CSE332: Data Abstractions

Lecture 17: Shortest Paths

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Single source shortest paths (from some specific point a)

- Done: BFS to find the minimum path length from **v** to **u** in $O(|E|)$

- Actually, can find the minimum path length from **v** to *every node*
  - Still $O(|E|)$
  - No faster way for a "distinguished" destination in the worst-case

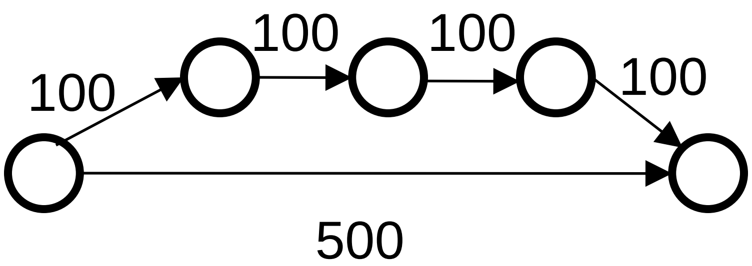
- Now: Weighted graphs

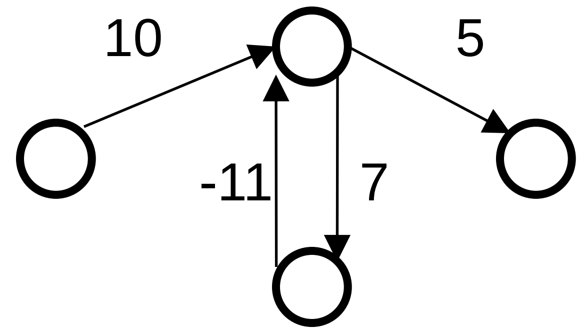
Given a weighted graph and node **v**,
find the minimum-cost path from **v** to every node

- As before, asymptotically no harder than for one destination
- Unlike before, BFS will not work

- Aside: We can find the shortest path from every vertex to every other vertex in $O(|V|^3)$

Not as easy





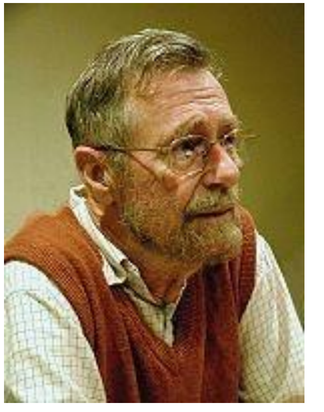
Why BFS won't work: Shortest path may not have the fewest edges

We will assume there are no negative weights

- Problem is ill-defined if there are negative-cost *cycles*
- What is the shortest path here?
- Even without negative cycles, can still get wrong answer if negative weights are involved

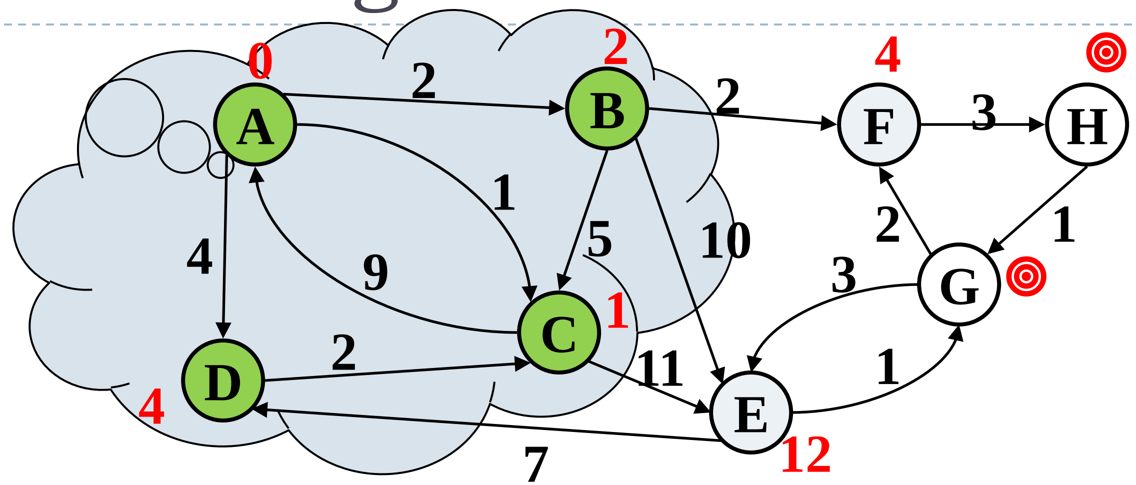
Dijkstra's Algorithm (for shortest paths)

- Named after its creator Edsger Dijkstra (1930-2002)
  - Truly one of the "founders" of computer science; this is just one of his many contributions
  - Quotation: "computer science is no more about computers than astronomy is about telescopes"



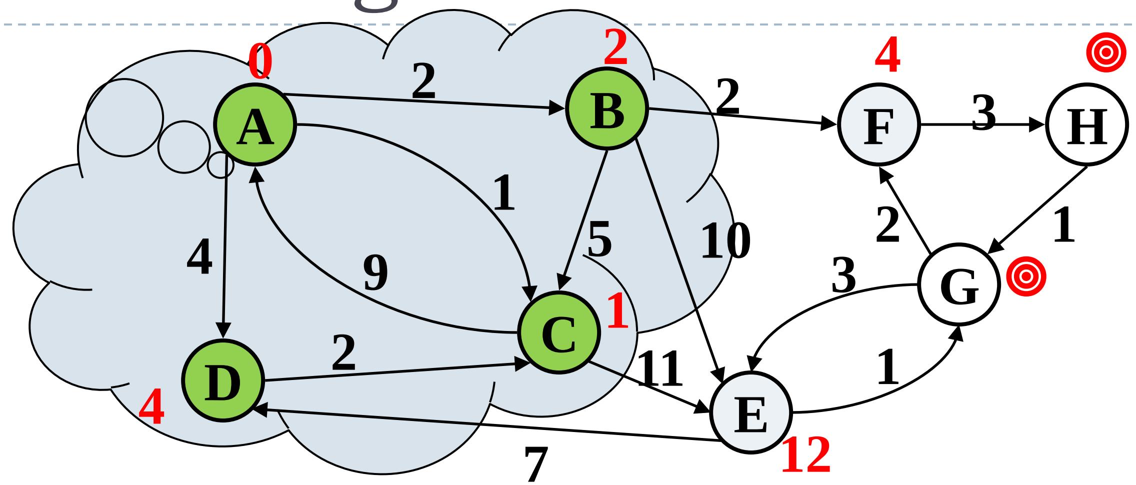
- The idea: reminiscent of BFS, but adapted to handle weights
  - A priority queue will prove useful for efficiency (later)
  - Will grow the set of nodes whose shortest distance has been computed
  - Nodes not in the set will have a "best distance so far"

Dijkstra's Algorithm: Idea



- Conceptually:
  - Grow our 'cloud' of known vertices by 1 each step
  - Pick a vertex outside the cloud, that's closest to our starting point
  - Guaranteed that we have the shortest path to everything within the cloud (more later)

Dijkstra's Algorithm: Idea



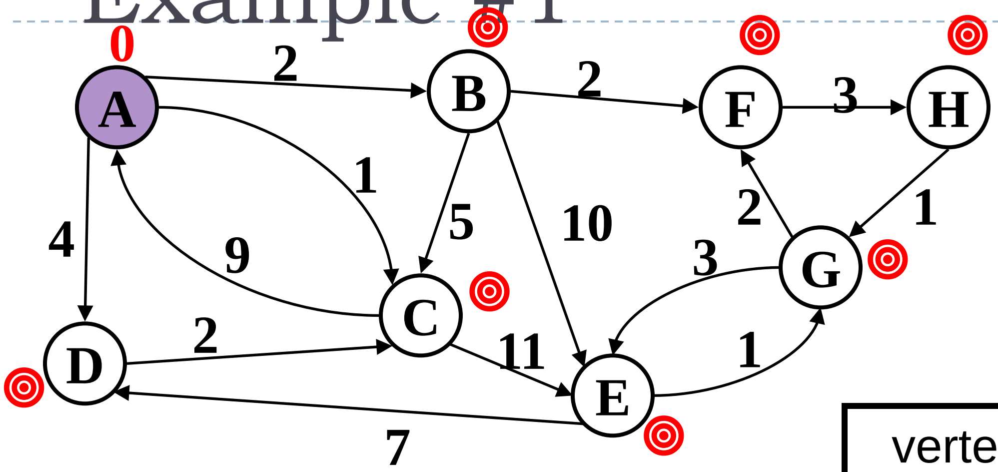
- Initially, start node has cost 0 and all other nodes have cost ∞
- Mark each vertex as 'unknown' (also referred to as 'unvisited', 'unexplored')
- At each step:
  - Pick closest unknown vertex **v** (will be start node for first step)
  - Add it to the "cloud" of known vertices
  - Update distances for nodes with edges from **v**
- That's it! (Have to prove it produces correct answers)

The Algorithm

1. For each node **`v`**, set **`v.cost=∞`** and **`v.known=false`**
2. Set **`source.cost = 0`**
3. While there are unknown nodes in the graph
   a) Select the unknown node **`v`** with lowest cost
   b) Mark **`v`** as known
   c) For each edge **`(v,u)`** with weight **`w`**,

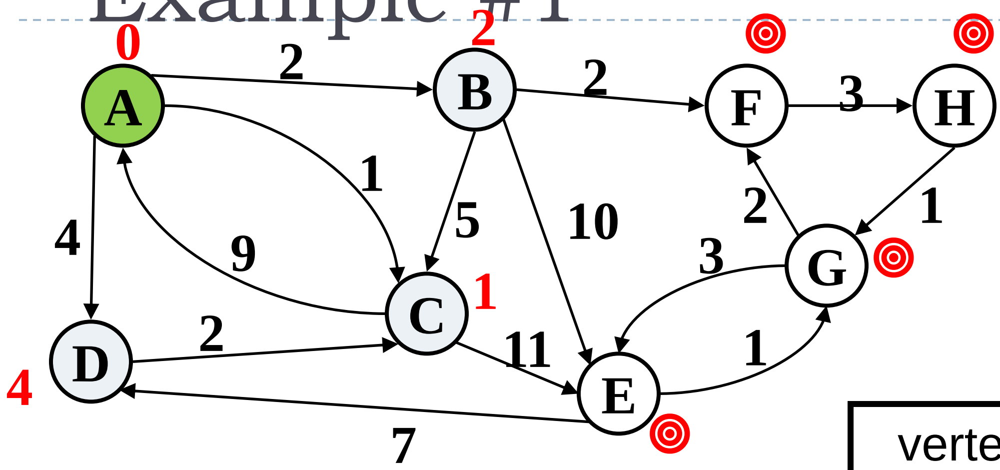
```
c1 = v.cost + w // cost of best path through v to u
c2 = u.cost     // cost of best path to u previously known
if(c1 < c2){ // if the path through v is better
   u.cost = c1
   u.path = v // for computing actual paths
}
```

Example #1



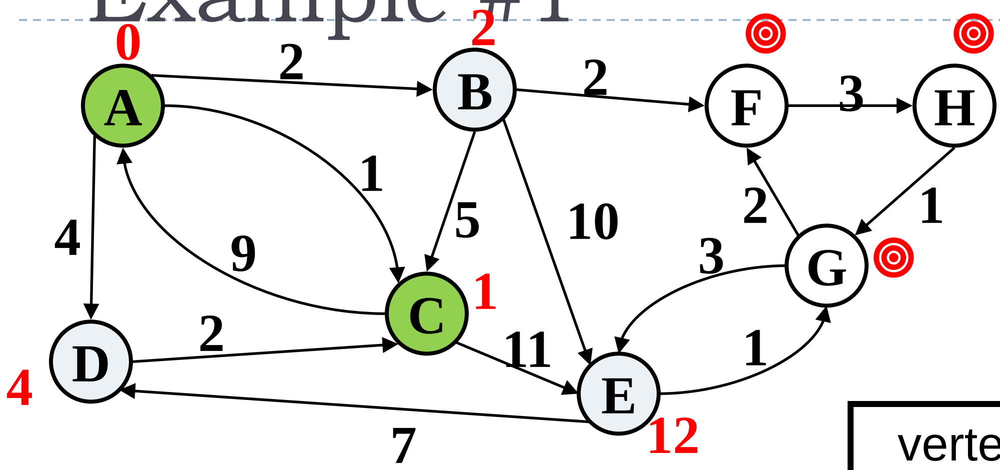
vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	
H		??	

Example #1



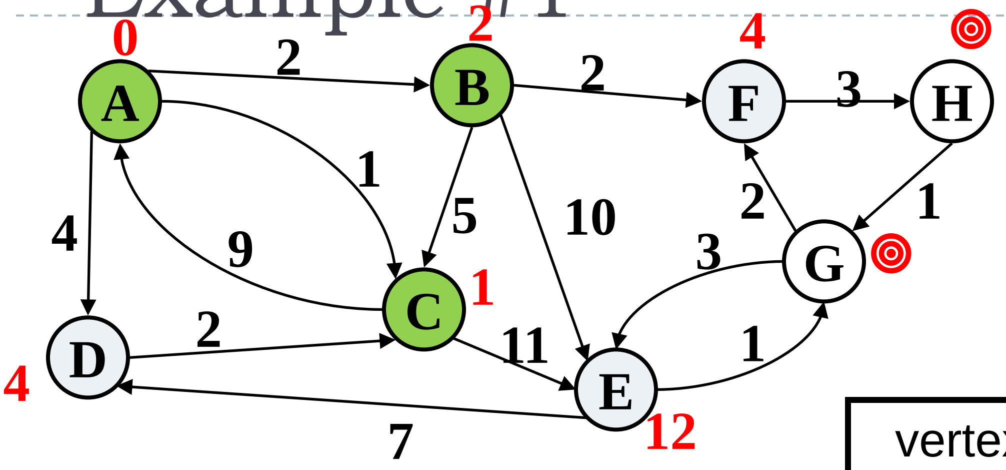
vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C		≤ 1	A
D		≤ 4	A
E		??	
F		??	
G		??	
H		??	

Example #1



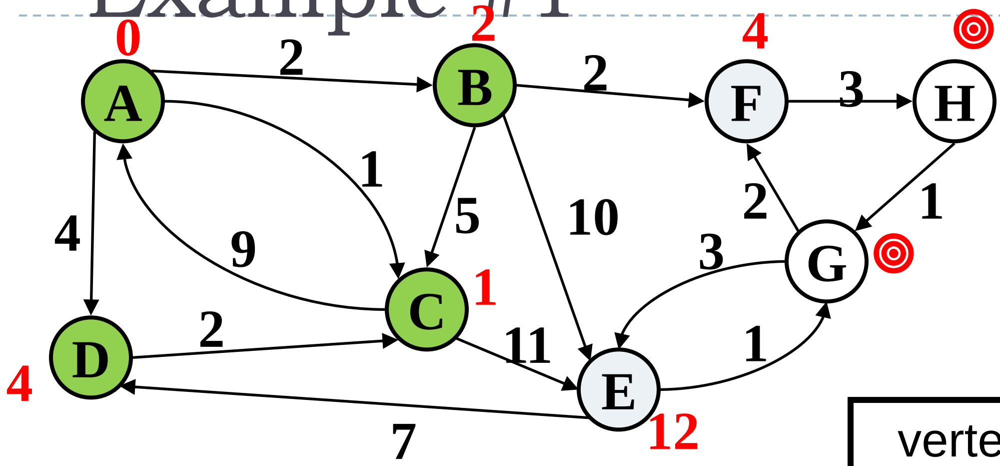
vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		??	
G		??	
H		??	

Example #1



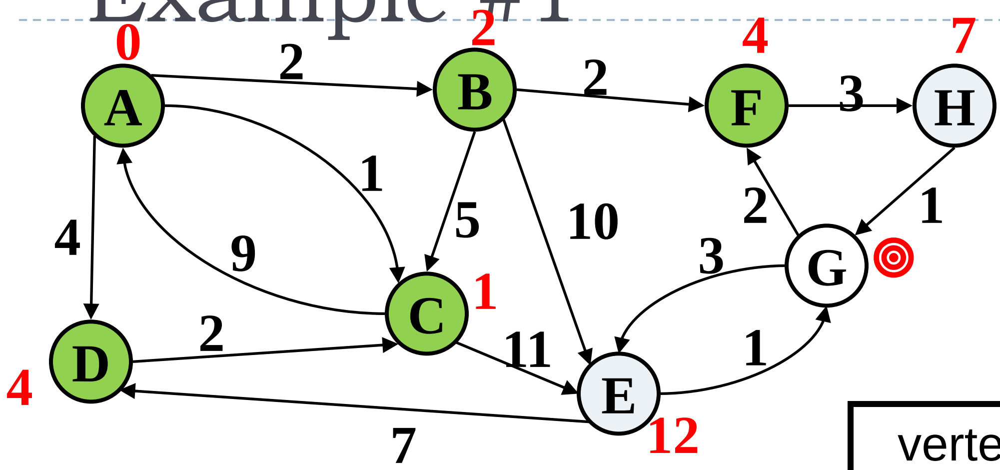
vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

Example #1



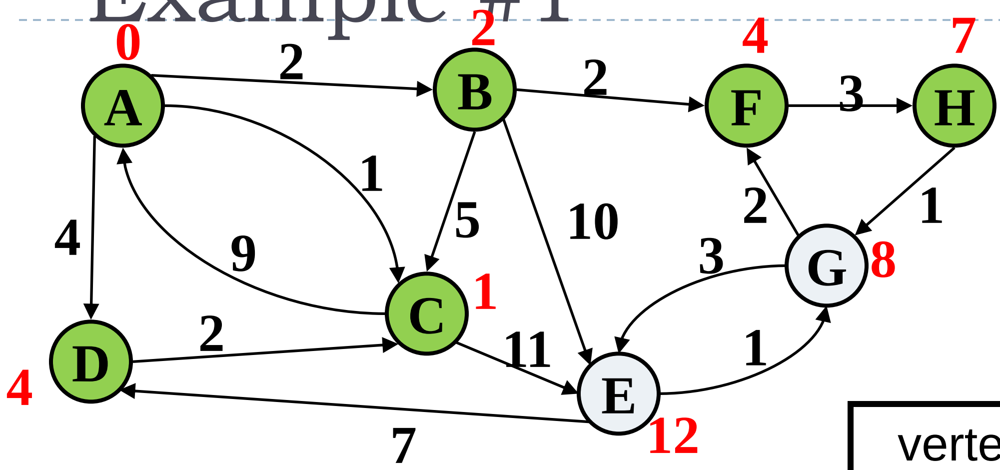
vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		$\leq$ 12	C
F		$\leq$ 4	B
G		??	
H		??	

Example #1



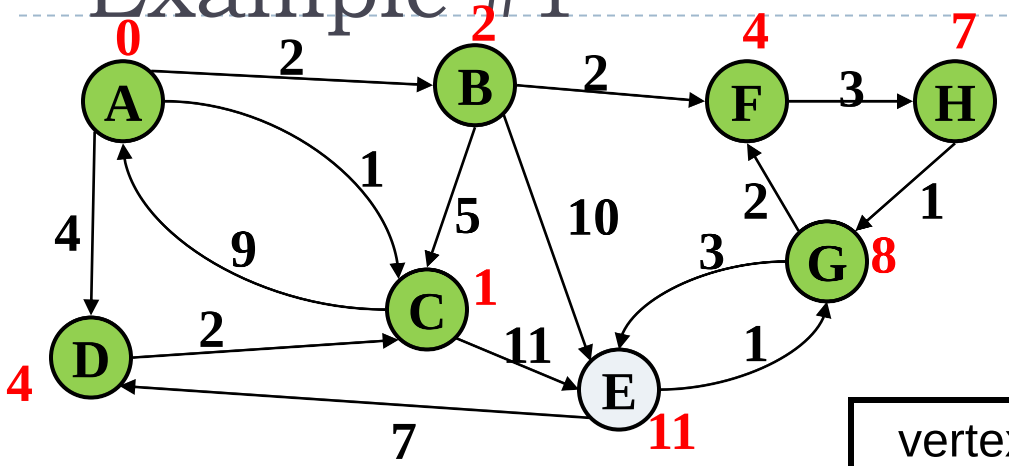
vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		??	
H		≤ 7	F

Example #1



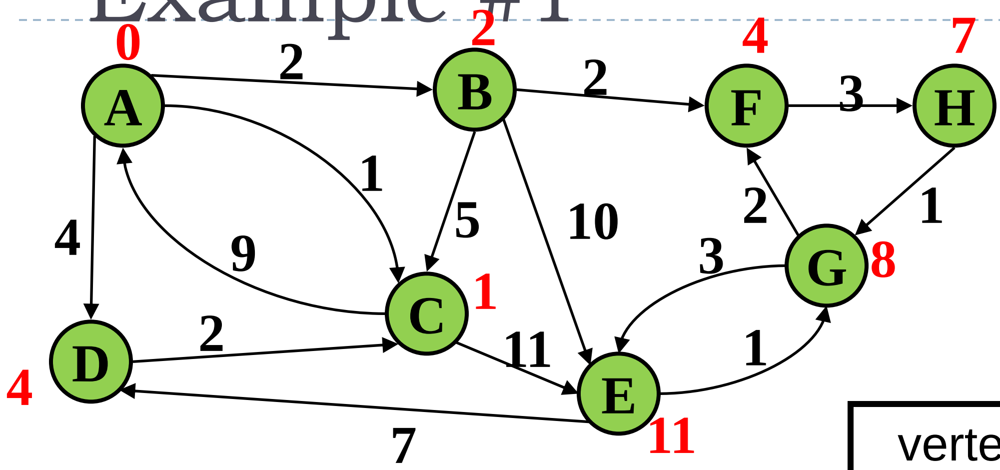
vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		≤ 8	H
H	Y	7	F

Example #1



vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Example #1



vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

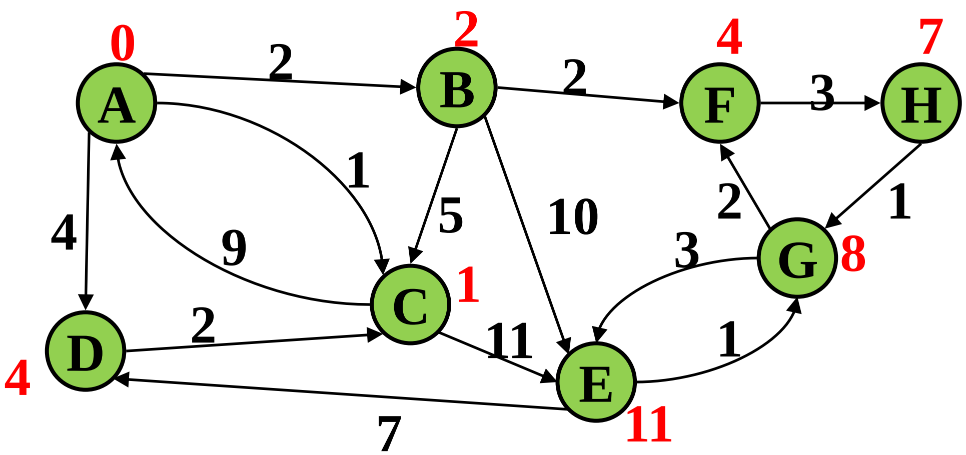
Important features

- Once a vertex is marked ‘known’, the cost of the shortest path to that node is known
  - As is the path itself

- While a vertex is still not known, another shorter path to it might still be found

Interpreting the results

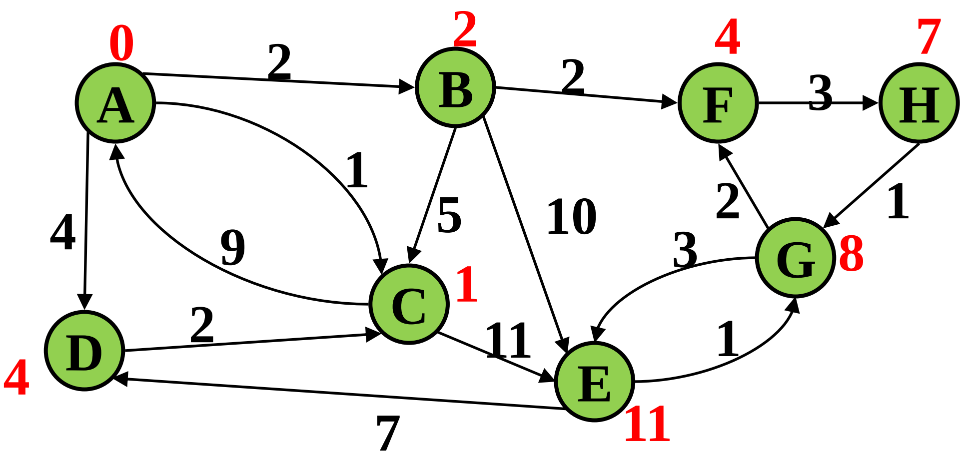
- Now that we're done, how do we get the path from, say, A to E?



vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

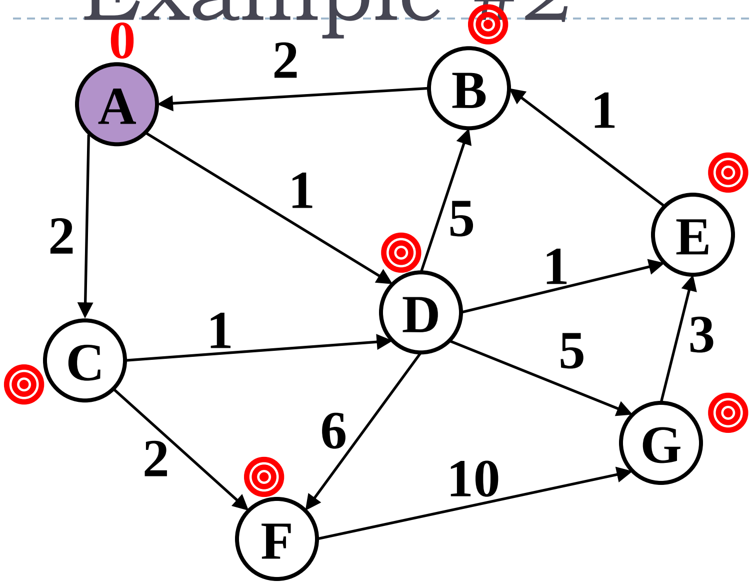
Stopping Short

- How would this have worked differently if we were only interested in the path from A to G?
  - A to E?



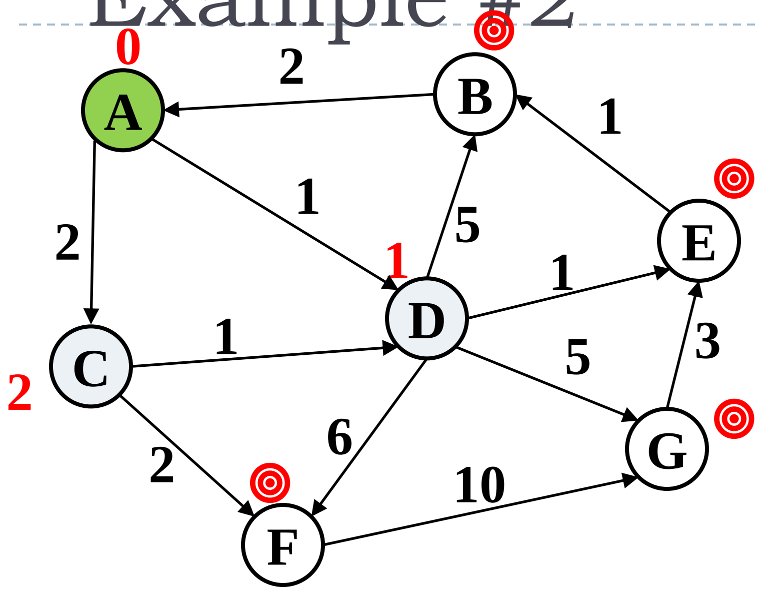
vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Example #2



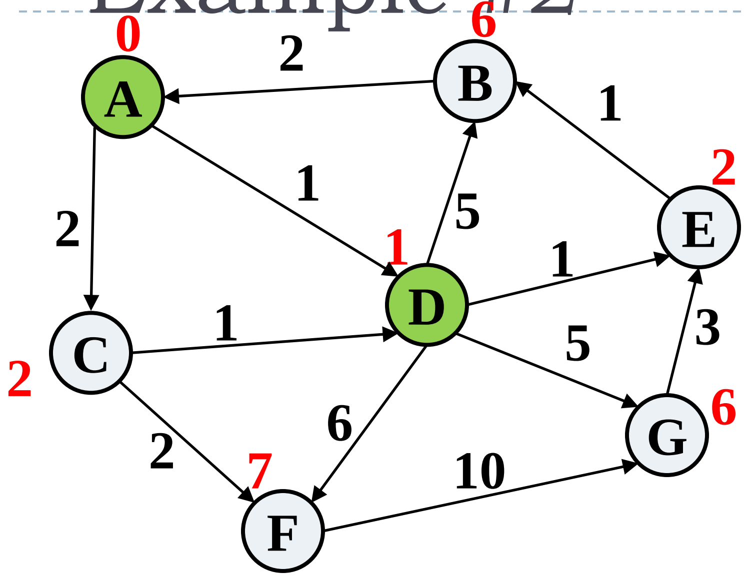
vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	

Example #2



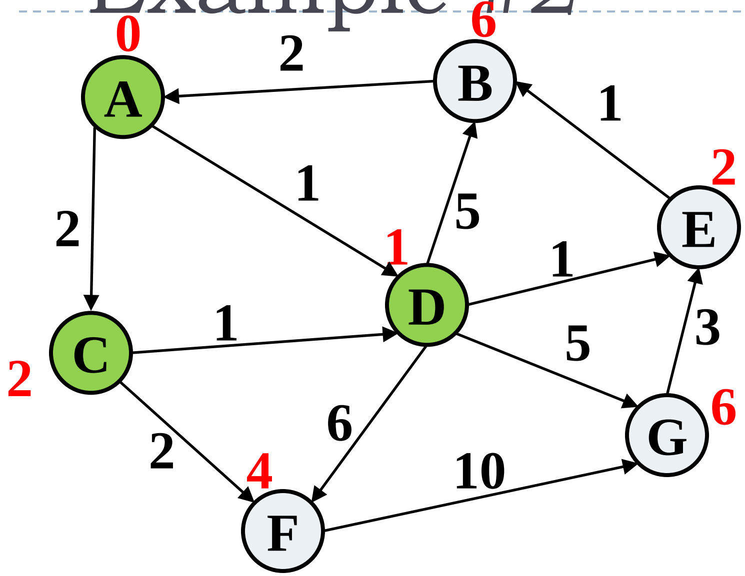
vertex	known?	cost	path
A	Y	0	
B		??	
C		≤ 2	A
D		≤ 1	A
E		??	
F		??	
G		??	

Example #2



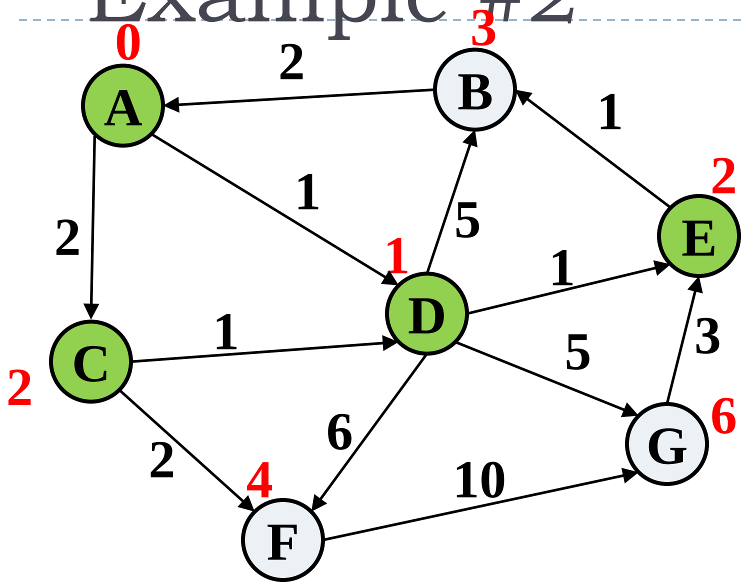
vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C		≤ 2	A
D	Y	1	A
E		≤ 2	D
F		≤ 7	D
G		≤ 6	D

Example #2



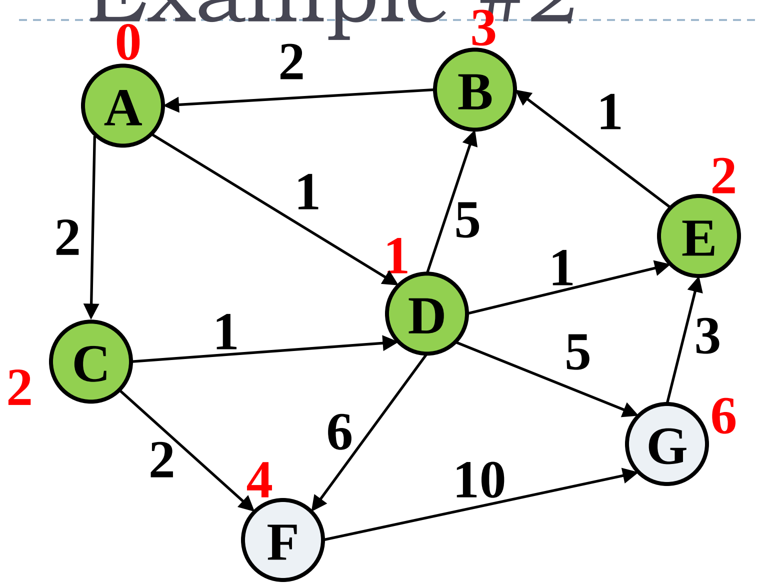
vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C	Y	2	A
D	Y	1	A
E		≤ 2	D
F		≤ 4	C
G		≤ 6	D

Example #2



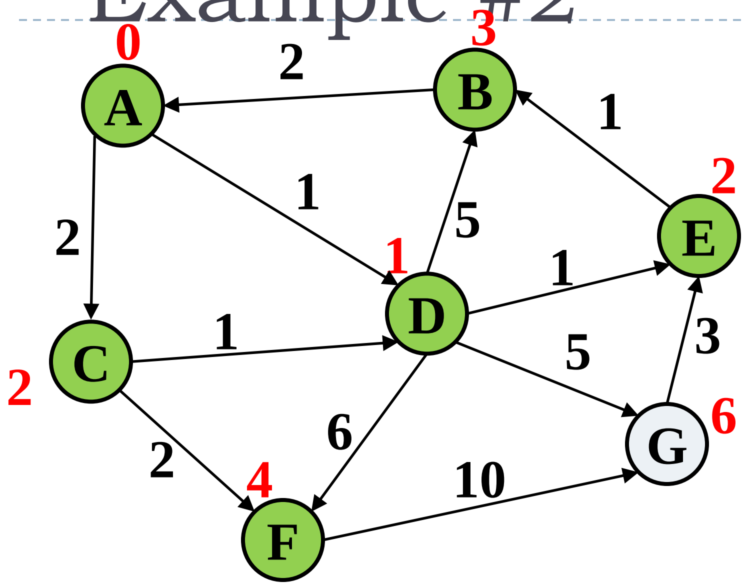
vertex	known?	cost	path
A	Y	0	
B		≤ 3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D

Example #2



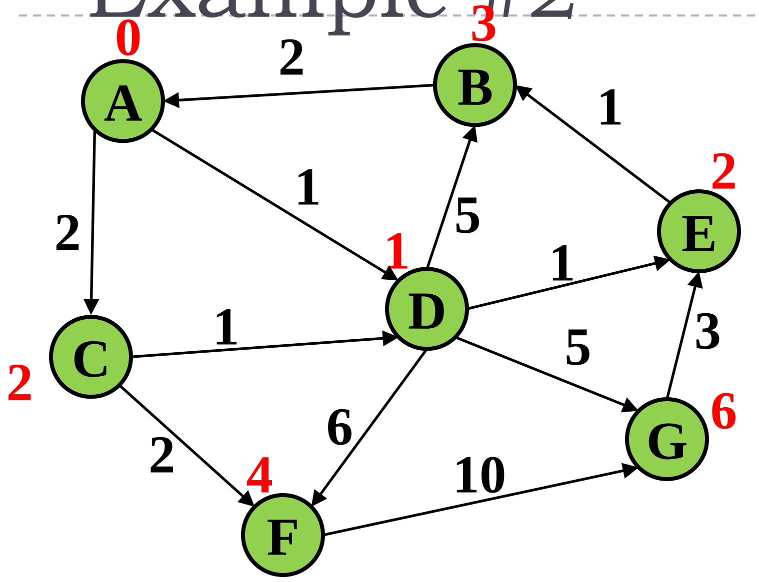
vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D

Example #2



vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G		≤ 6	D

Example #2

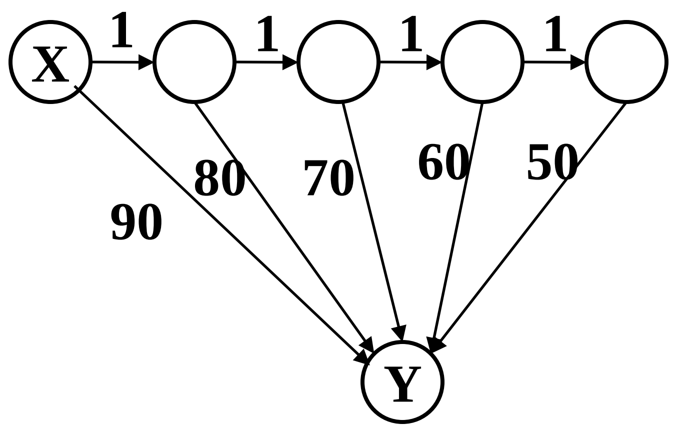


vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G	Y	6	D

True or false:

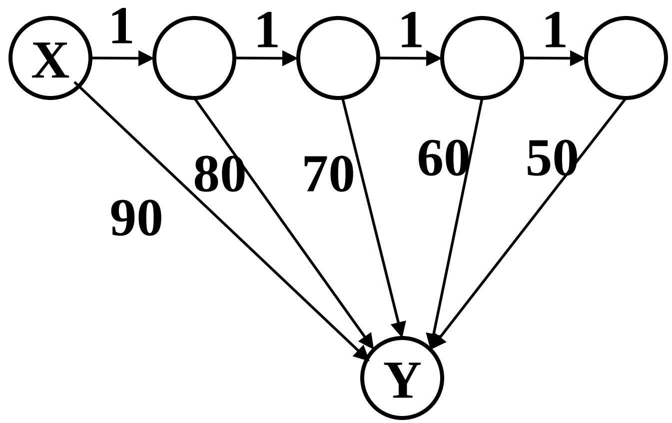
- If we were to count out all the edges ‘found’ by Dijkstra’s, we would have |V|-1 edges

Example #3



How will the best-cost-so-far from X to Y proceed?

Example #3



How will the best-cost-so-far from X to Y proceed?

90, 81, 72, 63, 54

Where are we?

- Have described Dijkstra’s algorithm
  - For single-source shortest paths in a weighted graph (directed or undirected) with no negative-weight edges
  - An example of a *greedy algorithm*: at each step, irrevocably does the best thing it can at that step
    - Because of the way the algorithm is structured, the ‘apparent best’ actually is the best

- What should we do after learning an algorithm?
  - Prove it is correct
    - Not obvious!
    - We will sketch the key ideas
  - Analyze its efficiency

Correctness: Intuition

Rough intuition:

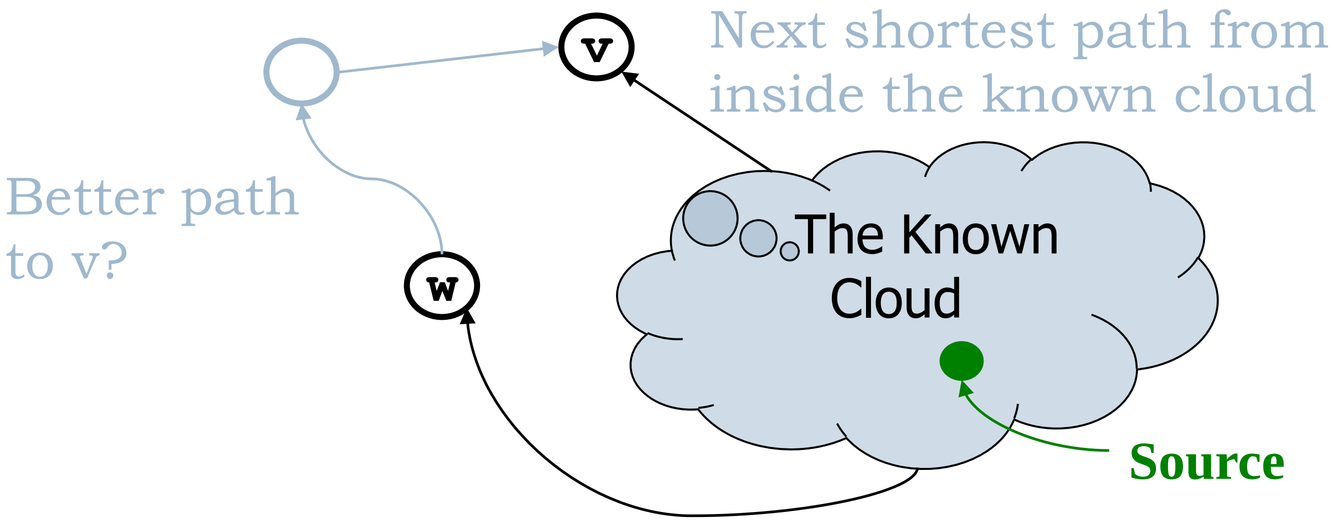
All the “known” vertices have the correct shortest path

- True initially: shortest path to start node has cost 0
- If it stays true every time we mark a node “known”, then by induction this holds and eventually everything is “known”

Key fact we need: When we mark a vertex “known” we won’t discover a shorter path later!

- This holds only because Dijkstra’s algorithm picks the node with the next shortest path-so-far
- The proof is by contradiction…

Correctness: The Cloud (Rough Idea)



Suppose **v** is the next node to be marked known (“added to the cloud”)

- The best-*known* path to **v** must have only nodes “in the cloud”
  - Since we’ve selected it, and we only know about paths through the cloud to a node right outside
- Assume the actual shortest path to **v** is different than the best-known
  - It won’t use only cloud nodes, or we would know about it; so it must use non-cloud nodes
  - Let **w** be the *first* non-cloud node on this ‘actual shortest path’
  - The part of the path up to **w** is already known and must be shorter than the best-known path to **v**. So **v** would not have been picked. Contradiction.

Efficiency, first approach

Use pseudocode to determine asymptotic run-time

- Notice each edge is processed only once

```
dijkstra(Graph G, Node start) {
  for each node: x.cost=infinity, x.known=false
  start.cost = 0
  while(not all nodes are known) {
    b = find unknown node with smallest cost
    b.known = true
    for each edge (b,a) in G
     if(!a.known)
       if(b.cost + weight((b,a)) < a.cost){
         a.cost = b.cost + weight((b,a))
         a.path = b
       }
}
```

$O(|V|)$

$O(|V|^2)$

$O(|E|)$

$O(|V|^2)$

Improving asymptotic running time

- So far: $O(|V|^2)$
- We had a similar "problem" with topological sort being $O(|V|^2)$ due to each iteration looking for the node to process next
  - We solved it with a queue of zero-degree nodes
  - But here we need the lowest-cost node and costs can change as we process edges
- Solution?
  - A priority queue holding all unknown nodes, sorted by cost
  - But must support `decreaseKey` operation
    - Must maintain a reference from each node to its position in the priority queue

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```
dijkstra(Graph G, Node start) {
  for each node: x.cost=infinity, x.known=false
  start.cost = 0
  build-heap with all nodes
  while(heap is not empty) {
    b = deleteMin()
    b.known = true
    for each edge (b,a) in G
     if(!a.known)
      if(b.cost + weight((b,a)) < a.cost){
        decreaseKey(a,"new cost – old cost")
        a.path = b
      }
}
```

$O(|V|)$

$O(|V|\log|V|)$

$O(|E|\log|V|)$

$O(|V|\log|V|+|E|\log|V|)$

Dense vs. sparse again

- First approach: $O(|V|^2)$
- Second approach: $O(|V|\log|V|+|E|\log|V|)$
- So which is better?
  - Sparse: $O(|V|\log|V|+|E|\log|V|)$
    - (if $|E| > |V|$, then $O(|E|\log|V|)$)
  - Dense: $O(|V|^2)$
- But, remember these are worst-case and asymptotic
  - Priority queue might have slightly worse constant factors
  - On the other hand, for "normal graphs", we might call **`decreaseKey`** rarely (or not percolate far), making $|E|\log|V|$ more like $|E|$