CSE332: Data Abstractions

Lecture 2: Math Review; Algorithm Analysis

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Proof via mathematical induction

Suppose *P(n)* is some rule involving n

- Example: $n \geq n/2 + 1$, for all $n \geq 2$

To prove *P(n)* for all integers $n \geq c$, it suffices to prove

1. *P(c)* – called the "basis" or "base case"
2. If *P(k)* then *P(k+1)* – called the "induction step" or "inductive case"

Why we will care:

To show an algorithm is correct or has a certain running time *no matter how big a data structure or input value is*

(Our "*n*" will be the data structure or input size.)

Example

P(n) = "the sum of the first *n* powers of 2 (starting at 2^0) is the next power of 2 minus 1"

Theorem: *P(n)* holds for all $n \geq 1$

1=2-1

1+2=4-1

1+2+4=8-1

So far so good…

Example

Theorem: *P(n)* holds for all *n* ≥ 1

Proof: By induction on *n*

- Base case, *n*=1: $2^0 = 1 = 2^1 - 1$
- Inductive case:
  - Inductive hypothesis: Assume the sum of the first *k* powers of 2 is 2^k-1
  - Show, given the hypothesis, that the sum of the first (*k*+1) powers of 2 is 2^{k+1}-1

From our inductive hypothesis we know:

$$1 + 2 + 4 + ... + 2^{k-1} = 2^k - 1$$

Add the next power of 2 to both sides…

$$1 + 2 + 4 + ... + 2^{k-1} + 2^k = 2^k - 1 + 2^k$$

We have what we want on the left; massage the right a bit

$$1 + 2 + 4 + ... + 2^{k-1} + 2^k = 2(2^k) - 1 = 2^{k+1} - 1$$

Note for homework

Proofs by induction will come up a fair amount on the homework

When doing them, be sure to state each part clearly:

- What you're trying to prove
- The base case
- The inductive case
- The inductive hypothesis
  - In many inductive proofs, you'll prove the inductive case by just starting with your inductive hypothesis, and playing with it a bit, as shown above

Powers of 2

- A bit is 0 or 1
- A sequence of *n* bits can represent 2^n distinct things
  - For example, the numbers 0 through 2^n-1
- 2^{10} is 1024 ("about a thousand", kilo in CSE speak)
- 2^{20} is "about a million", mega in CSE speak
- 2^{30} is "about a billion", giga in CSE speak

Java: an **`int`** is 32 bits and signed, so "max int" is "about 2 billion"

a **`long`** is 64 bits and signed, so "max long" is 2^{63}-1

Therefore…

We could give a unique id to…

- Every person in this room with 4 bits
- Every person in the U.S. with 29 bits
- Every person in the world with 33 bits
- Every person to have ever lived with 38 bits (estimate)
- Every atom in the universe with 250-300 bits

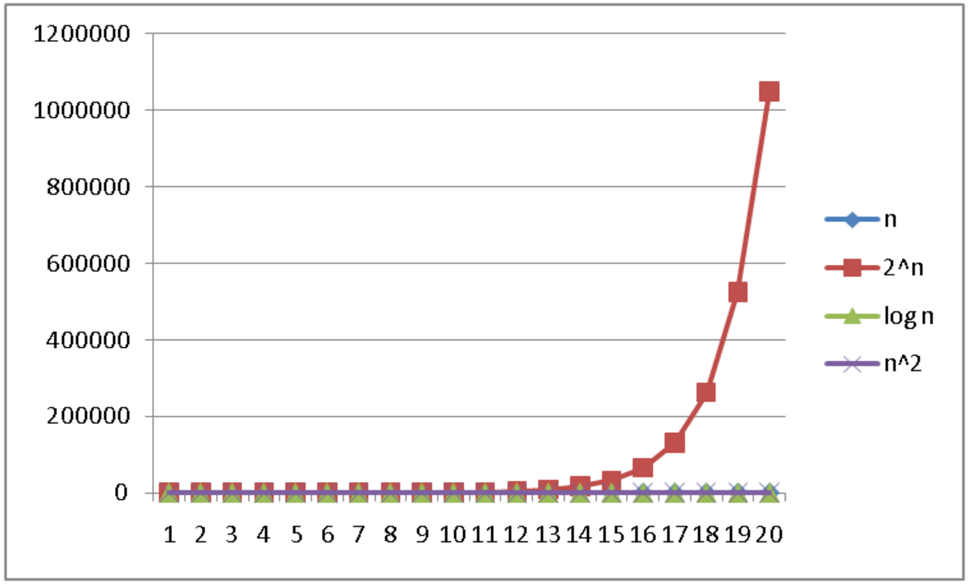
So if a password is 128 bits long and randomly generated, do you think you could guess it?

Logarithms and Exponents

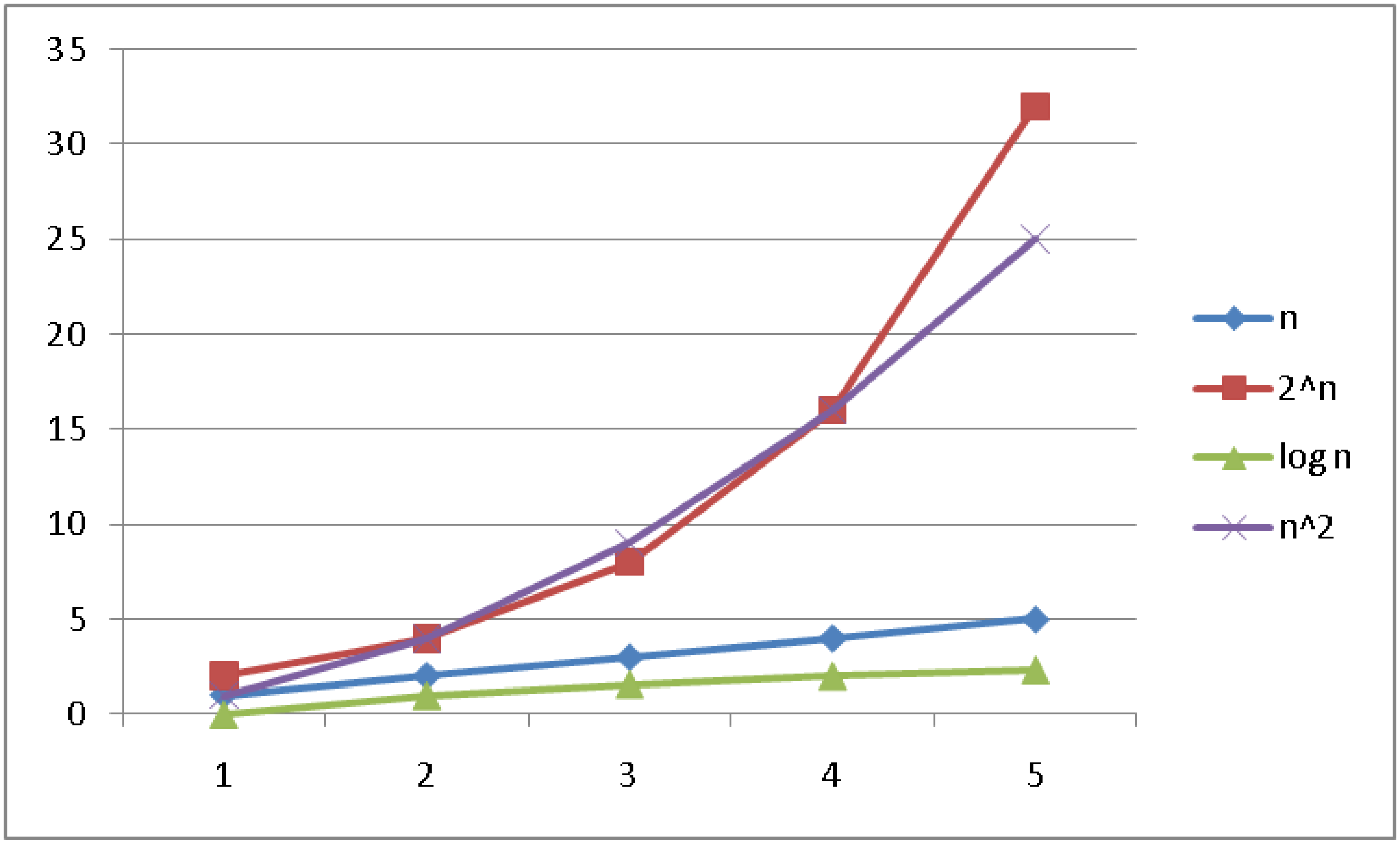
- Since so much is binary in CS, **`log`** almost always means $\log_2$
- Definition: $\log_2 x = y$ if $x = 2^y$
- So, $\log_2$ 1,000,000 = "a little under 20"

Just as exponents grow *very* quickly, logarithms grow *very* slowly

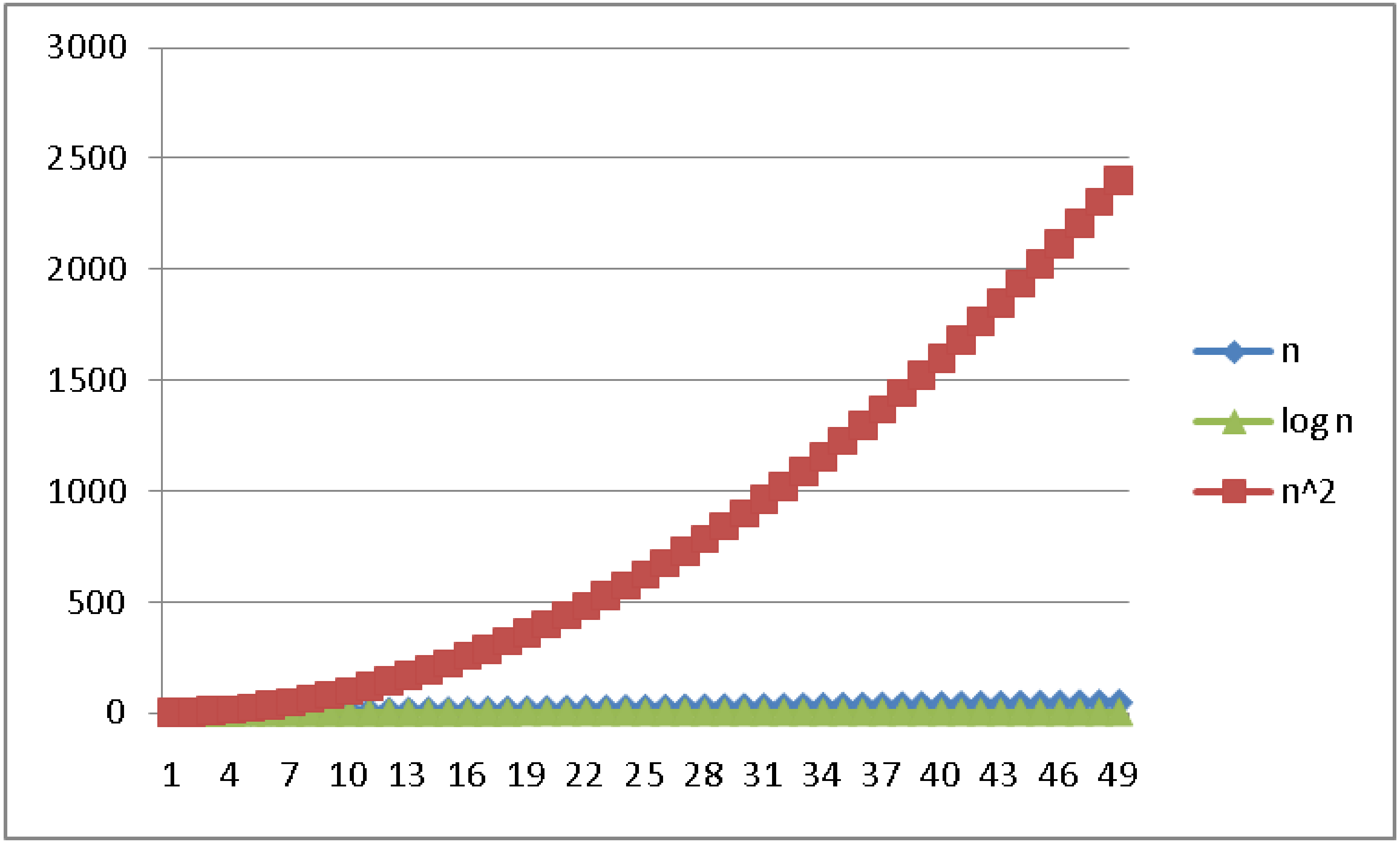
See Excel file for plot data – play with it!



Logarithms and Exponents



Logarithms and Exponents



Properties of logarithms

- **`log(A*B) = log A + log B`**
  - So **`log(N`**^{k}**`)= k log N`**
- **`log(A/B) = log A – log B`**
- **`x=`** $\log_2 2^x$
- **`log(log x)`** is written **`log log x`**
  - Grows as slowly as 2^{2^y} grows fast
  - Ex: $\log_2 \log_2 4billion \sim \log_2 \log_2 2^{32} = \log_2 32 = 5$
- **`(log x)(log x)`** is written **`log`**^{2}**`x`**
  - It is greater than **`log x`** for all **`x > 2`**

Log base doesn't matter (much)

"Any base *B* log is equivalent to base 2 log within a constant factor"

- And we are about to stop worrying about constant factors!
- In particular, $\log_2 x = 3.22 \log_{10} x$
- In general, we can convert log bases via a constant multiplier
- Say, to convert from base B to base A:

$$\log_B x = (\log_A x) / (\log_A B)$$

Algorithm Analysis

As the "size" of an algorithm's input grows (length of array to sort, size of queue to search, etc.):

- How much longer does the algorithm take (time)
- How much more memory does the algorithm need (space)

We are generally concerned about approximate runtimes

- Whether $T(n)=3n+2$ or $T(n)=n/4+8$, we say it runs in linear time
- Common categories:
  - Constant: $T(n)=1$
  - Linear: $T(n)=n$
  - Logarithmic: $T(n)=\log n$
  - Exponential: $T(n)=2^n$

Example

- First, what does this pseudocode return?

```
x := 0;
for i=1 to n do
  for j=1 to i do
     x := x + 3;
return x;
```

- For any n ≥ 0, it returns 3n(n+1)/2
- Proof: By induction on *n*
  - *P(n)* = after outer for-loop executes *n* times, **x** holds 3n(n+1)/2
  - Base: n=0, returns 0
  - Inductive case:
    - Inductive hypothesis: **x** holds 3k(k+1)/2 after *k* iterations.
  - Next iteration adds 3(k+1), for total of \
    3k(k+1)/2 + 3(k+1)=
    (3k(k+1) + 6(k+1))/2 =
    (k+1)(3k+6)/2 =
    3(k+1)(k+2)/2

Example

- How long does this pseudocode run?

```
x := 0;
for i=1 to n do
  for j=1 to i do
     x := x + 3;
return x;
```

- Find running time in terms of n, for any n ≥ 0
  - Assignments, additions, returns take "1 unit time"
    - Constant time
  - Loops take the sum of the time for their iterations

- So: 2 + 2*(number of times inner loop runs)
  - And how many times is that…

Example

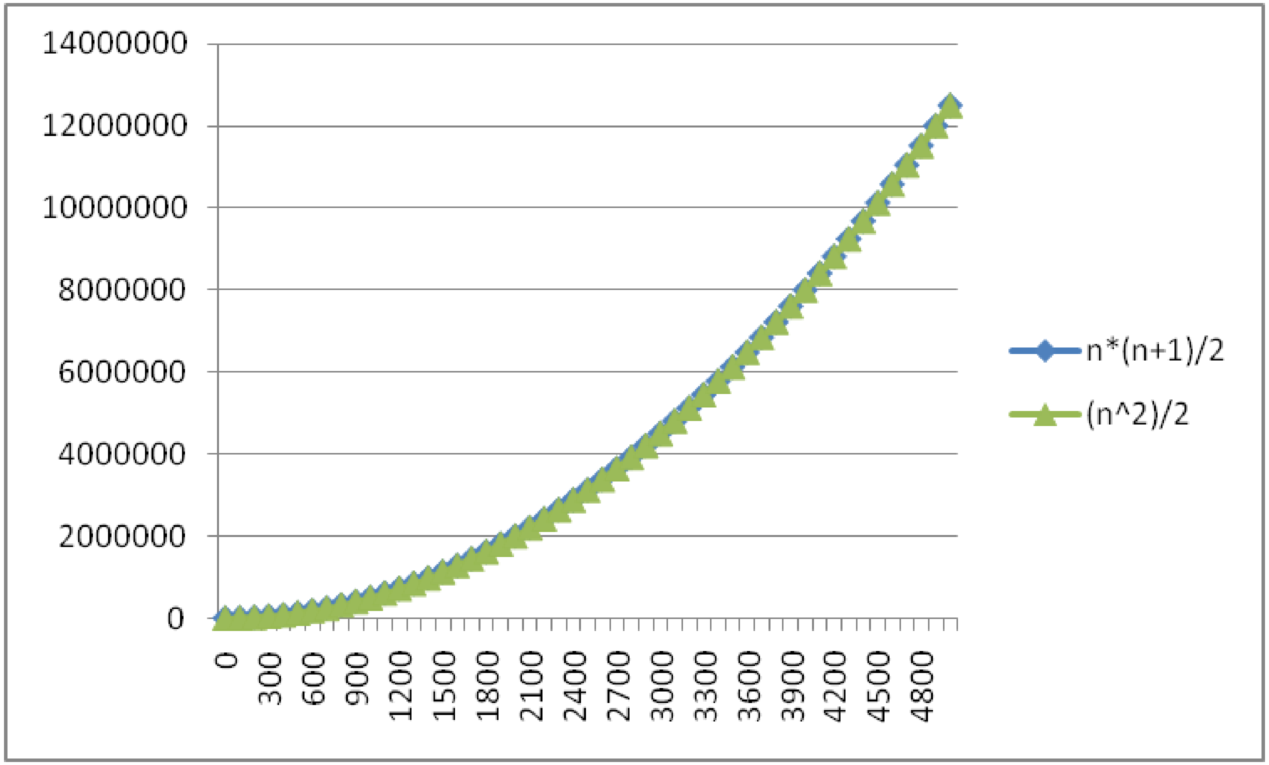
- How long does this pseudocode run?

```
x := 0;
for i=1 to n do
  for j=1 to i do
     x := x + 3;
return x;
```

- n=1 -> 1 time; n=2 -> 3 times; n=3 -> 6 times
- The total number of loop iterations is n*(n+1)/2
  - You'll get to prove it in the homework
  - This is *proportional to* n^2 , and we say $O(n^2)$, "big-Oh of"
    - For large enough n, the n and constant terms are irrelevant, as are the first assignment and return
    - See plot… n*(n+1)/2 vs. just $n^2/2$

Lower-order terms don't matter

n*(n+1)/2 vs. just $n^2/2$



Big Oh (also written Big-O)

- Big Oh is used for comparing asymptotic behavior of functions; which is ‘faster’?
- We’ll get into the definition later, but for now:
  - ‘f(n) is O(g(n))’ roughly means
    - The function f(n) is at least as small as g(n) as they go toward infinity
    - Think of it as ≤
  - BUT: Big Oh ignores constant factors
    - n+10 is O(n); we drop out the ‘+10’
    - 5n is O(n); we drop out the ‘x5’
    - The following is NOT true though: n^2 is O(n)
  - Note that ‘f(n) is O(g(n))’ gives an upper bound for f(n)
    - n is O(n^2)
    - 5 is O(n)

Big Oh: Common Categories

From fastest to slowest

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\log n)$	logarithmic
$O(n)$	linear
$O(n \log n)$	"n log *n*"
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where is k is an constant)
$O(k^n)$	exponential (where k is any constant > 1)

Usage note: "exponential" does not mean "grows really fast", it means "grows at rate proportional to k^n for some $k>1$"

- A savings account accrues interest exponentially (k=1.01?)