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This exam contains 9 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

Instructions:

- Closed book, closed notes, no cell phones, no calculators.
- You have **50 minutes** to complete the exam.
- Answer all problems on the exam paper.
- If you need extra space use the back of a page.
- Problems are not of equal difficulty; if you get stuck on a problem, move on.
- It may be to your advantage to read all the problems before beginning the exam.

| Problem | Points | Score |
|---------|--------|-------|
| 1 | 18 | |
| 2 | 18 | |
| 3 | 28 | |
| 4 | 16 | |
| Total: | 80 | |

The following function `findIndex` searches for a string in an array of strings that is promised to be sorted in **decreasing** order. In other words, we are promised that $A[0] \geq A[1] \geq \dots \geq A[n-1]$, where the ordering of strings is according to \geq in TypeScript, (reverse) alphabetical ordering.

```
/**
 * Finds the index where x appears in the given sorted array or where, if
 * it is not in the array, it could be inserted to maintain sorted order.
 * @param A Array of strings in *decreasing* order
 * @param x String to look for in a.
 * @returns an integer k such that  $A[j] > x$  for any  $0 \leq j < k$  and
 *  $x \geq A[j]$  for any  $k \leq j < A.length$ 
 */
function findIndex(A: string[], x: string): number
```

Suppose that the function returns k . If x is in the array, then we must have $A[k] = x$. If x is not in the array, then we must have ($k = n$ or $k \geq 0$) and $A[k] \neq x$.

For example, suppose that A is the array ["mouse", "dog", "dog", "cat"]. Then, the specification above tells us that

- A call to `findIndex(A, "zebra")` would return 0.
- A call to `findIndex(A, "dog")` would return 1 (not 2).
- A call to `findIndex(A, "cat")` would return 3.
- A call to `findIndex(A, "bat")` would return 4.
- A call to `findIndex(A, "kangaroo")` would return 1.

1. (18 points) **Loop, There It Is**

Consider the following code, which claims to implement `findIndex` from the prior page.

The precondition is that $A[j] \geq A[j + 1]$ for any $0 \leq j < n - 1$, where n is `A.length`.

```

let k: number = A.length;
{{ P1 : k = n }}
{{ Inv: x ≥ A[j] for any k ≤ j < n and k ≥ 0 }}
while (k !== 0 && x >= A[k - 1]) {
  {{ P2 : x ≥ A[j] for any k ≤ j < n and k ≥ 0 and k ≠ 0 and x ≥ A[k - 1] }}
  {{ Q2 : x ≥ A[j] for any k - 1 ≤ j < n and k - 1 ≥ 0 }}
  k = k - 1;
  {{ x ≥ A[j] for any k ≤ j < n and k ≥ 0 }}
}
{{ P3 : x ≥ A[j] for any k ≤ j < n and k ≥ 0 and (k = 0 or A[k - 1] > x) }}
{{ Q3 : A[j] > x for any 0 ≤ j < k and x ≥ A[j] for any k ≤ j < n }}
return k;

```

- (a) Use reasoning to fill in all blank assertions above. The ' P_i 's should be filled in with forward reasoning and the ' Q_i 's should be filled in with backward reasoning.
- (b) Prove that P_1 implies Inv.

Solution: Since $k = n$, Inv says that " $x \geq A[j]$ for any $n \leq j < n$ ". This is vacuously true since there are no such numbers j . We can also see that $k = n \geq 0$.

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(c) Prove that P_2 implies Q_2 .

Solution: $k \geq 0$ and $k \neq 0$ imply that $k \geq 1$, which is the second part.

All the facts of the first part are included in P_2 's first part except $x \geq A[k - 1]$, which is the last part, so all the facts of Q_2 are actually included.

(d) Prove that P_3 implies Q_3 .

Solution: The second part of Q_3 is included in P_3 .

For the first part, we argue by cases.

If $k = 0$, then the first part says " $A[j] > x$ for any $0 \leq j < 0$ ", which is vacuously true because there are no such j 's.

If $A[k - 1] > x$, then for any $0 \leq j < k$, we have $A[j] \geq A[k - 1] > x$ since A is sorted.

One of these cases must occur because of the "or" in P_3 , so Q_3 holds.

2. (18 points) **Give It Your Test Shot**

Fill in the body of the following unit test for `findIndex`. Include comments explaining the test cases, as we did in the coding homework problems.

```
it('findIndex', function() {
  // 0 times through the loop
  assert.deepStrictEqual(
    findIndex([], "zebra"),
    0);

  // 0 times through the loop
  assert.deepStrictEqual(
    findIndex(["mouse"], "zebra"),
    0);

  // 1 time through the loop
  assert.deepStrictEqual(
    findIndex(["mouse"], "cat"),
    1);

  // 1 time through the loop
  assert.deepStrictEqual(
    findIndex(["mouse", "cat"], "dog"),
    1);

  // many times through the loop
  assert.deepStrictEqual(
    findIndex(["mouse", "dog", "cat", "bat"], "aardvark"),
    4);

  // many times through the loop
  assert.deepStrictEqual(
    findIndex(["mouse", "mouse", "dog", "cat"], "cat"),
    3);
})
```

The remaining problems involve the implementation of the following ADT:

```
/** An array of strings with no duplicates. */
interface StringSet {

    /**
     * Returns a set that includes all the current elements and x also
     * @param x a string to insert into the set (if not already present)
     * @returns obj if contains(obj, x) = T
     *         L   if contains(obj, x) = F
     * where L = A ++ [x] ++ B with obj = A ++ B (i.e., L is an array
     * containing the strings from obj with x inserted somewhere)
     */
    insert(x: string): StringSet;

    /**
     * Returns the largest string in the set
     * @requires obj.length > 0
     * @returns max(obj), where max is defined on non-empty lists by
     *         max([y]) := y
     *         max(A ++ [y]) := max(A)   if y < max(A)
     *         max(A ++ [y]) := y       if y >= max(A)
     */
    max(): string;
}
}
```

We will implement it with the following class, whose concrete representation is an array sorted in decreasing order.

```
class ArrayStringSet implements StringSet {

    // RI: elems[j] > elems[j+1] for any 0 <= j < elems.length - 1
    // AF: obj = this.elems
    readonly elems: readonly string[];

    // @requires elems is sorted in decreasing order, with no duplicates
    constructor(elems: readonly string[]) {
        this.elems = elems;
    }

    ...
}
```

3. (28 points) **Run Array! Run Array!**

Fill in the missing parts of the implementation of `insert`. Your code must be correct with the **provided invariants**. (You do not need to turn in a proof, but it must be correct.)

```
insert = (x: string): StringSet => {
  const k = findIndex(this.elems, x);

  if (k < this.elems.length && this.elems[k] === x) {
    return this;
  }

  // Create an array one longer than this.elems.
  const E: string[] = new Array(this.elems.length + 1);

  // Define A := this.elems[0 .. k-1]

  let i: number = 0;

  // Inv: E[0 .. i - 1] = A[0 .. i - 1]
  while (i !== k) {
    E[i] = this.elems[i];
    i = i + 1;
  }

  // Now have E[0 .. i - 1] = A and i = k

  E[i] = x;
  i = i + 1;

  // Now have E[0 .. i - 1] = A ++ [x] and i = k + 1
```

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```

// Now have  $E[0 \dots i - 1] = A \text{ ++ } [x]$  and  $i = k + 1$  (from previous page)

// Define  $B := \text{this.elems}[k \dots \text{this.elems.length}-1]$  as shorthand
// With these definitions, we have  $\text{this.elems} = A \text{ ++ } B$ .

let j: number = 0;

// Inv:  $E[0 \dots i - 1] = A \text{ ++ } [x] \text{ ++ } B[0 \dots j - 1]$  and  $i = k + 1 + j$ 
while (k + j !== this.elems.length) {
  E[i] = this.elems[k + j];
  i = i + 1;
  j = j + 1;
}

// Now have  $E[0 \dots i - 1] = A \text{ ++ } [x] \text{ ++ } B$  and  $i = A.length + 1 + B.length$ ,
// so  $E = A \text{ ++ } [x] \text{ ++ } B$ 
return new ArrayStringSet(E);
};

```

4. (16 points) **Here Array, Gone Tomorrow**

(a) Fill in the implementation of `max` in `ArrayStringSet`.

```
max = (): string => {  
    return this.elems[0];
```

```
};
```

(b) Explain in clear English (or prove formally, if you prefer) why your code above is correct.

Solution: The precondition, together with the AF, says that `this.elems.length > 0`, so this array access is legal.

The invariant says that the first element is larger than every later element, so this array element is the largest.