CSE 331
Software Design & Implementation

Winter 2025

Section 8 – Trees and ADTs

Administrivia

- HW 8 released tonight, due Wed. March 5
  - Longer code section than recent weeks, so start doing it early and come to office hours

Proof By Calculation (Review)

- The goal of proof by calculation is to *show* that an assertion is true *given* facts that you already know
- You should ***start*** the proof with the left side of the assertion and ***end*** the proof with the right side of the assertion. Each symbol (=, >, <, etc.) connecting each line of the proof is that line's relationship to the previous line on the proof
- Only modify one side

Example:
Suppose we have the facts: $x = 3$, $y = 4$, $z > 5$ and we want to use proof by calculation to prove $x^2 + y^2 < z^2$. Our proof by calculation would look like this:

$$
\begin{aligned}
x^2 + y^2 &= 3^2 + y^2 && \text{since } x = 3 \\
&= 3^2 + 4^2 && \text{since } y = 4 \\
&= 25 \\
&= 5^2 \\
&< z^2 && \text{since } z > 5
\end{aligned}
$$

start with left side of assertion

note that each line shows the relationship to the previous line ONLY

end with right side of assertion

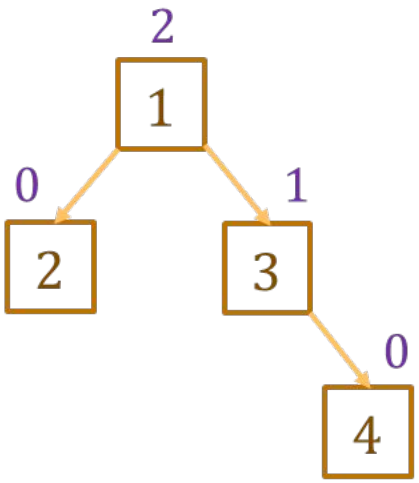
Trees

- Trees are inductive data types with a constructor that has 2+ recursive arguments

- These come up all the time…
  - no constructors with recursive arguments = “generalized enums”
  - constructor with 1 recursive arguments = “generalized lists”
  - constructor with 2+ recursive arguments = “generalized trees”

Height of a tree

- **Binary Tree:** a tree in which each node has at most 2 children
  - Not to be confused with Binary Search Tree, which also has the ordering property that (nodes in L) < x and (nodes in R) > x
- **type** Tree := empty | node(x: ℤ, L: Tree, R: Tree)



Mathematical definition of height

height : Tree → ℤ

height(empty) := −1

height(node(x, L, R)) := 1 + max(height(L), height(R))

Using Definitions in Calculations (example)

height : Tree → $\mathbb{Z}$

height(empty) := −1

height(node(x, L, R)) := 1 + max(height(L), height(R))

- Suppose "T = node(1, empty, node(2, empty, empty))"
- Prove that height(T) = 1

height(T) = height(node(1, empty, node(2, empty, empty)) since T = …

= 1 + max(height(empty), height(node(2, empty, empty))) def of height

= 1 + max(-1, height(node(2, empty, empty))) def of height

= 1 + max(-1, 1+ max(height(empty), height(empty))) def of height

= 1 + max(-1, 1+ max(-1, -1)) def of height (x 2)

= 1 + max(-1, 1+ -1) def of max

= 1 + max(-1, 0)

= 1 + 0 def of max

= 1

Task 1: One, Two, Tree…

The problem makes use of the following inductive type, representing a *left-leaning* binary tree

$$\begin{aligned} \textbf{type}\ \mathsf{Tree} := \quad & \mathsf{empty} \\ \mid \quad & \mathsf{node}(\mathsf{val} : \mathbb{Z},\ \mathsf{left} : \mathsf{Tree},\ \mathsf{right} : \mathsf{Tree}) \quad \text{with } \mathsf{height}(\mathsf{left}) \geqslant \mathsf{height}(\mathsf{right}) \end{aligned}$$

The "with" condition is an *invariant* of the node. Every node that is created must have this property,

$$\begin{aligned} \textbf{func}\ \mathsf{height}(\mathsf{empty}) &:= -1 \\ \mathsf{height}(\mathsf{node}(x, S, T)) &:= 1 + \mathsf{height}(S) \quad \text{for any } x : \mathbb{Z} \text{ and } S, T : \mathsf{Tree} \end{aligned}$$

Since height(S) >= height(T)

$$\mathsf{size} : \mathsf{Tree} \to \mathbb{N}$$

$$\begin{aligned} \mathsf{size}(\mathsf{empty}) &:= 0 \\ \mathsf{size}(\mathsf{node}(x, S, T)) &:= 1 + \mathsf{size}(S) + \mathsf{size}(T) \quad \text{for any } x : \mathbb{Z} \text{ and } S, T : \mathsf{Tree} \end{aligned}$$

Prove by structural induction that, for any left-leaning tree *T* we have

$$\mathsf{size}(T) \leqslant 2^{\mathsf{height}(T)+1} - 1$$

Task 2: How do I Love Tree

a Path tells you how to get to a particular node where each step along the path (item in the list) would be a direction pointing you to keep going down the LEFT or RIGHT branch of the tree.

$$\begin{aligned} \textbf{type}\ \text{BST} := \ & \text{empty} \\ | \ & \text{node}(x : \mathbb{Z},\ S : \text{BST},\ R : \text{BST}) \end{aligned}$$

$$\begin{aligned} \textbf{type}\ \text{Dir} &:= \text{LEFT} \mid \text{RIGHT} \\ \textbf{type}\ \text{Path} &:= \text{List}\langle \text{Dir} \rangle \end{aligned}$$

(a) Define a function "find(p : Path, T : BST)" that returns the node (a BST) at the path from the root of T or undefined if there is no such node.

(b) Define a function "remove(p : Path, T : BST)" that returns T except with the node at the given path replaced by empty.

Task 2: How do I Love Tree

(a) Define a function "find(p : Path, T : BST)" that returns the node (a BST) at the path from the root of T or undefined if there is no such node.

"undefined" sidebar

- If the end of the path cannot be reached within the tree (hit a dead-end before end of Path) → function should result in undefined
  - undefined just indicates invalid inputs
  - If an expression includes a call that results in undefined, then the *entire expression is undefined*
    - Similar to how an `Error` in code does not "return" but bubbles up to callers of the function with the error

Specifications for ADTs – Review

- New Terminology for specifying ADTs:
  - **Abstract State / Representation (Math)**
    - How clients should understand the object
    - Ex: `List(nil or cons)`
  - **Concrete State / Representation (Code)**
    - Actual fields of the record and the data stored
    - Ex: `{ list: List, last: bigint | undefined }`

- We've had different abstract and concrete types all along!
  - in our math, List is an inductive type (abstract)
  - in our code, `List` is a string or a record (concrete)

- Term "object" (or "obj") will refer to abstract state
  - "object" means mathematical object
  - "obj" is the mathematical value that the record represents

Documenting ADTs – Review

Abstract Function (AF) – defines what abstract state the field values represent

- Maps field values → the object they represent
- Output is math, this is a mathematical function

Representation Invariants (RI) – facts about the field values that must always be true

- Constructor must always make sure RI is true at runtime
- Can assume RI is true when reasoning about methods
- AF only needs to make sense when RI holds
- Must ensure that RI *always* holds

Documenting ADTs – Example

```
// A list of integers that can retrieve the last element in O(1)
export interface FastList {
/**
* Returns the object as a regular list
* @returns obj
*/
toList: () => List<bigint>
}

class FastLastList implements FastList {
  // RI: this.last = last(this.list);
  // AF: obj = this.list;

  // @ returns last(obj)
  getLast = (): bigint | undefined => {
    return this.last;
  };
}
```

Talk about functions in terms of the abstract state (obj)

Hide the representation details (i.e. real fields) from the client

Task 3: Let's Blow This Point

Suppose we had the following interface and implementation to represent a point in 2D space:

```
/** Represents a point with coordinates in (x,y) space. */
interface Point {
    /** @returns the x coordinate of the point */
    getX: () => number;

    /** @returns the y coordinate of the point */
    getY: () => number;

    /**
     * Returns the distance of this point to the origin.
     * @returns Math.sqrt(obj.x*obj.x + obj.y*obj.y)
     */
    distToOrigin: () => number;
}
```

```
class SimplePoint implements Point {
    // RI: <TODO>
    // AF: <TODO>
    readonly x: number;
    readonly y: number;
    readonly r: number;

    // Creates a point with the given coordinates
    constructor(x: number, y: number) {
     this.x = x;
     this.y = y;
     this.r = Math.sqrt(x*x + y*y);
    }

    getX = (): number => this.x;
    getY = (): number => this.y;
    distToOrigin = (): number => this.r;
}
```

(a) Define the representation invariant (RI) and abstraction function (AF) for the SimplePoint class.

Task 3: Let's Blow This Point

```
/** Represents a point with coordinates in (x,y) space. */
interface Point {
    /** @returns the x coordinate of the point */
    getX: () => number;

    /** @returns the y coordinate of the point */
    getY: () => number;

    /**
     * Returns the distance of this point to the origin.
     * @returns Math.sqrt(obj.x*obj.x + obj.y*obj.y)
     */
    distToOrigin: () => number;
}
```

```
class SimplePoint implements Point {
    // RI: <TODO>
    // AF: <TODO>
    readonly x: number;
    readonly y: number;
    readonly r: number;

    // Creates a point with the given coordinates
    constructor(x: number, y: number) {
     this.x = x;
     this.y = y;
     this.r = Math.sqrt(x*x + y*y);
    }

    getX = (): number => this.x;
    getY = (): number => this.y;
    distToOrigin = (): number => this.r;
}
```

(b) Use the RI or AF to prove that the distToOrigin method of the SimplePoint class is correct.

Task 3: Let's Blow This Point

(c) The following problem will make use of this math definition that rotates a point around the origin (x, y) by an angle θ:

$$\text{rotate} : (\text{Point}, \mathbb{R}) \to \text{Point}$$

$$\text{rotate}((x, y), \theta) = (x \cdot \cos(\theta) - y \cdot \sin(\theta),\ x \cdot \sin(\theta) + y \cdot \cos(\theta))$$

Suppose we have the following implementation of the rotate method:

/** @returns rotate(obj, θ) */

```
rotate = (theta: number): Point => {
    const newX = this.x * Math.cos(theta) - this.y * Math.sin(theta);
    const newY = this.x * Math.sin(theta) + this.y * Math.cos(theta);
    return new SimplePoint(newX, newY);
}
```

Prove that the rotate method is correct using the RI or AF.

Task 4: Going Back and Length

$$\text{len} : \text{List} \to \mathbb{N}$$

$$\begin{aligned} \text{len}(\text{nil}) &:= 0 \\ \text{len}(x :: L) &:= 1 + \text{len}(L) \end{aligned}$$

$$\text{rev} : \text{List} \to \text{List}$$

$$\begin{aligned} \text{rev}(\text{nil}) &:= \text{nil} \\ \text{rev}(x :: L) &:= \text{rev}(L) +\!+ [x] \end{aligned}$$

Lemma 1:
len(rev(*L*) :: *x*) =
len(rev(*L*)) + len(*x*::nil)
for any list *L* and element *x*

4. Prove by Structural Induction that len(rev(*L*)) = len(*L*) for any list *L*. You may use Lemma 1 in your proof.

*ok to work from top and bottom as long as only modifying right side!