# CSE332: Data Abstractions

# Lecture 3: Asymptotic Analysis

Dan Grossman
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## *Comparing algorithms*

When is one *algorithm* (not *implementation*) better than another?

- Various possible answers (clarity, security, …)
- But a big one is *performance*: for sufficiently large inputs, runs in less time (our focus) or less space

Large inputs because probably any algorithm is "plenty good" for small inputs (if *n* is 10, probably anything is fast)

Answer will be *independent* of CPU speed, programming language, coding tricks, etc.

Answer is general and rigorous, complementary to "coding it up and timing it on some test cases"

- Can do analysis before coding!



## *Analyzing code ("worst case")*

Basic operations take "some amount of" constant time

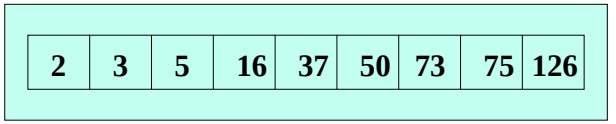
- Arithmetic (fixed-width)
- Assignment
- Access one Java field **or array index**
- Etc.

(This is an *approximation of reality*: a very useful "lie".)

| | |
|---|---|
| Consecutive statements | Sum of times |
| Conditionals | Time of test plus slower branch |
| Loops | Sum of iterations |
| Calls | Time of call's body |
| Recursion | Solve *recurrence equation* |



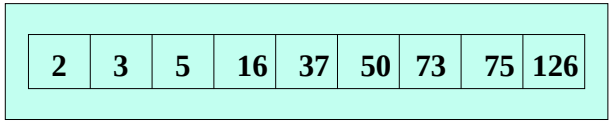
## *Example*




Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   ???
}
```

## Linear search




Find an integer in a *sorted* array

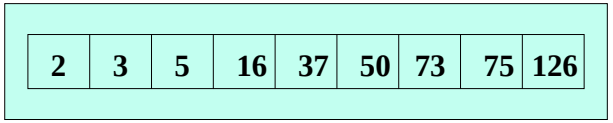
```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case: 6ish steps = $O(1)$

Worst case: 6ish*(arr.length)
= $O$(arr.length)



## Binary search




Find an integer in a *sorted* array

- Can also be done non-recursively but "doesn't matter" here

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2; //i.e., lo+(hi-lo)/2
   if(lo==hi)       return false;
   if(arr[mid]==k)  return true;
   if(arr[mid]< k)  return help(arr,k,mid+1,hi);
   else             return help(arr,k,lo,mid);
}
```



## Binary search

Best case: 8ish steps = $O(1)$

Worst case: $T(n)$ = 10ish + $T(n/2)$ where $n$ is **`hi-lo`**

- $O($**`log`** $n)$ where $n$ is **`array.length`**
- Solve *recurrence equation* to know that…

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2;
   if(lo==hi)       return false;
   if(arr[mid]==k)  return true;
   if(arr[mid]< k)  return help(arr,k,mid+1,hi);
   else             return help(arr,k,lo,mid);
}
```



## Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
   - $T(n) = 10 + T(n/2)$ $T(1) = 8$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.
   - $T(n) = 10 + 10 + T(n/4)$
     $= 10 + 10 + 10 + T(n/8)$
     $= \ldots$
     $= 10k + T(n/(2^k))$
3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case
   - $n/(2^k) = 1$ means $n = 2^k$ means $k = \log_2 n$
   - So $T(n) = 10 \log_2 n + 8$ (get to base case and do it)
   - So $T(n)$ is $O($**`log`** $n)$

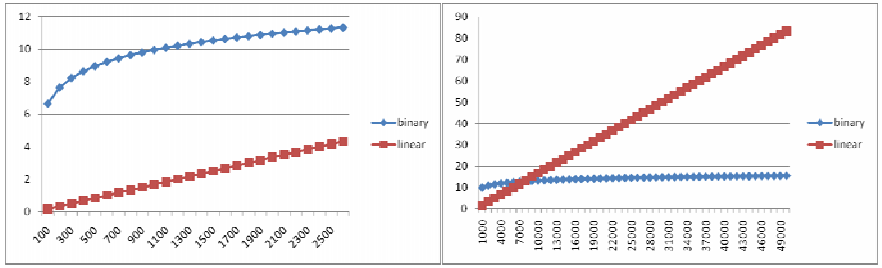
## *Ignoring constant factors*

- So binary search is $O(\log n)$ and linear is $O(n)$
  - But which is faster
- Could depend on constant factors
  - How *many* assignments, additions, etc. for each $n$
  - And could depend on size of $n$
- But there exists some $n_0$ such that for all $n > n_0$ binary search wins
- Let's play with a couple plots to get some intuition…



## *Example*

- Let's try to "help" linear search
  - Run it on a computer 100x as fast (say 2010 model vs. 1990)
  - Use a new compiler/language that is 3x as fast
  - Be a clever programmer to eliminate half the work
  - So doing each iteration is 600x as fast as in binary search
- Note: 600x still helpful for problems without logarithmic algorithms!






## *Another example: sum array*

Two "obviously" linear algorithms: $T(n) = O(1) + T(n-1)$

Iterative:

```
int sum(int[] arr){
  int ans = 0;
  for(int i=0; i<arr.length; ++i)
     ans += arr[i];
  return ans;
}
```

Recursive:

- Recurrence is $k + k + \ldots + k$ for $n$ times

```
int sum(int[] arr){
  return help(arr,0);
}
int help(int[]arr,int i) {
  if(i==arr.length)
    return 0;
  return arr[i] + help(arr,i+1);
}
```



## *What about a binary version?*

```
int sum(int[] arr){
   return help(arr,0,arr.length);
}
int help(int[] arr, int lo, int hi) {
   if(lo==hi)    return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

Recurrence is $T(n) = O(1) + 2T(n/2)$

- $1 + 2 + 4 + 8 + \ldots$ for $\log n$ times
- $2^{(\log n)} - 1$ which is proportional to $n$ (definition of logarithm)

Easier explanation: it adds each number once while doing little else

"Obvious": You can't do better than $O(n)$ – have to read whole array

## Parallelism teaser

- But suppose we could do two recursive calls *at the same time*
  - *Like having a friend do half the work for you!*

```
int sum(int[]arr){
   return help(arr,0,arr.length);
}
int help(int[]arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

- If you have as many "friends of friends" as needed the recurrence is now $T(n) = O(1) + 1T(n/2)$
  - $O(\log n)$ : same recurrence as for **find**



## Really common recurrences

Should know how to solve recurrences but also recognize some really common ones:

| | |
|---|---|
| $T(n) = O(1) + T(n-1)$ | linear |
| $T(n) = O(1) + 2T(n/2)$ | linear |
| $T(n) = O(1) + T(n/2)$ | logarithmic |
| $T(n) = O(1) + 2T(n-1)$ | exponential |
| $T(n) = O(n) + T(n-1)$ | quadratic (see previous lecture) |
| $T(n) = O(n) + T(n/2)$ | linear |
| $T(n) = O(n) + 2T(n/2)$ | $O(n \log n)$ |

Note big-Oh can also use more than one variable

- Example: can sum all elements of an $n$-by-$m$ matrix in $O(nm)$



## Asymptotic notation

About to show formal definition, which amounts to saying:

1. Eliminate low-order terms
2. Eliminate coefficients

Examples:

- $4n + 5$
- $0.5n \log n + 2n + 7$
- $n^3 + 2^n + 3n$
- $n \log (10n^2)$



## Big-Oh relates functions

We use $O$ on a function f($n$) (for example $n^2$) to mean *the set of functions with asymptotic behavior* ***less than or equal to*** f($n$)

So $(3n^2+17)$ **is in** $O(n^2)$

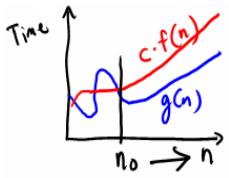
- $3n^2+17$ and $n^2$ have the same asymptotic behavior

Confusingly, we also say/write:

- $(3n^2+17)$ **is** $O(n^2)$
- $(3n^2+17) = O(n^2)$

But we would never say $O(n^2) = (3n^2+17)$

## Formally Big-Oh (Dr? Ms? Mr? ☺)




Definition:

g($n$) is in O( f($n$) ) if there exist constants $c$ and such that g($n$) ≤ $c$ f($n$) for all $n \geq n_0$

- To show g($n$) is in O( f($n$) ), pick a $c$ large enough to "cover the constant factors" and $n_0$ large enough to "cover the lower-order terms"
  - Example: Let g($n$) = $3n^2+17$ and f($n$) = $n^2$
    $c$=5 and $n_0$=10 is more than good enough
- This is "less than or equal to"
  - So $3n^2+17$ is also $O(n^5)$ and $O(2^n)$ etc.



## More Asymptotic Notation

- Upper bound: $O$( f($n$) ) is the set of all functions asymptotically less than or equal to f($n$)
  - g($n$) is in $O$( f($n$) ) if there exist constants $c$ and $n_0$ such that g($n$) ≤ $c$ f($n$) for all $n \geq n_0$
- Lower bound: Ω( f($n$) ) is the set of all functions asymptotically greater than or equal to f($n$)
  - g($n$) is in Ω( f($n$) ) if there exist constants $c$ and $n_0$ such that g($n$) ≥ $c$ f($n$) for all $n \geq n_0$
- Tight bound: θ( f($n$) ) is the set of all functions asymptotically equal to f($n$)
  - Intersection of $O$( f($n$) ) and Ω( f($n$) ) (use *different* $c$ values)



## Correct terms, in theory

A common error is to say $O$( f($n$) ) when you mean θ( f($n$) )

- Since a linear algorithm is also $O(n^5)$, it's tempting to say "this algorithm is exactly $O(n)$"
- That doesn't mean anything, say it is θ($n$)
- That means that it is not, for example $O$(**log** $n$)

Less common notation:

- "little-oh": like "big-Oh" but strictly less than
  - Example: sum is $o(n^2)$ but not $o(n)$
- "little-omega": like "big-Omega" but strictly greater than
  - Example: sum is ω(**log** $n$) but not ω($n$)



## What we are analyzing

- The most common thing to do is give an $O$ or θ bound to the worst-case running time of an algorithm
- Example: binary-search algorithm
  - Common: θ(**log** $n$) running-time in the worst-case
  - Less common: θ(1) in the best-case (item is in the middle)
  - Less common: Algorithm is Ω(**log log** $n$) in the worst-case (it is not really, really, really fast asymptotically)
  - Less common (but very good to know): the find-in-sorted-array ***problem*** is Ω(**log** $n$) in the worst-case
    - *No* algorithm can do better (without parallelism)
    - A ***problem*** cannot be $O$(f($n$)) since you can always find a slower algorithm, but can mean ***there exists*** an algorithm

## Other things to analyze

- Space instead of time
  - Remember we can often use space to gain time
- Average case
  - Sometimes only if you assume something about the distribution of inputs
    - See CSE312 and STAT391
  - Sometimes uses randomization in the algorithm
    - Will see an example with sorting; also see CSE312
  - Sometimes an *amortized guarantee*
    - Will discuss in a later lecture



## Summary

Analysis can be about:

- The problem or the algorithm (usually algorithm)
- Time or space (usually time)
  - Or power or dollars or …
- Best-, worst-, or average-case (usually worst)
- Upper-, lower-, or tight-bound (usually upper or tight)



## Usually asymptotic is valuable

- Asymptotic complexity focuses on behavior for large $n$ and is independent of any computer / coding trick
- But you can "abuse" it to be misled about trade-offs
- Example: $n^{1/10}$ vs. **`log`** $n$
  - Asymptotically $n^{1/10}$ grows more quickly
  - But the "cross-over" point is around $5 * 10^{17}$
  - So if you have input size less than $2^{58}$, prefer $n^{1/10}$