Week 9 Definitions

This week, we will work with locations on a 2D map. We will represent locations as follows:

$$\textbf{type}\ \text{Location} := \{x\colon \mathbb{R},\ y\colon \mathbb{R}\}$$

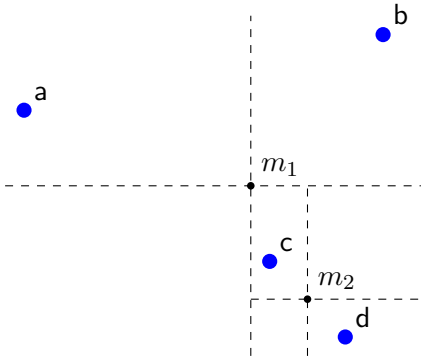
A basic operation on locations is calculating the distance between them. This can be done using the function dist : (Location, Location) $\to \mathbb{R}$, which is defined as follows:

$$\text{dist}(\ell, r) := \sqrt{(\ell.x - r.x)^2 + (\ell.y - r.y)^2}$$

We will be interested in finding the location, from amongst a list of possibilities, that is closest to a given target location. The simplest way to do this would be to calculate the distance of each location in the list to the target location. However, we can do this more efficiently by using a better data structure, namely, a tree, similar to the one we used in HW8, that recursively break up the points into smaller rectangles until each rectangle is either empty or contains a single point.

Trees of Locations

For example, with the four blue points below, we would start by splitting them at point "m_1" (the centroid of those locations). The NW and NE regions then contain only a single point and the SW region is empty. The SE region, however, still contains two points, so we would split them again at point "m_2", which leaves four regions containing at most one point.



We can represent this tree with the following inductive data type:

type LocTree := empty
| single(loc : Location)
| split(at : Location, nw : LocTree, ne : LocTree, sw : LocTree, se : LocTree)

The constructor "empty" represents a region that is empty, while "single(s)" represents a region containing a single location at location s. The final constructor "split(m, nw, ne, sw, se)" represents a region that splits at location m into the four regions to the NW, NE, SW, and SE, respectively.

With those definitions, the example above would be represented with the following tree:

$$\text{split}(m_1, \text{single}(a), \text{single}(b), \text{empty},\\ \text{split}(m_2, \text{single}(c), \text{empty}, \text{empty}, \text{single}(d)))$$

Regions

Note that, while each node represents some rectangular region in space, the bounds of that region are not recorded in any field. That said, we can easily calculate the bounds of the region as we work down to that node, recursively, through the tree.

Specifically, we can store a rectangular region as an instance of the following type

$$\textbf{type}\ \text{Region} := \{\text{x1}\colon \mathbb{R},\ \text{x2}\colon \mathbb{R},\ \text{y1}\colon \mathbb{R},\ \text{y2}\colon \mathbb{R}\}$$

The region R contains the location ℓ iff $R.\text{x1} \leqslant \ell.x \leqslant R.\text{x2}$ and $R.\text{y1} \leqslant \ell.y \leqslant R.\text{y2}$. Thus, the following region includes every point in the plane:

$$\text{EVERYWHERE} := \{\text{x1}\colon -\infty,\ \text{x2}\colon \infty,\ \text{y1}\colon -\infty,\ \text{y2}\colon \infty\}$$

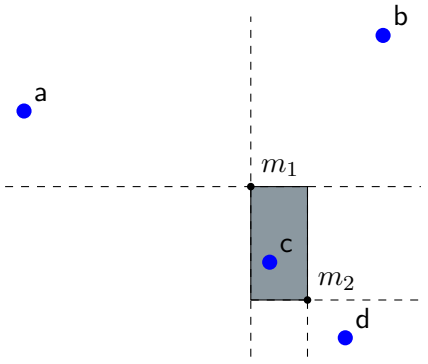
The following functions $\text{NW}, \ldots, \text{SE} : (\text{Location}, \text{Region}) \to \text{Region}$ return the intersection of the region passed in with one of the quadrants of a node that was split at the location passed in. For example, $\text{NW}(m, R)$ would return the region that includes only the area that is both inside of R and northwest of m (i.e., an x coordinate that is $< m.x$ and a y coordinate that is $< m.y$).

These functions assume that m falls within the region R, and they are defined as follows:

$$\begin{aligned}
\text{NW}(m, R) &:= \{\text{x1}\colon R.\text{x1},\ \text{x2}\colon m.x,\ \text{y1}\colon R.\text{y1},\ \text{y2}\colon m.y\} \\
\text{NE}(m, R) &:= \{\text{x1}\colon m.x,\ \text{x2}\colon R.\text{x2},\ \text{y1}\colon R.\text{y1},\ \text{y2}\colon m.y\} \\
\text{SW}(m, R) &:= \{\text{x1}\colon R.\text{x1},\ \text{x2}\colon m.x,\ \text{y1}\colon m.y,\ \text{y2}\colon R.\text{y2}\} \\
\text{SE}(m, R) &:= \{\text{x1}\colon m.x,\ \text{x2}\colon R.\text{x2},\ \text{y1}\colon m.y,\ \text{y2}\colon R.\text{y2}\}
\end{aligned}$$

For example, $NW(m, R)$ leaves the left side of the region at $R.\text{x1}$, but it extends the region to the right to $m.x$. Any point further right than that is outside the region to the NW of m.

Returning to our example from before



the shaded region, which is the region that lies in the SE quadrant of the split at m_1 and then, within that, the NW quadrant of the split at m_2, can be calculated as $\text{NW}(m_2, \text{SE}(m_1, \text{EVERYWHERE}))$.