about the distribution of inputs
    - See CSE312 and STAT391
  - Sometimes uses randomization in the algorithm
    - Will see an example with sorting; also see CSE312
  - Sometimes an *amortized guarantee*
    - Will discuss in a later lecture

Summary

Analysis can be about:

- The problem or the algorithm (usually algorithm)
- Time or space (usually time)
  - Or power or dollars or …
- Best-, worst-, or average-case (usually worst)
- Upper-, lower-, or tight-bound (usually upper or tight)

Big-Oh Caveats

- Asymptotic complexity (Big-Oh) focuses on behavior for **large *n*** and is independent of any computer / coding trick
  - But you can "abuse" it to be misled about trade-offs
  - Example: $n^{1/10}$ vs. `log` n
    - Asymptotically $n^{1/10}$ grows more quickly
    - But the "cross-over" point is around $5 * 10^{17}$
    - So if you have input size less than 2^{58}, prefer $n^{1/10}$
- Comparing O() for ***small n*** values can be misleading
  - Quicksort: O(nlogn) (expected)
  - Insertion Sort: $O(n^2)$ (expected)
  - Yet in reality Insertion Sort is faster for small n's
  - We'll learn about these sorts later

Addendum: Timing vs. Big-Oh?

- At the core of CS is a backbone of theory & mathematics
  - Examine the algorithm itself, mathematically, not the implementation
  - Reason about performance as a function of n
  - Be able to mathematically prove things about performance
- Yet, timing has its place
  - In the real world, we do want to know whether implementation A runs faster than implementation B on data set C
  - Ex: Benchmarking graphics cards
  - We will do some timing in project 3 (and in 2, a bit)
- Evaluating an algorithm? Use asymptotic analysis
- Evaluating an implementation of hardware/software? Timing can be useful