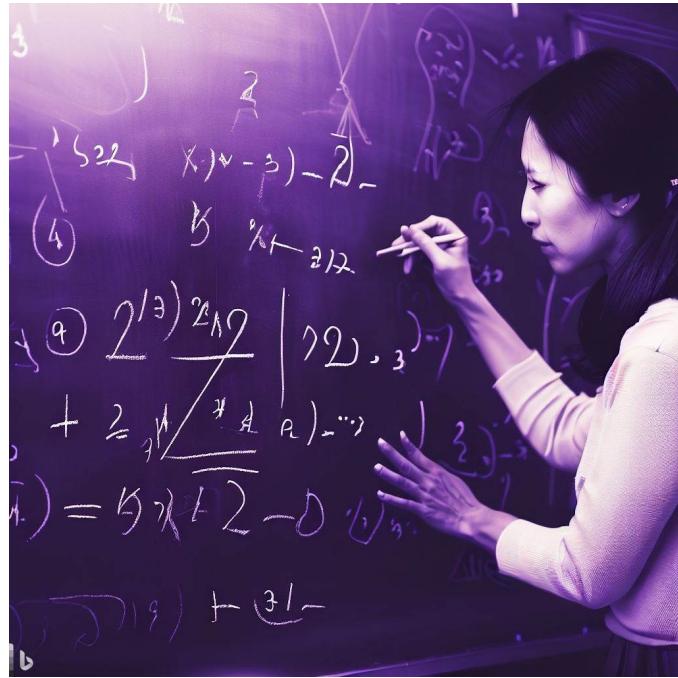


# CSE 331

## Reasoning About Functional Code

James Wilcox and Kevin Zatloukal



# Recall: Code Without Mutation

---

- Our math notation includes only...
  - expressions
  - conditionals (pattern matching & side conditions)
  - recursion
- This is all we need, mathematically
  - can write anything computable with just these
- Can do these in Java as well...

# Java Code Without Mutation

---

- **Code without mutation consists of...**
  - straight-line code variable declarations and `return`
  - conditionals `if` statements
  - recursion
- **Code that only uses these properties is "functional"**
  - we will limit ourselves to functional code *initially*
- **Can translate any of our math functions into this**
- **Data is not so easy**
  - compound types need to be faked with classes

# Inductive Data Types in Java

---

- Java does not natively support these data types  
other languages support these as well as tuples and records
- Can fake them with classes, e.g.:

```
public class List {                                type List := nil | cons_hd : Z, tl : List
    final int hd;
    final List tl;

    public static final List nil = null;

    public static List cons(int hd, List tl) {
        return new List(hd, tl);
    }

    private List(int hd, List tl) {                cons(1, cons(2, nil));
        this.hd = hd;
        this.tl = tl;
    }
}
```

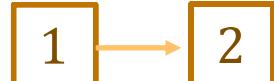
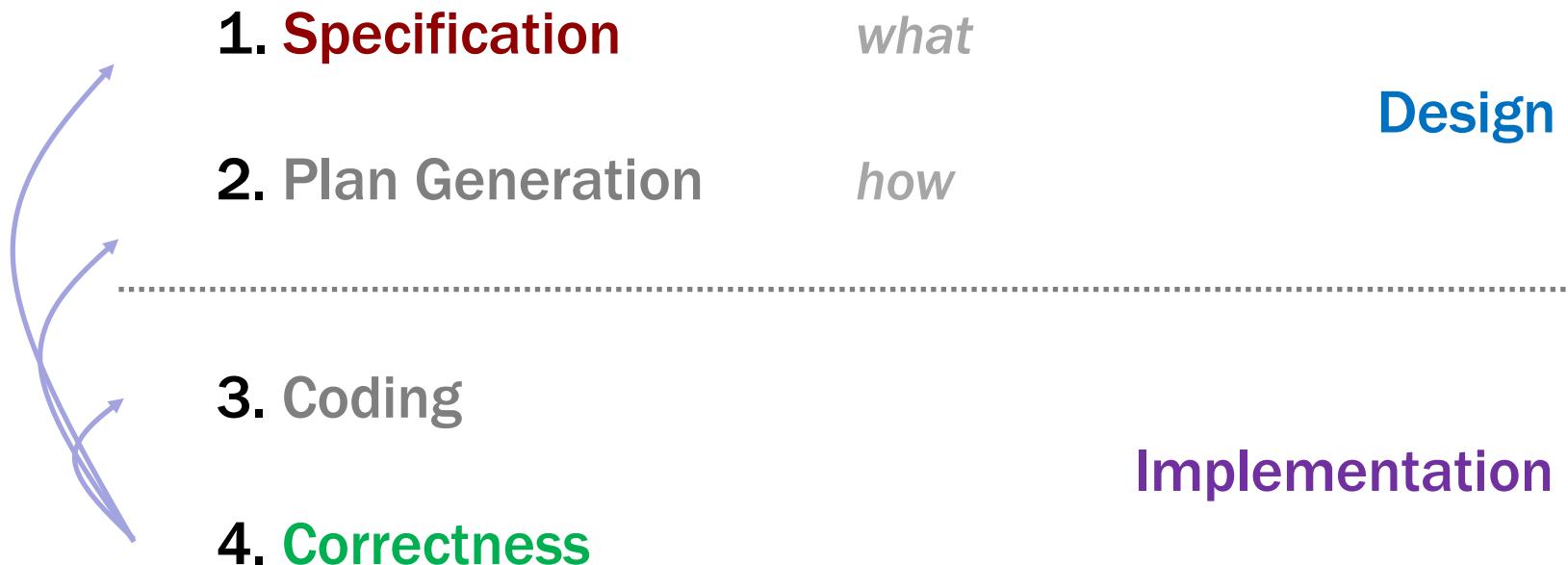


Diagram illustrating a cons cell structure. Two orange-bordered boxes are labeled '1' and '2'. An orange arrow points from box '1' to box '2', representing the cons cell structure.

# Recall: Software Development Process

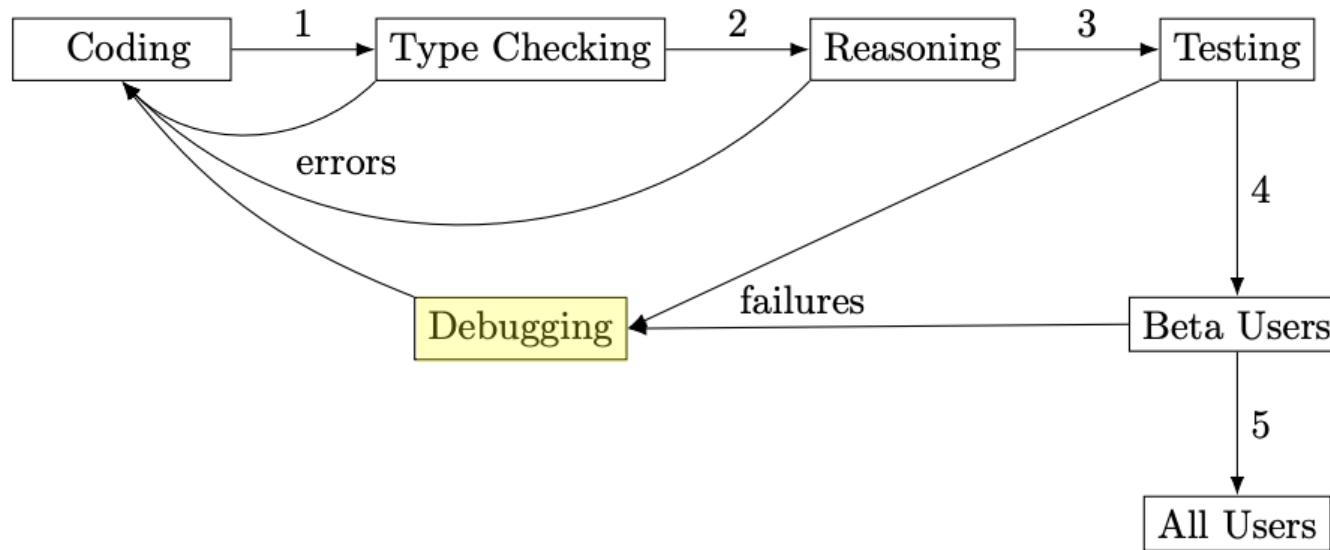
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- an **imperative spec comes with a plan**  
can translate this directly into Java code
- a **declarative spec does not**

# Software Implementation

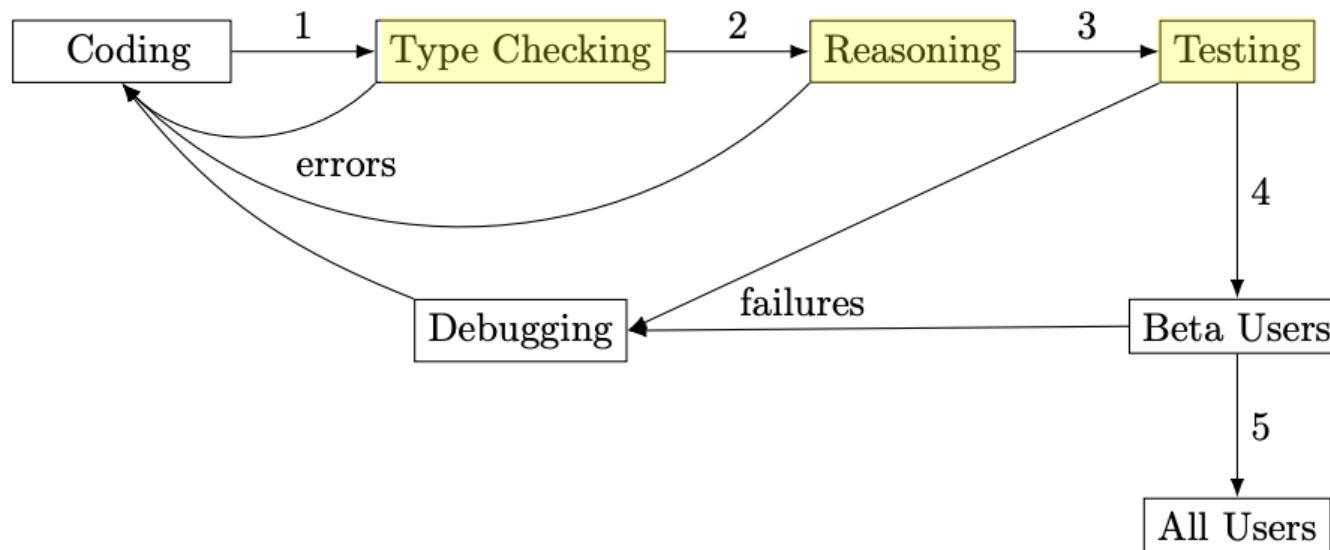
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- **Debugging is the search from failure to bug**
  - harder the more code that must be searched
  - easiest possible case is a unit test failure
  - time required is *random with a long tail*

# Software Implementation

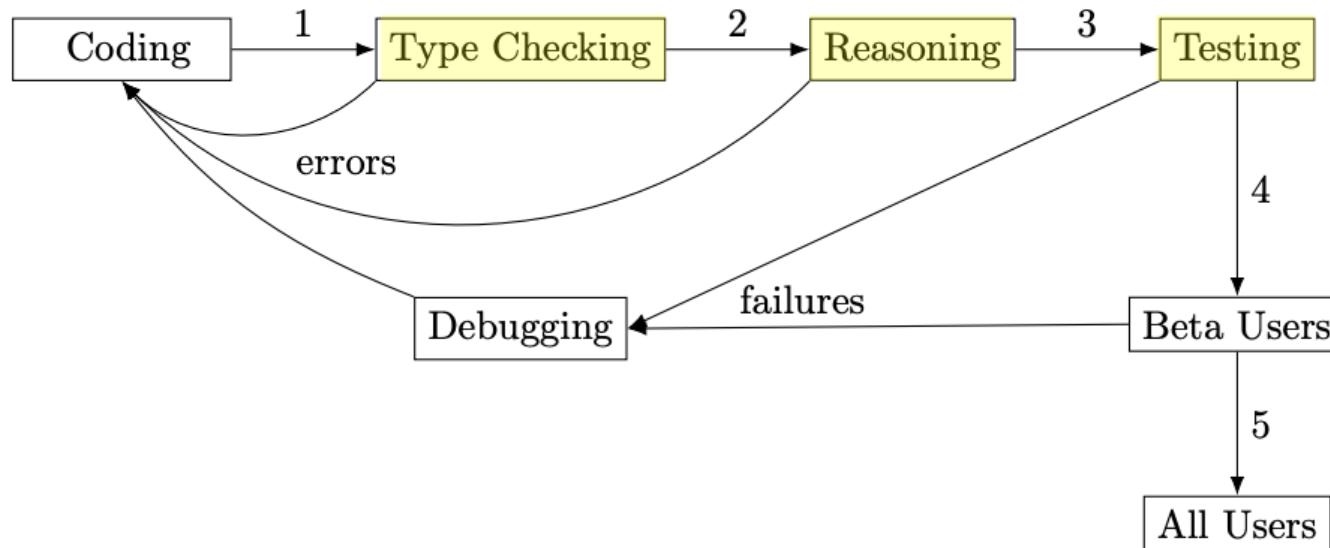
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- **Three techniques used to check correctness**
  - each removes  $\sim 2/3^{\text{rd}}$  of the bugs present
  - but each tends to find *different* bugs
  - need all three techniques to get 99+% assurance

# Software Implementation

---



- Three techniques used to check correctness
  - type checking is familiar (and more coming later)
  - already discussed testing
  - focus now on reasoning

# Reasoning

---

- “Thinking through” what the code does on all inputs
  - ensuring what it does is correct in *all* cases
- Type checking does **not** do this
  - only checks that return values have the right type
  - e.g., ensures that an `int` but not that it is `1`
- Testing does **not** do this
  - only verifies a correct output on *some* inputs

# Reasoning

---

- “Thinking through” what the code does on all inputs
  - neither testing nor type checking can do this
- Can be done formally or informally
  - most professionals reason *informally*
  - we will start with formal reasoning and move to informal
    - formal reasoning is a steppingstone to informal reasoning (same core ideas)
    - formal reasoning still needed for the **hardest** problems
- Definition of correctness comes from the specification...

# Correctness Requires a Specification

---

**Specification contains two sets of facts**

**Precondition:**

facts we are *promised* about the inputs

**Postcondition:**

facts we are required to *ensure* for the output

**Correctness (satisfying the spec):**

for every input satisfying the precondition,  
the output will satisfy the postcondition

# Facts

---

- Starting point for reasoning is “facts” about the code
  - things we know to be true about the variables
    - these hold for all inputs (no matter what value the variable has)
  - typically, “=” or “ $\leq$ ”

```
// @param n a natural number
int f(int n) {
    final int m = 2 * n;
    return (m + 1) * (m - 1);
}
```

find facts by reading along path  
from top to return statement

- At the return statement, we know these facts:
  - $n \geq 0$
  - $m = 2n$

note: these hold for all valid inputs

# Facts

---

- Starting point for reasoning is “facts” about the code
  - things we know to be true about the variables
    - these hold for all inputs (no matter what value the variable has)
  - typically, “=” or “ $\leq$ ”

```
// @param n a natural number
int f(int n) {
    final int m = 2 * n;
    return (m + 1) * (m - 1);
}
```

- No need to include the fact that  $n$  is an integer ( $n : \mathbb{Z}$ )
  - that is true, but the type checker takes care of that
  - no need to repeat reasoning done by the type checker

# Finding Facts at a Return Statement

---

- Consider this code:

```
// Inputs a and b can be any integers.  
// Returns a non-negative integer.  
int f(int a, int b) {  
    final List L = cons(a, cons(b, nil));  
    if (a >= 0 && b >= 0)  
        return sum(L);  
    ...
```

find facts by reading along path  
from top to return statement

facts are math statements about the code

- Known facts include “ $a \geq 0$ ”, “ $b \geq 0$ ”, and “ $L = \text{cons}(\dots)$ ”
- Remains to prove that “ $\text{sum}(L) \geq 0$ ”

# Implications

---

- **We can use the facts we know to prove more facts**
  - if we can prove R using facts P and Q,  
we say that R “follows from” or “is implied by” P and Q
  - proving this fact is proving an “**implication**”
- **We will see how to do this shortly...**  
will be familiar to those who have taken 311

# Checking Correctness of Functional Code

---

- Steps for checking correctness of functional code:
  1. Collect facts at each return statement
  2. Ensure those facts **imply** each fact of the **postcondition**
- Checking correctness requires proving **implications**
  - need to prove facts about the **return** values
  - return values must satisfy the facts of the **postcondition**
- If the known facts do not imply the postcondition, then the code is **wrong**
  - some valid input does not satisfy the postcondition
    - the code will be correct in 99% of our examples,
    - but this is the reason why we do reasoning: to find mistakes

# Collecting Facts

---

- Saw how to collect facts in code consisting of
  - "final" variable declarations
  - "if" statements
  - collect facts by reading along path from top to return
- Those elements cover all code without mutation
  - covers everything describable by our math notation
  - we can calculate interesting values with *recursion*
- Will need more tools to handle code with mutation...

# Mutation Makes Reasoning Harder

---

Description	Testing	Tools	Reasoning	
no mutation	full coverage	type checker	calculation induction	HW2
local variable mutation	"	"	Floyd logic	HW4
heap state mutation	"	"	rep invariants	HW6
array mutation		"	for-any facts	HW8

# Correctness with No Mutation

---

- **Proving implications is the core step of reasoning**
  - other techniques output implications for us to prove
- **Facts are written in our math notation**
  - we will use math tools to prove implications
- **Core technique is "proof by calculation"**
- **Other techniques we will need:**
  - proof by cases
  - structural induction

# Proof by Calculation

# Proof by Calculation

---

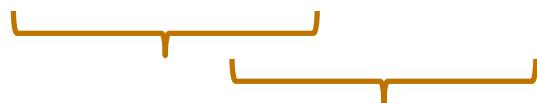
- Proves an **implication**
  - fact to be shown is an equation or **inequality**
- **Uses known facts and definitions**
  - latter includes, e.g., the fact that  $\text{len}(\text{nil}) = 0$

# Example Proof by Calculation

---

- Given  $x = y$  and  $z \leq 10$ , prove that  $x + z \leq y + 10$ 
  - show the third fact follows from the first two
- Start from the left side of the inequality to be proved

$$x + z = y + z \leq y + 10$$



A diagram showing a bracket under the term  $y + z$ . Two orange arrows point from this bracket to the text "since  $x = y$ " and "since  $z \leq 10$ " respectively.

since  $x = y$    since  $z \leq 10$

All together, this tells us that  $x + z \leq y + 10$

# Example Proof by Calculation

---

- Given  $x = y$  and  $z \leq 10$ , prove that  $x + z \leq y + 10$ 
  - show the third fact follows from the first two
- Start from the left side of the inequality to be proved

$$\begin{aligned} x + z \\ = y + z & \quad \text{since } x = y \\ \leq y + 10 & \quad \text{since } z \leq 10 \end{aligned}$$

- easier to read when split across lines
- “calculation block”, includes explanations in right column
  - proof by calculation means using a calculation block
- “=” or “ $\leq$ ” relates that line to the previous line

# Calculation Blocks

---

- Chain of “=” shows first = last

$$\begin{aligned} a \\ = b \\ = c \\ = d \end{aligned}$$

- proves that  $a = d$
- all 4 of these are the same number

# Calculation Blocks

---

- **Chain of “=” and “≤” shows first ≤ last**

$$\begin{aligned} x + z & \\ = y + z & \quad \text{since } x = y \\ \leq y + 10 & \quad \text{since } z \leq 10 \\ = y + 3 + 7 & \\ \leq w + 7 & \quad \text{since } y + 3 \leq w \end{aligned}$$

- **each number is equal or strictly larger than previous**  
last number is equal or strictly larger than the first number
- **analogous for “≥”**

# Using Calculation to Prove Correctness

---

```
// Inputs x and y are positive integers
// Returns a positive integer.
int f(int x, int y) => {
    return x + y;
};
```

- Known facts “ $x \geq 1$ ” and “ $y \geq 1$ ”
- Correct if the return value is a positive integer

$$x + y$$

# Using Calculation to Prove Correctness

---

```
// Inputs x and y are positive integers
// Returns a positive integer.
int f(int x, int y) => {
    return x + y;
};
```

- Correct if the return value is a positive integer

$$\begin{aligned}x + y \\ \geq x + 1 & \quad \text{since } y \geq 1 \\ \geq 1 + 1 & \quad \text{since } x \geq 1 \\ = 2 \\ \geq 1\end{aligned}$$

- calculation shows that  $x + y \geq 1$

# Using Calculation to Prove Correctness

---

```
// Inputs x and y with x >= 9 and y >= -8
// Returns a positive integer.
int f(int x, int y) => {
    return x + y;
};
```

- Known facts “ $x \geq 9$ ” and “ $y \geq -8$ ”
- Correct if the return value is a positive integer

$$x + y$$

# Using Calculation to Prove Correctness

---

```
// Inputs x and y with x >= 9 and y >= -8
// Returns a positive integer.
int f(int x, int y) => {
    return x + y;
};
```

- Known facts “ $x \geq 9$ ” and “ $y \geq -8$ ”
- Correct if the return value is a positive integer

$$\begin{aligned} x + y & \\ \geq x + -8 & \quad \text{since } y \geq -8 \\ \geq 9 - 8 & \quad \text{since } x \geq 9 \\ = 1 & \end{aligned}$$

# Using Calculation to Prove Correctness

---

```
// Inputs x and y with x > 8 and y > -9
// Returns a positive integer.
int f(int x, int y) => {
    return x + y;
};
```

- Known facts “ $x > 8$ ” and “ $y > -9$ ”
- Correct if the return value is a positive integer

$$\begin{aligned} x + y \\ > x + -9 & \quad \text{since } y > -9 \\ > 8 - 9 & \quad \text{since } x > 8 \\ = -1 \end{aligned}$$

warning: avoid using “ $>$ ” (or “ $<$ ”) *multiple* times in a calculation block

# Using Calculation to Prove Correctness

---

```
// Inputs x and y with x > 3 and y > 4
// Returns an integer that is 10 or larger.
int f(int x, int y) => {
    return x + y;
};
```

- Known facts “ $x \geq 4$ ” and “ $y \geq 5$ ”
- Correct if the return value is 10 or larger

$$x + y$$

# Using Calculation to Prove Correctness

---

```
// Inputs x and y with x > 3 and y > 4
// Returns an integer that is 10 or larger.
int f(int x, int y) => {
    return x + y;
};
```

- Known facts “ $x \geq 4$ ” and “ $y \geq 5$ ”
- Correct if the return value is 10 or larger

$$\begin{aligned} & x + y \\ & \geq x + 5 && \text{since } y \geq 5 \\ & \geq 4 + 5 && \text{since } x \geq 4 \\ & = 9 \end{aligned}$$

proof doesn't work because the code is wrong!

# Using Definitions in Calculations

---

- **Most useful with function calls**
  - cite the definition of the function to get the return value
- **For example:**

$$\begin{aligned} \text{sum}(\text{nil}) &:= 0 \\ \text{sum}(x :: L) &:= x + \text{sum}(L) \end{aligned}$$

- **Can cite facts such as**
  - $\text{sum}(\text{nil}) = 0$
  - $\text{sum}(a :: b :: \text{nil}) = a + \text{sum}(b :: \text{nil})$

second case of definition with  $x = a$  and  $L = b :: \text{nil}$

# Recall: Finding Facts at a Return Statement

---

- Consider this code

```
// Inputs a and b must be integers.  
// Returns a non-negative integer.  
int f(int a, int b) {  
    final List L = cons(a, cons(b, nil));  
    if (a >= 0 && b >= 0)  
        return sum(L);  
    ...
```

find facts by reading along path  
from top to return statement

- Known facts include “ $a \geq 0$ ”, “ $b \geq 0$ ”, and “ $L = \text{cons}(\dots)$ ”
- Must prove that  $\text{sum}(L) \geq 0$

# Using Definitions in Calculations

---

$\text{sum}(\text{nil}) := 0$

$\text{sum}(x :: L) := x + \text{sum}(L)$

- **Know** “ $a \geq 0$ ”, “ $b \geq 0$ ”, **and** “ $L = a :: b :: \text{nil}$ ”
- **Prove that**  $\text{sum}(L) \geq 0$

$\text{sum}(L)$

# Using Definitions in Calculations

---

$$\text{sum}(\text{nil}) := 0$$
$$\text{sum}(x :: L) := x + \text{sum}(L)$$

- Know “ $a \geq 0$ ”, “ $b \geq 0$ ”, and “ $L = a :: b :: \text{nil}$ ”
- Prove that  $\text{sum}(L) \geq 0$

$$\begin{aligned} \text{sum}(L) &= \text{sum}(a :: b :: \text{nil}) && \text{since } L = a :: b :: \text{nil} \\ &= a + \text{sum}(b :: \text{nil}) && \text{def of sum} \\ &= a + b + \text{sum}(\text{nil}) && \text{def of sum} \\ &= a + b && \text{def of sum} \\ &\geq 0 + b && \text{since } a \geq 0 \\ &\geq 0 && \text{since } b \geq 0 \end{aligned}$$

# Proving Correctness with Conditionals

---

```
// Inputs x and y are integers.  
// Returns a number less than x.  
int f(int x, int y) {  
    if (y < 0) {  
        return x + y;  
    } else {  
        return x - 1;  
    }  
};
```

- Known fact in then (top) branch: “ $y \leq -1$ ”

$$x + y$$

# Proving Correctness with Conditionals

---

```
// Inputs x and y are integers.  
// Returns a number less than x.  
int f(int x, int y) {  
    if (y < 0) {  
        return x + y;  
    } else {  
        return x - 1;  
    }  
}
```

- Known fact in then (top) branch: “ $y \leq -1$ ”

$$\begin{aligned} & x + y \\ & \leq x + -1 && \text{since } y \leq -1 \\ & < x + 0 && \text{since } -1 < 0 \\ & = x \end{aligned}$$

# Proving Correctness with Conditionals

---

```
// Inputs x and y are integers.  
// Returns a number less than x.  
int f(int x, int y) {  
    if (y < 0) {  
        return x + y;  
    } else {  
        return x - 1;  
    }  
};
```

- Known fact in else (bottom) branch: “ $y \geq 0$ ”

$x - 1$

# Proving Correctness with Conditionals

---

```
// Inputs x and y are integers.  
// Returns a number less than x.  
int f(int x, int y) {  
    if (y < 0) {  
        return x + y;  
    } else {  
        return x - 1;  
    }  
};
```

- Known fact in else (bottom) branch: “ $y \geq 0$ ”

$$\begin{aligned} & x - 1 \\ & < x + 0 && \text{since } -1 < 0 \\ & = x \end{aligned}$$

# Proving Correctness with Multiple Claims

---

- Need to check the claim from the spec at each `return`
- If spec claims multiple facts, then we must prove that each of them holds

```
// Inputs x and y are integers with x < y - 1
// Returns a number less than y and greater than x.
int f(int x, int y) { ... };
```

- multiple claims to prove:  $x < r$  and  $r < y$   
where “r” is the return value
- requires two calculation blocks

# Example Correctness with Conditionals

---

```
// Returns r with (r=a or r=b) and r >= a and r >= b
int max(int a, int b) {
    if (a >= b) {
        return a;
    } else {declarative spec of max
        return b;
    }
};
```

- Three different facts to prove at each `return`
- Two known facts in each branch (return value is “r”):
  - then branch:  $a \geq b$  and  $r = a$
  - else branch:  $a < b$  and  $r = b$
  - prove an “or” holds by proving one of the two options holds

# Proof By Cases

# Proof By Cases

---

- **Sometimes necessary split a proof into cases**
  - fact may be hard to prove for all values at once
- **Example: can't prove it for all  $x$  at once, but can prove it for  $x \geq 0$  and  $x < 0$** 
  - will see an example next
- **If we can prove it in those two cases, it holds for all  $x$** 
  - follows since the cases are exhaustive  
(don't need to be exclusive in this case)
  - can pick any cases we want, not just cases in the code

# Example Proof By Cases

---

$f : \mathbb{Z} \rightarrow \mathbb{Z}$

$$\begin{aligned} f(m) &:= 2m + 1 && \text{if } m \geq 0 \\ f(m) &:= 0 && \text{if } m < 0 \end{aligned}$$

- Want to prove that  $f(m) > m$
- Doesn't seem possible as is
  - can't even apply the definition of  $f$
  - need to know if  $m < 0$  or  $m \geq 0$
- Split our analysis into these two separate cases...

# Proof By Cases

---

$$\begin{aligned} f(m) &:= 2m + 1 & \text{if } m \geq 0 \\ f(m) &:= 0 & \text{if } m < 0 \end{aligned}$$

- Prove that  $f(m) > m$

**Case**  $m \geq 0$ :

$$f(m) =$$

$$> m$$

# Proof By Cases

---

$$f(m) := 2m + 1$$

$$f(m) := 0$$

if  $m \geq 0$

if  $m < 0$

- Prove that  $f(m) > m$

Case  $m \geq 0$ :

$$\begin{aligned} f(m) &= 2m + 1 \\ &= m + m + 1 \\ &\geq m + 1 \\ &> m \end{aligned}$$

side condition requires a "since"

def of  $f$  (since  $m \geq 0$ )

since  $m \geq 0$

since  $1 > 0$

# Proof By Cases

---

$$\begin{aligned} f(m) &:= 2m + 1 && \text{if } m \geq 0 \\ f(m) &:= 0 && \text{if } m < 0 \end{aligned}$$

- Prove that  $f(m) > m$

**Case**  $m \geq 0$ :

$$f(m) = \dots > m$$

**Case**  $m < 0$ :

$$\begin{aligned} f(m) &= 0 && \text{def of } f \text{ (since } m < 0\text{)} \\ &> m && \text{since } m < 0 \end{aligned}$$

Since these two cases are exhaustive,  $f(m) > m$  holds in general.

# Recall: Correctness with No Mutation

---

- **Proving implications is the core step of reasoning**
  - other techniques output implications for us to prove
- **Core technique is "proof by calculation"**
  - other techniques break down into calculations also
- **Other techniques we will need:**
  - proof by cases
  - **structural induction**

# **Structural Induction**

# Proof by Calculation

---

- Our proofs so far have used **fixed-length lists**
  - e.g.,  $\text{sum}(a :: b :: \text{nil}) \geq 0$
- Would like to prove facts about any length list L
- For example...

# Example: Repeating List Elements

---

- Consider the following function:

```
echo(nil)      := nil
echo(x :: L)   := x :: x :: echo(L)
```

- Produces a list where every element is repeated twice

```
echo(1 :: 2 :: nil)
  = 1 :: 1 :: echo(2 :: nil)           def of echo
  = 1 :: 1 :: 2 :: 2 :: echo(nil)     def of echo
  = 1 :: 1 :: 2 :: 2 :: nil          def of echo
```

# Example: Repeating List Elements

---

```
echo(nil)      := nil
echo(x :: L)   := x :: x :: echo(L)
```

- Suppose we have the following code:

```
final int m = len(S);           // S is some List
final List R = echo(S);
...
return 2*m; // = len(echo(S))
```

- spec says to return `len(echo(S))` but code returns  $2 \cdot \text{len}(S)$
- Need to prove that  $\text{len}(\text{echo}(S)) = 2 \cdot \text{len}(S)$

# Proof by Calculation

---

- Our proofs so far have used **fixed-length lists**
  - e.g.,  $\text{sum}(a :: b :: \text{nil}) \geq 0$
- Would like to prove facts about any length list  $L$
- Need more tools for this...
  - **structural recursion** *calculates* on inductive types
  - **structural induction** *reasons* about inductive types
    - both tools are specific to **inductive types**

# Structural Induction

---

Let  $P(S)$  be the claim “ $\text{len}(\text{echo}(S)) = 2 \text{ len}(S)$ ”

To prove  $P(S)$  holds for any list  $S$ , prove two implications

**Base Case:** prove  $P(\text{nil})$

- use any known facts and definitions

**Inductive Step:** prove  $P(x :: L)$

- $x$  and  $L$  are variables
- use any known facts and definitions plus one more fact...
- make use of the fact that  $L$  is also a List

# Structural Induction

---

To prove  $P(S)$  holds for any list  $S$ , prove two implications

**Base Case:** prove  $P(\text{nil})$

- use any known facts and definitions

**Inductive Hypothesis:** assume  $P(L)$  is true

- use this in the inductive step, but not anywhere else

**Inductive Step:** prove  $P(x :: L)$

- use known facts and definitions and Inductive Hypothesis

# Why This Works

---

**With Structural Induction, we prove two facts**

$P(\text{nil})$

$\text{len}(\text{echo}(\text{nil})) = 2 \text{ len}(\text{nil})$

$P(x :: L)$

$\text{len}(\text{echo}(x :: L)) = 2 \text{ len}(x :: L)$

(second assuming  $\text{len}(\text{echo}(L)) = 2 \text{ len}(L)$ )

**Why is this enough to prove  $P(S)$  for any  $S : \text{List}$ ?**

# Why This Works

---

Build up an object using constructors:

nil

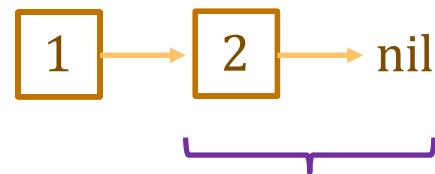
first constructor

`2 :: nil`

second constructor

`1 :: 2 :: nil`

second constructor



nil already exists when building `2 :: nil`



2 :: nil already exists when building `1 :: 2 :: nil`

# Why This Works

---

**Build up a proof the same way we built up the object**

$P(\text{nil})$

$\text{len}(\text{echo}(\text{nil})) = 2 \text{ len}(\text{nil})$

$P(x :: L)$

$\text{len}(\text{echo}(x :: L)) = 2 \text{ len}(x :: L)$

(second assuming  $\text{len}(\text{echo}(L)) = 2 \text{ len}(L)$ )



$P(\text{nil})$

$P(\text{nil})$  already proven when proving  $P(2 :: \text{nil})$



$P(2 :: \text{nil})$  already proven when proving  $P(1 :: 2 :: \text{nil})$

# Example: Repeating List Elements

---

```
echo(nil)      := nil
echo(x :: L)   := x :: x :: echo(L)
```

- **Prove that  $\text{len}(\text{echo}(S)) = 2 \text{len}(S)$  for any  $S : \text{List}$**

**Base Case** (nil):

**Need to prove that  $\text{len}(\text{echo}(\text{nil})) = 2 \text{len}(\text{nil})$**

$$\text{len}(\text{echo}(\text{nil}))$$
$$=$$
$$= 2 \text{len}(\text{nil})$$

```
len(nil)      := 0
```

```
len(x :: L)   := 1 + len(L)
```

# Example: Repeating List Elements

---

```
echo(nil)      := nil
echo(x :: L)   := x :: x :: echo(L)
```

- **Prove that  $\text{len}(\text{echo}(S)) = 2 \text{len}(S)$  for any  $S : \text{List}$**

**Base Case** (nil):

$$\begin{aligned} \text{len}(\text{echo}(\text{nil})) &= \text{len}(\text{nil}) && \text{def of echo} \\ &= 0 && \text{def of len} \\ &= 2 \cdot 0 \\ &= 2 \text{len}(\text{nil}) && \text{def of len} \end{aligned}$$

# Example: Repeating List Elements

---

```
echo(nil)      := nil
echo(x :: L)   := x :: x :: echo(L)
```

- **Prove that  $\text{len}(\text{echo}(S)) = 2 \text{len}(S)$  for any  $S : \text{List}$**

**Inductive Step** ( $x :: L$ ):

**Need to prove that  $\text{len}(\text{echo}(x :: L)) = 2 \text{len}(x :: L)$**

**Get to assume claim holds for  $L$ , i.e., that  $\text{len}(\text{echo}(L)) = 2 \text{len}(L)$**

# Example: Repeating List Elements

---

```
echo(nil)      := nil
echo(x :: L)   := x :: x :: echo(L)
```

- **Prove that  $\text{len}(\text{echo}(S)) = 2 \text{len}(S)$  for any  $S : \text{List}$**

**Inductive Hypothesis:** assume that  $\text{len}(\text{echo}(L)) = 2 \text{len}(L)$

**Inductive Step** ( $x :: L$ ):

$\text{len}(\text{echo}(x :: L))$

$\text{len}(\text{nil})$	$:= 0$	
$\text{len}(x :: L)$	$:= 1 + \text{len}(L)$	$= 2 \text{len}(x :: L)$

# Example: Repeating List Elements

---

```
echo(nil)      := nil
echo(x :: L)   := x :: x :: echo(L)
```

- **Prove that  $\text{len}(\text{echo}(S)) = 2 \text{len}(S)$  for any  $S : \text{List}$**

**Inductive Hypothesis:** assume that  $\text{len}(\text{echo}(L)) = 2 \text{len}(L)$

**Inductive Step** ( $x :: L$ ):

$$\begin{aligned} \text{len}(\text{echo}(x :: L)) &= \text{len}(x :: x :: \text{echo}(L)) && \text{def of echo} \\ &= 1 + \text{len}(x :: \text{echo}(L)) && \text{def of len} \\ &= 2 + \text{len}(\text{echo}(L)) && \text{def of len} \\ &= 2 + 2 \text{len}(L) && \text{Ind. Hyp.} \\ &= 2(1 + \text{len}(L)) \\ &= 2 \text{len}(x :: L) && \text{def of len} \end{aligned}$$

$\text{len}(\text{nil}) := 0$

$\text{len}(x :: L) := 1 + \text{len}(L)$

# Structural Induction in General

---

- General case: assume  $P$  holds for constructor *arguments*

`type T := A | B(x : Z) | C(y : Z, t : T) | D(z : Z, u : T, v : T)`

- To prove  $P(t)$  for any  $t$ , we need to prove:
  - $P(A)$
  - $P(B(x))$  for any  $x : Z$
  - $P(C(y, t))$  for any  $y : Z$  and  $t : T$  assuming  $P(t)$  is true
  - $P(D(z, u, v))$  for any  $z : Z$  and  $u, v : T$  assuming  $P(u)$  and  $P(v)$
- These four facts are enough to prove  $P(t)$  for any  $t$ 
  - for each constructor, have proof that it produces an object satisfying  $P$

# Structural Induction in General

---

- Each inductive type has its own form of induction
  - special way to reason about that type
- Lists are defined like this:

```
type List := nil | cons(x : Z, L : List)
```

- To prove  $P(S)$  for any list  $S$ , we need to prove:
  - $P(\text{nil})$
  - $P(x :: L)$  for any  $x : Z$  and  $L : \text{List}$       **assuming**  $P(L)$  is true

## Example 2: Repeating List Elements

---

```
echo(nil)      := nil
echo(x :: L)   := x :: x :: echo(L)
```

- Suppose we have the following code:

```
final int y = sum(S);           // S is some List
final List R = echo(S);
...
return 2*y; // = sum(echo(S))
```

- spec says to return  $\text{sum}(\text{echo}(S))$  but code returns  $2 \text{ sum}(S)$
- Need to prove that  $\text{sum}(\text{echo}(S)) = 2 \text{ sum}(S)$

## Example 2: Repeating List Elements

---

```
echo(nil)      := nil
echo(x :: L)   := x :: x :: echo(L)
```

- **Prove that  $\text{sum}(\text{echo}(S)) = 2 \text{ sum}(S)$  for any  $S : \text{List}$**

**Base Case** (nil):

$$\begin{aligned} \text{sum}(\text{echo}(\text{nil})) &= \\ &= 2 \text{ sum}(\text{nil}) \end{aligned}$$

---

```
sum(nil)      := 0
sum(x :: L)   := x + sum(L)
```

## Example 2: Repeating List Elements

---

```
echo(nil)      := nil
echo(x :: L)   := x :: x :: echo(L)
```

- **Prove that  $\text{sum}(\text{echo}(S)) = 2 \text{ sum}(S)$  for any  $S : \text{List}$**

**Base Case** (nil):

$$\begin{aligned} \text{sum}(\text{echo}(\text{nil})) &= \text{sum}(\text{nil}) && \text{def of echo} \\ &= 0 && \text{def of sum} \\ &= 2 \cdot 0 \\ &= 2 \text{ sum}(\text{nil}) && \text{def of sum} \end{aligned}$$

**Inductive Step** ( $x :: L$ ):

**Need to prove that  $\text{sum}(\text{echo}(x :: L)) = 2 \text{ sum}(x :: L)$**

**Get to assume claim holds for  $L$ , i.e., that  $\text{sum}(\text{echo}(L)) = 2 \text{ sum}(L)$**

## Example 2: Repeating List Elements

---

```
echo(nil)      := nil
echo(x :: L)   := x :: x :: echo(L)
```

- **Prove that  $\text{sum}(\text{echo}(S)) = 2 \text{sum}(S)$  for any  $S : \text{List}$**

**Inductive Hypothesis:** assume that  $\text{sum}(\text{echo}(L)) = 2 \text{sum}(L)$

**Inductive Step (** $x :: L$ **):**

$$\begin{aligned} \text{sum}(\text{echo}(x :: L)) &= \\ &= 2 \text{sum}(x :: L) \end{aligned}$$

---

```
sum(nil)      := 0
sum(x :: L)   := x + sum(L)
```

## Example 2: Repeating List Elements

---

```
echo(nil)      := nil
echo(x :: L)   := x :: x :: echo(L)
```

- **Prove that  $\text{sum}(\text{echo}(S)) = 2 \text{ sum}(S)$  for any  $S : \text{List}$**

**Inductive Hypothesis:** assume that  $\text{sum}(\text{echo}(L)) = 2 \text{ sum}(L)$

**Inductive Step** ( $x :: L$ ):

$$\begin{aligned} \text{sum}(\text{echo}(x :: L)) &= \text{sum}(x :: x :: \text{echo}(L)) && \text{def of echo} \\ &= x + \text{sum}(x :: \text{echo}(L)) && \text{def of sum} \\ &= 2x + \text{sum}(\text{echo}(L)) && \text{def of sum} \\ &= 2x + 2 \text{ sum}(L) && \text{Ind. Hyp.} \\ &= 2(x + \text{sum}(L)) \\ &= 2 \text{ sum}(x :: L) && \text{def of sum} \end{aligned}$$

# Recall: Concatenating Two Lists

---

- Mathematical definition of  $\text{concat}(S, R)$

$\text{concat}(\text{nil}, R) := R$

important operation  
abbreviated as " $\#$ "

$\text{concat}(x :: L, R) := x :: \text{concat}(L, R)$

- Puts all the elements of L before those of R

$\text{concat}(1 :: 2 :: \text{nil}, 3 :: 4 :: \text{nil})$

def of concat

$= 1 :: \text{concat}(2 :: \text{nil}, 3 :: 4 :: \text{nil})$

def of concat

$= 1 :: 2 :: \text{concat}(\text{nil}, 3 :: 4 :: \text{nil})$

def of concat

$= 1 :: 2 :: 3 :: 4 :: \text{nil}$

## Example 3: Length of Concatenated Lists

---

```
concat(nil, R)      := R
concat(x :: L, R)   := x :: concat(L, R)
```

- Suppose we have the following code:

```
final int m = len(S);           // S is some List
final int n = len(R);           // R is some List
...
return m + n; // = len(concat(S, R))
```

- **spec returns**  $\text{len}(\text{concat}(S, R))$  **but code returns**  $\text{len}(S) + \text{len}(R)$
- Need to prove that  $\text{len}(\text{concat}(S, R)) = \text{len}(S) + \text{len}(R)$

# Example 3: Length of Concatenated Lists

---

$\text{concat}(\text{nil}, R) := R$

$\text{concat}(x :: L, R) := x :: \text{concat}(L, R)$

- **Prove that  $\text{len}(\text{concat}(S, R)) = \text{len}(S) + \text{len}(R)$** 
  - prove by induction on  $S$
  - prove the claim for any choice of  $R$  (i.e.,  $R$  is a variable)

**Base Case** ( $\text{nil}$ ):

$\text{len}(\text{concat}(\text{nil}, R)) =$

$= \text{len}(\text{nil}) + \text{len}(R)$

# Example 3: Length of Concatenated Lists

---

$$\begin{aligned}\text{concat}(\text{nil}, R) &:= R \\ \text{concat}(x :: L, R) &:= x :: \text{concat}(L, R)\end{aligned}$$

- **Prove that  $\text{len}(\text{concat}(S, R)) = \text{len}(S) + \text{len}(R)$** 
  - **prove by induction on S**
  - **prove the claim for any choice of R (i.e., R is a variable)**

**Base Case** (nil):

$$\begin{aligned}\text{len}(\text{concat}(\text{nil}, R)) &= \text{len}(R) && \text{def of concat} \\ &= 0 + \text{len}(R) \\ &= \text{len}(\text{nil}) + \text{len}(R) && \text{def of len}\end{aligned}$$

# Example 3: Length of Concatenated Lists

---

$\text{concat}(\text{nil}, R) := R$

$\text{concat}(x :: L, R) := x :: \text{concat}(L, R)$

- **Prove that**  $\text{len}(\text{concat}(S, R)) = \text{len}(S) + \text{len}(R)$

**Inductive Step** ( $x :: L$ ):

**Need to prove that**

$$\text{len}(\text{concat}(x :: L, R)) = \text{len}(x :: L) + \text{len}(R)$$

**Get to assume claim holds for  $L$ , i.e., that**

$$\text{len}(\text{concat}(L, R)) = \text{len}(L) + \text{len}(R)$$

## Example 3: Length of Concatenated Lists

---

$\text{concat}(\text{nil}, R) := R$

$\text{concat}(x :: L, R) := x :: \text{concat}(L, R)$

- **Prove that**  $\text{len}(\text{concat}(S, R)) = \text{len}(S) + \text{len}(R)$

**Inductive Hypothesis:** assume that  $\text{len}(\text{concat}(L, R)) = \text{len}(L) + \text{len}(R)$

**Inductive Step** ( $x :: L$ ):

$$\text{len}(\text{concat}(x :: L, R)) =$$

$$= \text{len}(x :: L) + \text{len}(R)$$

# Example 3: Length of Concatenated Lists

---

$$\begin{aligned}\text{concat}(\text{nil}, R) &:= R \\ \text{concat}(x :: L, R) &:= x :: \text{concat}(L, R)\end{aligned}$$

- **Prove that**  $\text{len}(\text{concat}(S, R)) = \text{len}(S) + \text{len}(R)$

**Inductive Hypothesis:** assume that  $\text{len}(\text{concat}(L, R)) = \text{len}(L) + \text{len}(R)$

**Inductive Step** ( $x :: L$ ):

$$\begin{aligned}\text{len}(\text{concat}(x :: L, R)) &= \text{len}(x :: \text{concat}(L, R)) && \text{def of concat} \\ &= 1 + \text{len}(\text{concat}(L, R)) && \text{def of len} \\ &= 1 + \text{len}(L) + \text{len}(R) && \text{Ind. Hyp.} \\ &= \text{len}(x :: L) + \text{len}(R) && \text{def of len}\end{aligned}$$

# Comparing Reasoning vs Testing

---

```
static List concat(List S, List R) {  
    if (S == nil) {  
        return R;  
    } else {  
        return cons(S.hd, concat(S.tl, R));  
    }  
};
```

- **Testing:** 3 cases
  - loop coverage requires 0, 1, and many recursive calls
  - *does not guarantee the code works on all inputs*
- **Reasoning:** 2 calculations