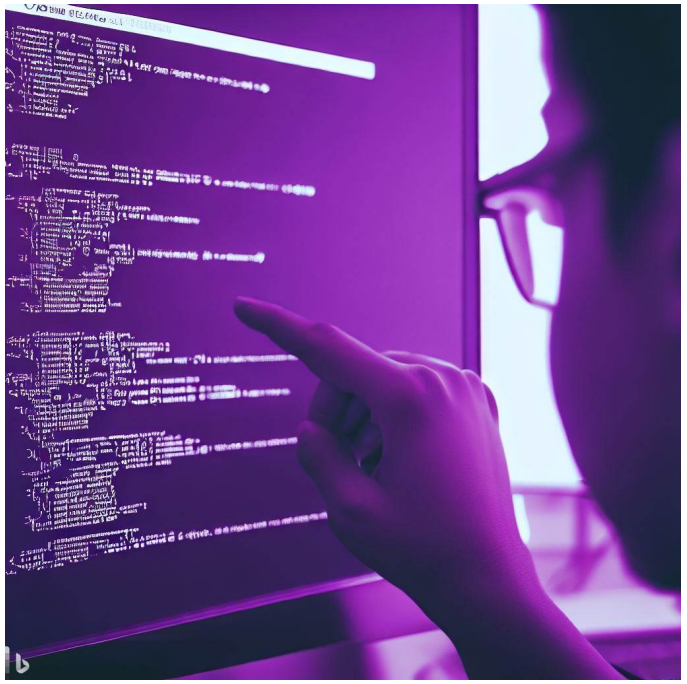


CSE 331

Floyd Logic

James Wilcox and Kevin Zatloukal

Reasoning So Far

- Code so far made up of three elements
  - straight-line code (just variable declarations and returns)
  - conditionals
  - recursion

- All code without mutation looks like this

- Proving correctness is proving implications
  - check that known facts imply the required facts

Recall: Finding Facts at a Return Statement

- **Consider this code**

```
// Inputs a and b must be integers.
// Returns a non-negative integer.
public int f(int a, int b) {
  if (a >= 0 && b >= 0) {
    final List L = cons(a, cons(b, nil));
    return sum(L);
  }
  …
```

find facts by reading along path from top to return statement

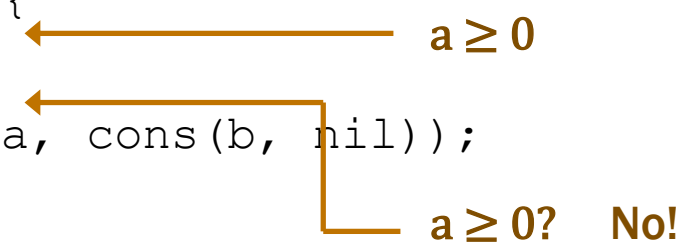
- **Known facts include** "$a \geq 0$", "$b \geq 0$", **and** "$L = \text{cons}(...)$"

- **Prove that postcondition holds:** "$\text{sum}(L) \geq 0$"

Finding Facts at a Return Statement

- Consider this code

```
// Inputs a and b must be integers.
// Returns a non-negative integer.
public int f(int a, int b) => {
  if (a >= 0 && b >= 0) {
    a = a - 1;
    final List L = cons(a, cons(b, nil));
    return sum(L);
  }
  …
```



- Facts no longer hold <u>throughout</u> straight-line code
- When we state a fact, we have to say <u>where</u> it holds

Correctness Levels

Description	Testing	Tools	Reasoning
no mutation	coverage	type checking	calculation induction
local variable mutation	"	"	Floyd logic
heap state mutation	"	"	rep invariants
array mutation	"	"	for-any facts

Finding Facts at a Return Statement

```
// Inputs a and b must be integers.
// Returns a non-negative integer.
public int f(int a, int b) {
  if (a >= 0 && b >= 0) {
    {{ a ≥ 0 }}
    a = a - 1;
    {{ a ≥ -1 }}
    final List L = cons(a, cons(b, nil));
    return sum(L);
  }
```

- When we state a fact, we have to say <u>where</u> it holds
- {{ .. }} notation indicates facts true at that point
  - cannot assume those are true anywhere else

Finding Facts at a Return Statement

```
// Inputs a and b must be integers.
// Returns a non-negative integer.
public int f(int a, int b) {
  if (a >= 0 && b >= 0) {
    {{ a ≥ 0 }}
    a = a - 1;
    {{ a ≥ -1 }}
    final List L = cons(a, cons(b, nil));
    return sum(L);
  }
```

- There are <u>mechanical</u> tools for moving facts around
  - "forward reasoning" says how they change as we move down
  - "backward reasoning" says how they change as we move up

Finding Facts at a Return Statement

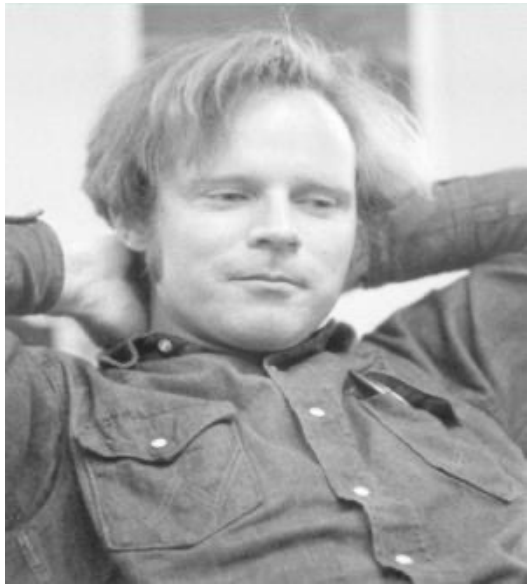
```
// Inputs a and b must be integers.
// Returns a non-negative integer.
public int f(int a, int b) {
  if (a >= 0 && b >= 0) {
    {{ a ≥ 0 }}
    a = a - 1;
    {{ a ≥ -1 }}
    final List L = cons(a, cons(b, nil));
    return sum(L);
  }
```

- Professionals are *insanely* good at forward reasoning
  - "programmers are the Olympic athletes of forward reasoning"
  - you'll have an edge by learning backward reasoning too

Floyd Logic

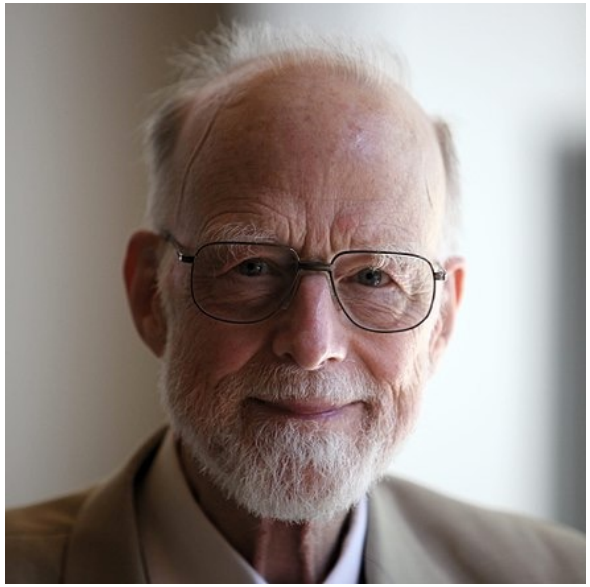
Floyd Logic

- Invented by Robert Floyd and Sir Anthony Hoare
  - Floyd won the Turing award in 1978
  - Hoare won the Turing award in 1980



Robert Floyd

picture from Wikipedia



Tony Hoare

Floyd Logic Terminology

- The **program state** is the values of the variables

- An **assertion** (in {{ .. }}) is a T/F claim about the state
  - an assertion "holds" if the claim is true
  - assertions are *math* not code

    (we do our reasoning in math)

- Most important assertions:
  - **precondition**: claim about the state when the function starts
  - **postcondition**: claim about the state when the function ends

Hoare Triples

- A **Hoare triple** has two assertions and some code

$$\{\{ P \}\}$$
$$S$$
$$\{\{ Q \}\}$$

  - P is the precondition, Q is the postcondition
  - S is the code

- Triple is "**valid**" if the code is correct:
  - S takes *any* state satisfying P into a state satisfying Q

    does not matter what the code does if P does not hold initially
  - otherwise, the triple is invalid

Correctness Example

```
/**
 * @param n an integer with n >= 1
 * @returns an integer m with m >= 10
 */
public int f(int n) {
  n = n + 3;
  return n * n;
};
```

Correctness Example

```
/**
 * @param n an integer with n >= 1
 * @returns an integer m with m >= 10
 */
public int f(int n) {
  {{ n ≥ 1 }}
  n = n + 3;
  {{ n² ≥ 10 }}
  return n * n;
};
```

- Precondition and postcondition come from spec
- Remains to check that the triple is valid

Hoare Triples with No Code

- Code could be empty:

$$\{\{\ P\ \}\}$$
$$\{\{\ Q\ \}\}$$

- When is such a triple valid?
  - valid iff P implies Q
  - we already know how to check validity in this case: prove each fact in Q by calculation, using facts from P

Hoare Triples with No Code

- **Code could be empty:**

$$\{\{\ a \geq 0,\ b \geq 0,\ L = \text{cons}(a, \text{cons}(b, \text{nil}))\ \}\}$$
$$\{\{\ \text{sum}(L) \geq 0\ \}\}$$

- **Check that P implies Q by calculation**

sum(L)	= sum(cons(a, cons(b, nil)))	since L = …
	= a + sum(cons(b, nil))	def of sum
	= a + b + sum(nil)	def of sum
	= a + b	def of sum
	≥ 0 + b	since a ≥ 0
	≥ 0 + 0	since b ≥ 0
	= 0	

Hoare Triples with Code

- Code with code:

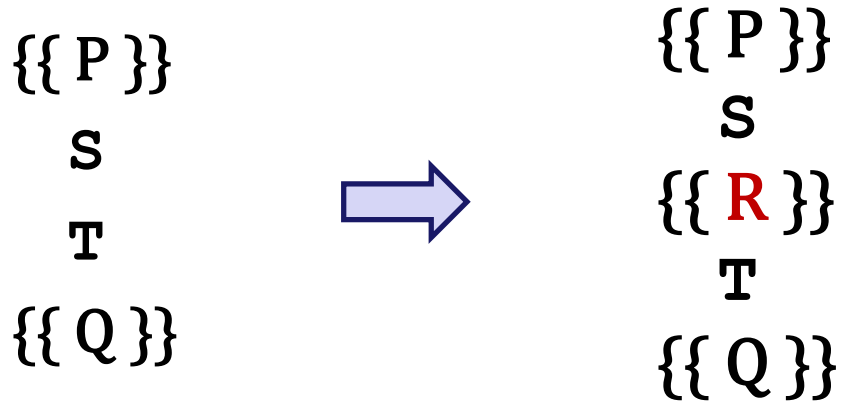
$$\{\{ P \}\}$$
$$s$$
$$\{\{ Q \}\}$$

- Easy if s is empty, but what if not?

- We can use forward & backward reasoning
  - move the assertions toward each other until they meet
  - then we have a triple with no code

Hoare Triples with Multiple Lines of Code

- Code with multiple lines:

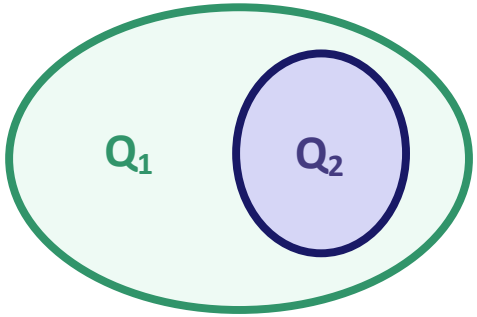


- Valid iff there exists an R making both triples valid
  - i.e., {{ P }} S {{ R }} is valid and {{ R }} T {{ Q }} is valid

- Will see next how to put these to good use…

Recall: Stronger Assertions

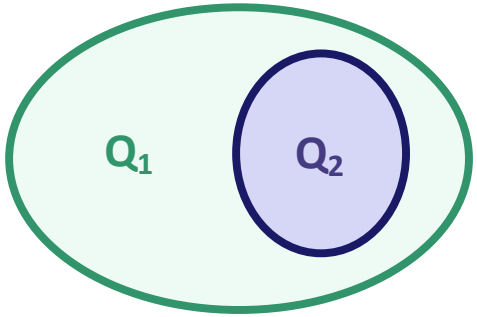
- **Assertion** is stronger iff it holds in a subset of states



- Stronger assertion implies the weaker one
  - stronger is a synonym for “implies”
  - weaker is a synonym for “is implied by”

Recall: Stronger Assertions

- **Assertion** is **stronger** iff it holds in a subset of states



- **Weakest** possible assertion is "true" (all states)
  - an empty assertion ("") also means "true"

- **Strongest** possible assertion is "false" (no states!)

Mechanical Reasoning Tools

- Forward / backward reasoning fill in assertions
  - mechanically create valid triples

- **Forward** reasoning fills in postcondition

$$\{\{ P \}\}\ s\ \{\{ \_\_ \}\}$$

  - gives *strongest* postcondition making the triple valid

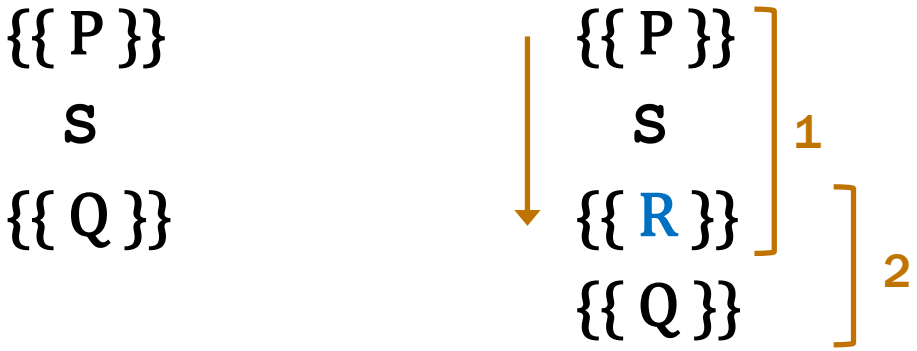
- **Backward** reasoning fills in precondition

$$\{\{ \_\_ \}\}\ s\ \{\{ Q \}\}$$

  - gives *weakest* precondition making the triple valid

Correctness via Forward Reasoning

- Apply forward reasoning



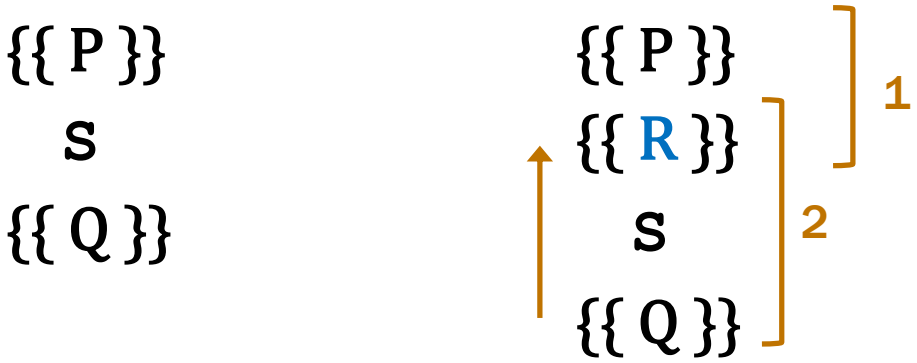
  - first triple is always valid
  - only need to check second triple

    just requires proving an implication (since no code is present)

- If second triple is invalid, the code is **incorrect**
  - true because R is the strongest assertion possible here

Correctness via Backward Reasoning

- Apply backward reasoning



  - second triple is always valid
  - only need to check first triple

    just requires proving an implication (since no code is present)

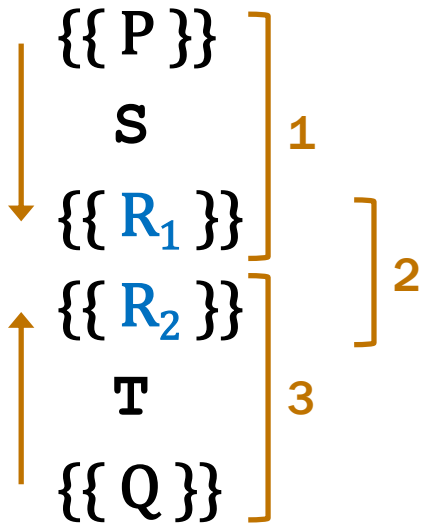
- If first triple is invalid, the code is **incorrect**
  - true because R is the weakest assertion possible here

Mechanical Reasoning Tools

- Forward / backward reasoning fill in assertions
  - mechanically create valid triples
- Reduce correctness to proving implications (again)
  - this was already true for functional code
  - will soon have the same for imperative code
- Implication will be false if the code is **incorrect**
  - reasoning can verify correct code
  - reasoning will never accept incorrect code

Correctness via Forward & Backward

- Can use both types of reasoning on longer code



  - first and third triples is always valid
  - only need to check second triple

    verify that R_1 implies R_2

Forward & Backward Reasoning

Forward and Backward Reasoning

- Imperative code made up of
  - assignments (mutation)
  - conditionals
  - loops

- Anything can be rewritten with just these

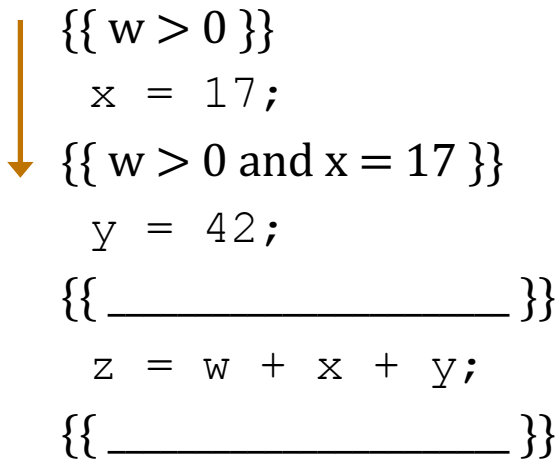
- We will learn forward / backward rules to handle them
  - will also learn a rule for function calls
  - once we have those, we are done

Example Forward Reasoning through Assignments

```
{{ w > 0 }}
 x = 17;
{{ ____________________ }}
 y = 42;
{{ ____________________ }}
 z = w + x + y;
{{ ____________________ }}
```

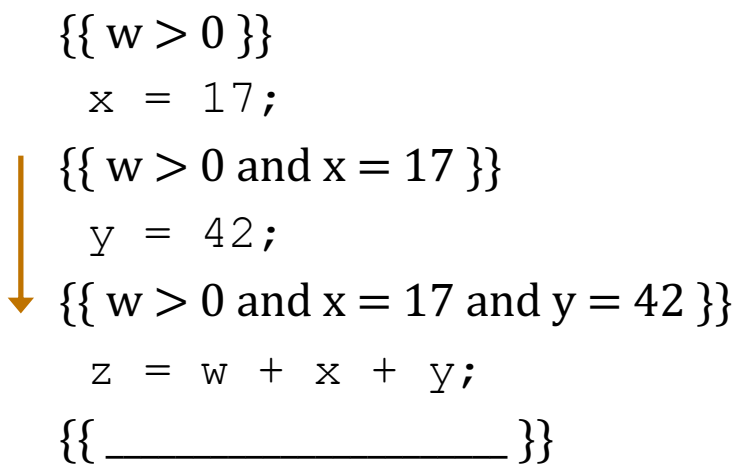
- What do we know is true after `x = 17` ?
  - want the strongest postcondition (most precise)

Example Forward Reasoning through Assignments



- **What do we know is true after `x = 17` ?**
  - **`w` was not changed, so $w > 0$ is still true**
  - **`x` is now 17**
- **What do we know is true after `y = 42` ?**

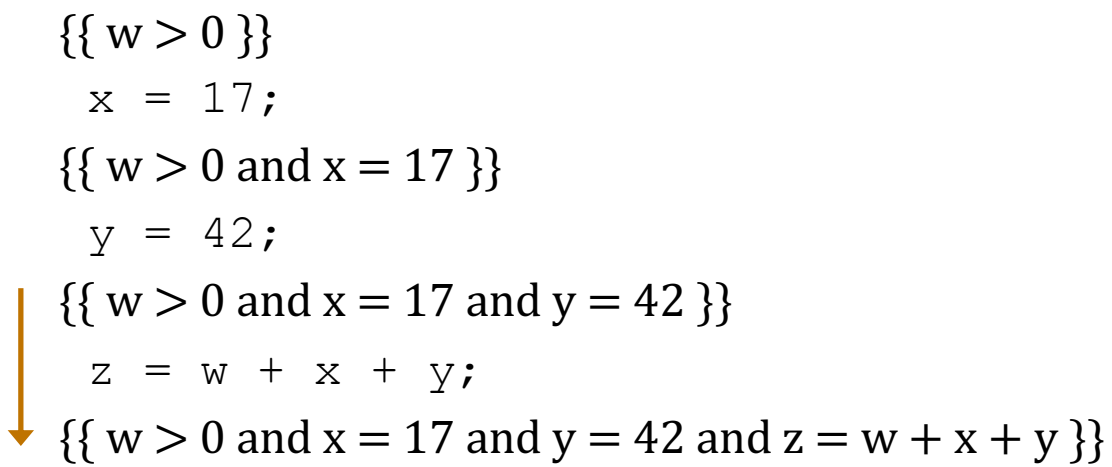
Example Forward Reasoning through Assignments



- **What do we know is true after `y = 42` ?**
  - `w` **and** `x` **were not changed, so previous facts still true**
  - `y` **is now** `42`

- **What do we know is true after `z = w + x + y` ?**

Example Forward Reasoning through Assignments



- **What do we know is true after** `z = w + x + y` **?**
  - w, x, **and y were not changed, so previous facts still true**
  - z **is now** $w + x + y$
- **Could also write** $z = w + 59$ **(since** $x = 17$ **and** $y = 42$**)**

Example Forward Reasoning through Assignments

{{ $w > 0$ }}

```
x = 17;
```

{{ $w > 0$ and $x = 17$ }}

```
y = 42;
```

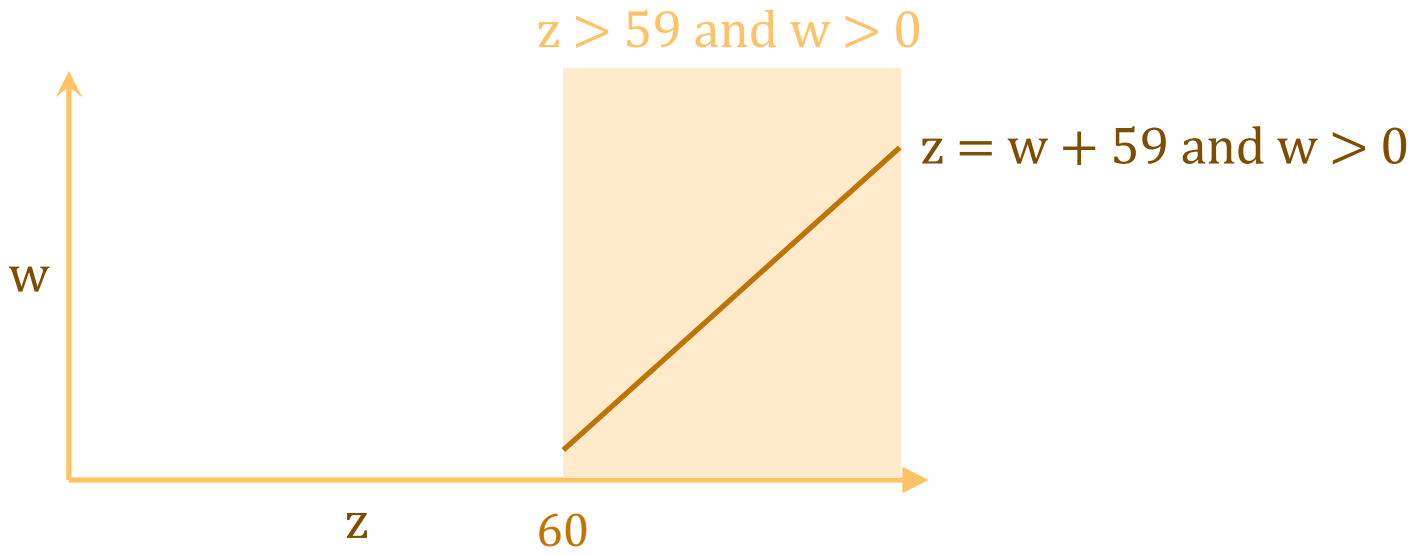
{{ $w > 0$ and $x = 17$ and $y = 42$ }}

```
z = w + x + y;
```

{{ $w > 0$ and $x = 17$ and $y = 42$ and $z = w + x + y$ }}

- Could write $z = w + 59$, but <u>do not</u> write $z > 59$!
  - that is true since $w > 0$, but…

Example Forward Reasoning through Assignments



- **Could write $z = w + 59$, but <u>do not</u> write $z > 59$!**
  - **that is true but it is not the strongest postcondition**

    correctness check could now fail even if the code is right

Code Example of Forward Reasoning

```
// @param w an integer > 0
// @returns an integer z > 59
public int f(int w) {
  int x = 17;
  int y = 42;
  int z = w + x + y;
  return z;
};
```

- Let's check correctness using Floyd logic...

Code Example of Forward Reasoning

```
// @param w an integer > 0
// @returns an integer z > 59
public int f(int w) {
  {{ w > 0 }}
  int x = 17;
  int y = 42;
  int z = w + x + y;
  {{ z > 59 }}
  return z;
};
```

- Reason forward...

Code Example of Forward Reasoning

```
// @param w an integer > 0
// @returns an integer z > 59
public int f(int w) {
  {{ w > 0 }}
  int x = 17;
  int y = 42;
  int z = w + x + y;
  {{ w > 0 and x = 17 and y = 42 and z = w + x + y }}
  {{ z > 59 }}
  return z;
};
```

- **Check implication:**

$$
\begin{aligned}
z &= w + x + y \\
&= w + 17 + y && \text{since } x = 17 \\
&= w + 59 && \text{since } y = 42 \\
&> 59 && \text{since } w > 0
\end{aligned}
$$

Code Example of Forward Reasoning

```
// @param w an integer > 0
// @returns an integer z > 59
public int f(int w) {
  int x = 17;
  int y = 42;
  int z = w + x + y;
  return z;
};
```

find facts by reading along path from top to return statement

- How about if we use our old approach?
- Known facts: $w > 0$, $x = 17$, $y = 42$, and $z = w + x + y$
- Prove that postcondition holds: $z > 59$

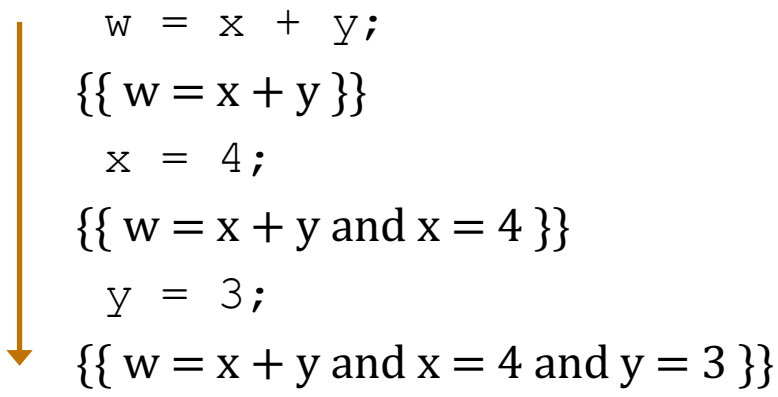
Code Example of Forward Reasoning

```
// @param w an integer > 0
// @returns an integer z > 59
public int f(int w) {
  int x = 17;
  int y = 42;
  int z = w + x + y;
  return z;
};
```

- We've been doing forward reasoning already!
  - forward reasoning is (only) "and" with *no mutation*
- Line-by-line facts are for mutation (not "`final`")

Forward Reasoning through Assignments

- Forward reasoning is trickier with mutation
  - gets harder if we mutate a variable

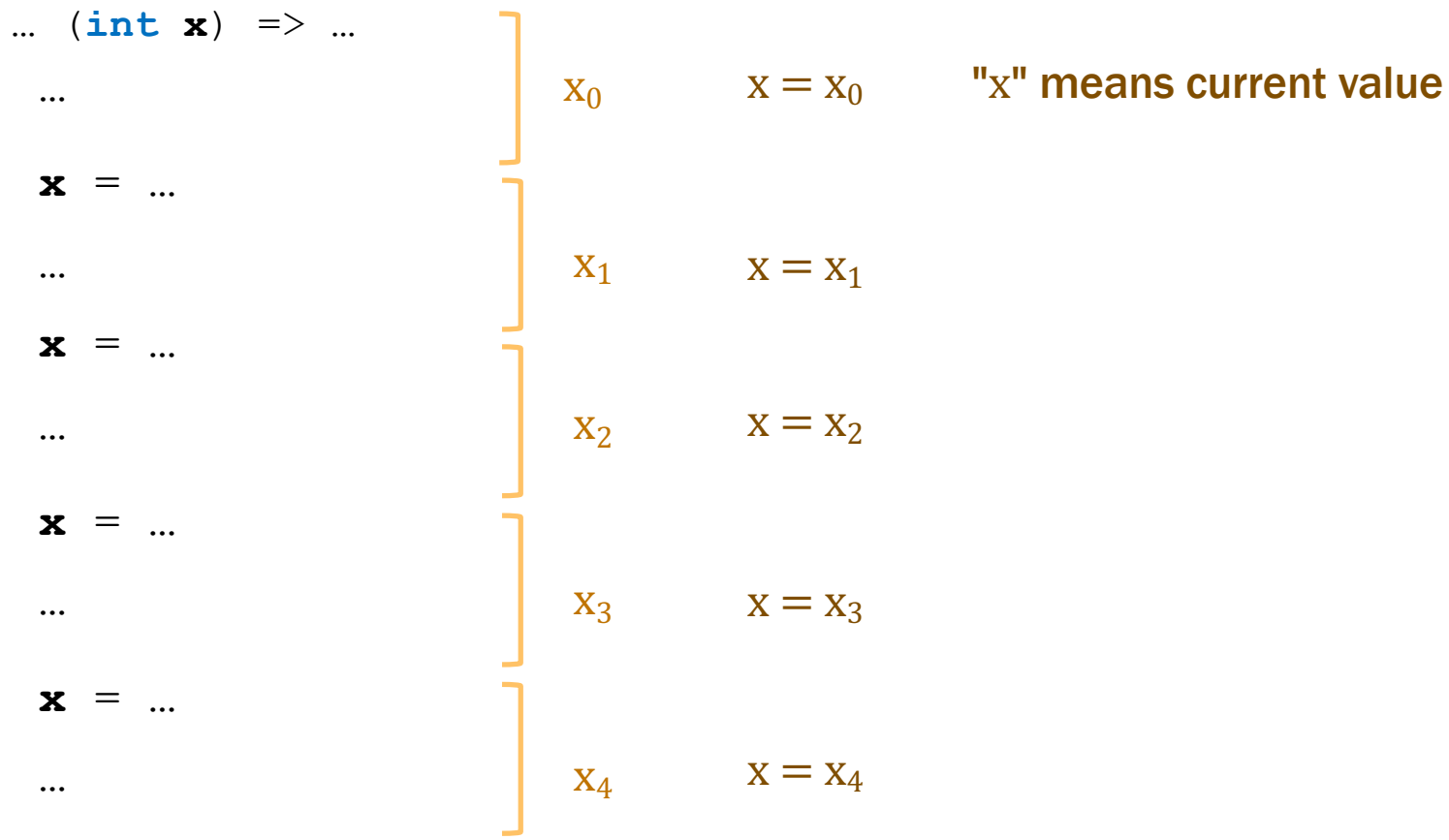


- Final assertion is not necessarily true
  - $w = x + y$ is true with their old values, not the new ones
  - changing the value of "x" can invalidate facts about x

    facts refer to the old value, not the new value
  - avoid this by using different names for old and new values

Forward Reasoning through Assignments

- Can use subscripts to refer to values at different times



Forward Reasoning through Assignments

- Rewrite existing facts to use names of earlier values
  - will use "x" and "y" to refer to current values
  - can use "x_0" and "y_0" (or other subscripts) for earlier values

$\{\{ w = x + y \}\}$

```
x = 4;
```

$\{\{ w = \mathbf{x_0} + y \text{ and } \mathbf{x} = 4 \}\}$

```
y = 3;
```

$\{\{ w = x_0 + \mathbf{y_0} \text{ and } x = 4 \text{ and } \mathbf{y} = 3 \}\}$

- Final assertion is now accurate
  - w is equal to the sum of the initial values of x and y

Forward Reasoning through Assignments

- For assignments, general forward reasoning rule is

{{ P }}
  x = y;
{{ $P[x \mapsto x_k]$ and $x = y[x \mapsto x_k]$ }}

  - replace all "x"s in P and y with "x_k"s

- This process can be simplified in many cases
  - no need for x_0 if we can write it in terms of new value
  - e.g., if "$x = x_0 + 1$", then "$x_0 = x - 1$"
  - assertions will be easier to read without old values

    (Technically, this is weakening, but it's usually fine

    Postconditions usually do not refer to old values of variables.)

Forward Reasoning through Assignments

- For assignments, general forward reasoning rule is

$\{\{ P \}\}$

`x = y;`

$\{\{ P[x \mapsto x_k] \text{ and } x = y[x \mapsto x_k] \}\}$

x_k is name of previous value

- If $x_0 = f(x)$, then we can simplify this to

$\{\{ P \}\}$

`x = ... x ...;`

$\{\{ P[x \mapsto f(x)] \}\}$

no need for, e.g., "and $x = x_0 + 1$"

  - if assignment is "$x = x_0 + 1$", then "$x_0 = x - 1$"
  - if assignment is "$x = 2x_0$", then "$x_0 = x/2$"
  - does not work for integer division (an un-invertible operation)

Correctness Example by Forward Reasoning

```
/**
 * @param n an integer with n >= 1
 * @returns an integer m with m >= 10
 */
public int f = (int n) {
```

{{ $n \geq 1$ }}

```
  n = n + 3;
```

$n = n_0 + 3$ **means** $n - 3 = n_0$

{{ $n - 3 \geq 1$ }}

{{ $n^2 \geq 10$ }}

check this implication

```
  return n * n;
};
```

$n^2 \geq 4^2$ since $n - 3 \geq 1$ (i.e., $n \geq 4$)

$= 16$

> 10

This is the preferred approach.
Avoid subscripts when possible.

Mutation in Straight-Line Code

- Alternative ways of writing this code:

```
n = n + 3;
return n * n;
```

```
final int n1 = n + 3;
return n1 * n1;
```

- Mutation in *straight-line* code is unnecessary
  - can always use different names for each value

- Why would we prefer the former?
  - seems like it might save memory…
  - but it doesn't!

    most compilers will turn the left into the right on their own (SSA form)

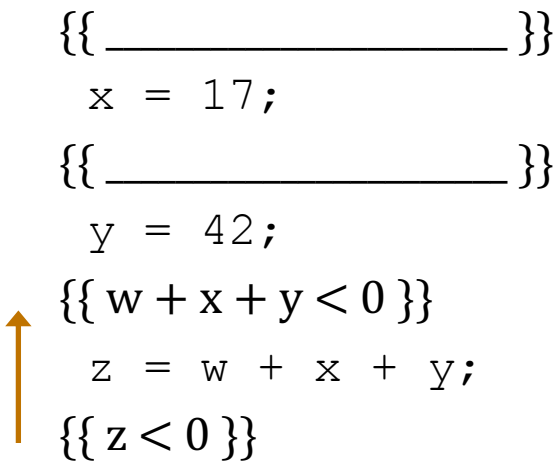
    it's better at saving memory than you are, so it does it itself

Example Backward Reasoning with Assignments

```
{{ _____________ }}
 x = 17;
{{ _____________ }}
 y = 42;
{{ _____________ }}
 z = w + x + y;
{{ z < 0 }}
```

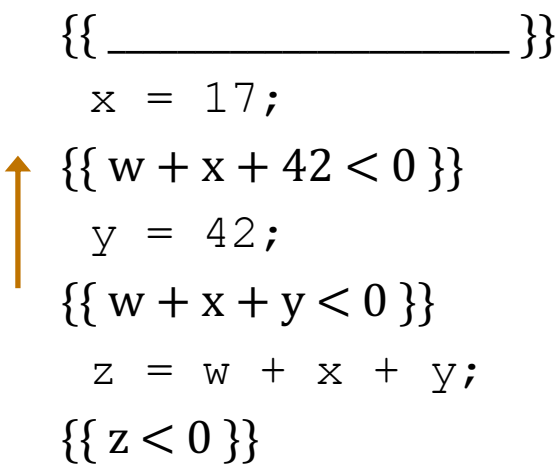
- **What must be true before** `z = w + x + y` **so** $z < 0$ **?**
  - **want the weakest precondition (most allowed states)**

Example Backward Reasoning with Assignments



- **What must be true before** $z = w + x + y$ **so** $z < 0$ **?**
  - **must have** $w + x + y < 0$ **beforehand**
- **What must be true before** $y = 42$ **for** $w + x + y < 0$ **?**

Example Backward Reasoning with Assignments



- **What must be true before** $y = 42$ **for** $w + x + y < 0$ **?**
  - **must have** $w + x + 42 < 0$ **beforehand**

- **What must be true before** $x = 17$ **for** $w + x + 42 < 0$ **?**

Example Backward Reasoning with Assignments

$\{\{ w + 17 + 42 < 0 \}\}$

```
x = 17;
```

$\{\{ w + x + 42 < 0 \}\}$

```
y = 42;
```

$\{\{ w + x + y < 0 \}\}$

```
z = w + x + y;
```

$\{\{ z < 0 \}\}$

- **What must be true before** $x = 17$ **for** $w + x + 42 < 0$ **?**
  - **must have** $w + 59 < 0$ **beforehand**
- **All we did was substitute right side for the left side**
  - **e.g., substitute "**$w + x + y$**" for "**z**" in "**$z < 0$**"**
  - **e.g., substitute "**42**" for "**y**" in "**$w + x + y < 0$**"**
  - **e.g., substitute "**17**" for "**x**" in "**$w + x + 42 < 0$**"**

Backward Reasoning through Assignments

- For assignments, backward reasoning is substitution

$\{\{\ Q[x \mapsto y]\ \}\}$

```
  x = y;
```

$\{\{\ Q\ \}\}$

  - just replace all the "x"s with "y"s
  - we will denote this substitution by $Q[x \mapsto y]$

- Mechanically simpler than forward reasoning
  - no need for subscripts

Correctness Example by Backward Reasoning

```
/**
 * @param n an integer with n >= 1
 * @returns an integer m with m >= 10
 */
public int f(int n) {
  {{ n ≥ 1 }}
  n = n + 3;
  {{ n² ≥ 10 }}
  return n * n;
};
```

- Code is correct if this triple is valid...

Correctness Example by Backward Reasoning

```
/**
 * @param n an integer with n >= 1
 * @returns an integer m with m >= 10
 */
public int f(int n) {
   {{ n ≥ 1 }}
   {{ (n + 3)² ≥ 10 }}
   n = n + 3;
   {{ n² ≥ 10 }}
   return n * n;
};
```

check this implication

$$(n+3)^2 \geq (1+3)^2 \quad \text{since } n \geq 1$$
$$= 16$$
$$> 10$$

Correctness Example by Forward Reasoning

```
/**
 * @param n an integer with n >= 1
 * @returns an integer m with m >= 10
 */
public int f(int n) {
   {{ n ≥ 1 }}
   n = n + 3;
   {{ n – 3 ≥ 1 }}
   {{ n² ≥ 10 }}
   return n * n;
};
```

check this implication

$n^2 \geq 4^2$ since $n - 3 \geq 1$ (i.e., $n \geq 4$)

$= 16$

> 10

Forward reasoning produces known facts.
Backward reasoning produces fact to prove.

Conditionals

Conditionals in Functional Programming

```
// Inputs a and b must be integers.
// Returns a non-negative integer.
public int f(int a, int b) {
  if (a >= 0 && b >= 0) {
    final List L = cons(a, cons(b, nil));
    return sum(L);
  }
  …
```

- Prior reasoning also included *conditionals*
  - what does that look like in Floyd logic?

Conditionals in Floyd Logic

```
// Inputs a and b must be integers.
// Returns a non-negative integer.
public int f(int a, int b) {
  {{ }}
  if (a >= 0 && b >= 0) {
    {{ a ≥ 0 and b ≥ 0 }}
    final List L = cons(a, cons(b, nil));
    return sum(L);
  }
  …
```

- Conditionals introduce extra facts in forward reasoning
  - simple “and” since nothing is mutated

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
public int g(int n) {
  int m;
  if (n >= 0) {
    m = 2 * n + 1;
  } else {
    m = 0;
  }
  return m;
}
```

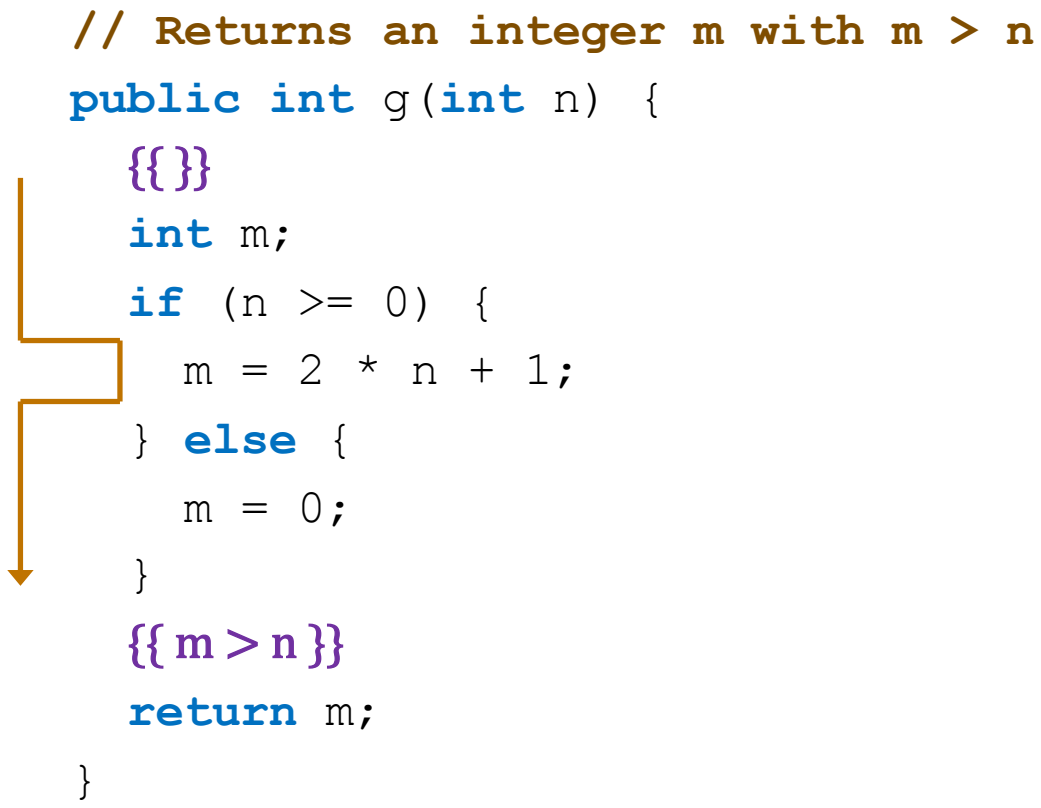
- Code like this was impossible without mutation
  - cannot write to a "`final`" after its declaration
- How do we handle it now?

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
public int g(int n) {
  int m;
  if (n >= 0) {
    m = 2 * n + 1;
  } else {
    m = 0;
  }
  return m;
}
```

- Reason *separately* about each path to a `return`
  - handle each path the same as before
  - but now there can be multiple paths to one `return`

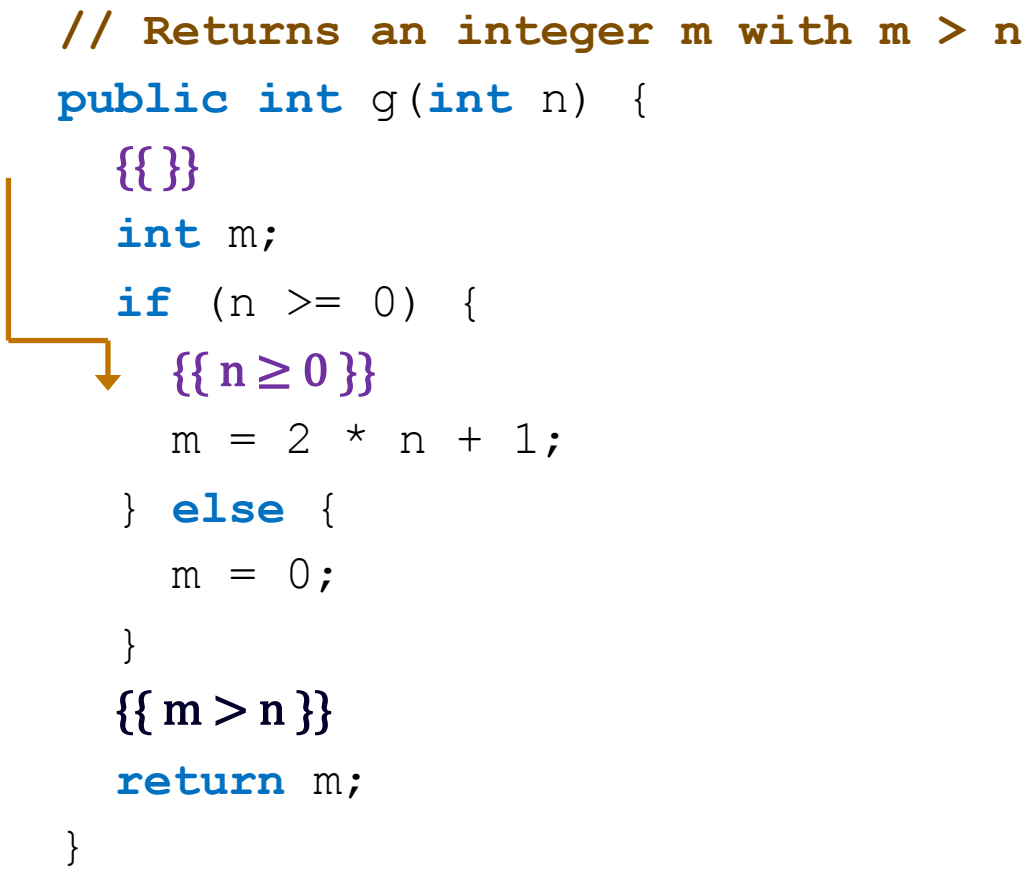
Conditionals in Floyd Logic



```
// Returns an integer m with m > n
public int g(int n) {
  {{ }}
  int m;
  if (n >= 0) {
    m = 2 * n + 1;
  } else {
    m = 0;
  }
  {{ m > n }}
  return m;
}
```

- Check correctness path through “then” branch

Conditionals in Floyd Logic



Conditionals in Floyd Logic

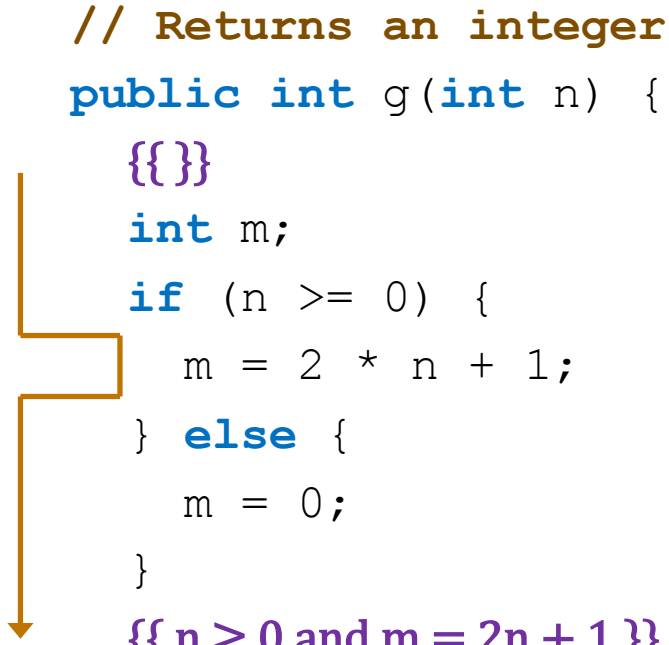
```
// Returns an integer m with m > n
public int g(int n) {
  {{ }}
  int m;
  if (n >= 0) {
    {{ n ≥ 0 }}
    m = 2 * n + 1;
    {{ n ≥ 0 and m = 2n + 1}}
  } else {
    m = 0;
  }
  {{ m > n }}
  return m;
}
```

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
public int g(int n) {
  {{ }}
  int m;
  if (n >= 0) {
    {{ n ≥ 0 }}
    m = 2 * n + 1;
    {{ n ≥ 0 and m = 2n + 1}}
  } else {
    m = 0;
  }
  {{ n ≥ 0 and m = 2n + 1 }}
  {{ m > n }}
  return m;
}
```

$$
\begin{aligned}
m &= 2n+1 \\
&> 2n && \text{since } 1 > 0 \\
&\geq n && \text{since } n \geq 0
\end{aligned}
$$

Conditionals in Floyd Logic



```
// Returns an integer m with m > n
public int g(int n) {
  {{ }}
  int m;
  if (n >= 0) {
    m = 2 * n + 1;
  } else {
    m = 0;
  }
  {{ n ≥ 0 and m = 2n + 1 }}
  {{ m > n }}
  return m;
}
```

- Note: no mutation, so we can do this in our head
  - read along the path, and collect all the facts

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
public int g(int n) {
  {{ }}
  int m;
  if (n >= 0) {
    m = 2 * n + 1;
  } else {
    m = 0;
  }
  {{ n < 0 and m = 0 }}
  {{ m > n }}
  return m;
}
```

$m = 0$

$> n$ since $0 > n$

- Check correctness path through "else" branch
  - note: no mutation, so we can do this in our head

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
public int g(int n) {
  {{ }}
  int m;
  if (n >= 0) {
    m = 2 * n + 1;
    {{ n ≥ 0 and m = 2n + 1 }}
  } else {
    m = 0;
    {{ n < 0 and m = 0 }}
  }
  {{ ____________________________________________ }}
  {{ m > n }}
  return m;
}
```

What do we know is true even if we don't know which branch was taken?

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
public int g(int n) {
  {{ }}
  int m;
  if (n >= 0) {
    m = 2 * n + 1;
  } else {
    m = 0;
  }
  {{ (n ≥ 0 and m = 2n + 1) or (n < 0 and m = 0) }}
  {{ m > n }}
  return m;
}
```

- The “or” means we must reason by cases anyway!

Conditionals in Floyd Logic

```
{{ P }}
if (cond) {
    {{ P and cond }}
    S1
    {{ A }}
} else {
    {{ P and not cond }}
    S2
    {{ B }}
}
{{ A or B }}
{{ Q }}
```

- Postcondition is of the form {{ A or B }}
  - A being what we know if we had taken the `if` branch
  - B being what we know if we had taken the `else`

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
public int g(int n) {
  {{ }}
  int m;
  if (n >= 0) {
    m = 2 * n + 1;
  } else {
    return 0;
  }
  {{ (n ≥ 0 and m = 2n + 1) or (n < 0 and ??) }}
  {{ m > n }}
  return m;
}
```

- What is the state after a "`return`"?

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
public int g(int n) {
  {{ }}
  int m;
  if (n >= 0) {
    m = 2 * n + 1;
  } else {
    return 0;
  }
  {{ (n ≥ 0 and m = 2n + 1) or (n < 0 and false) }}
  {{ m > n }}
  return m;
}
```

simplifies to just $n \geq 0$ and $m = 2n + 1$

- State after a "`return`" is false (no states)

Conditionals With Returns

- Latter rule for "`if` .. `return`" is useful:

```
{{ P }}
if (cond)
  return something;
{{ P and not cond }}
…
return something else;
```

- Only reach the line after the "`if`" if `cond` was false

- Only one path to each "`return`" statement
  - forward reason to the "`return`" inside the "`if`"
  - forward reason to the "`return`" after the "`if`"

Conditionals in Floyd Logic

```
// Returns an integer m, with m > 0
public int h(int x) {
  {{ }}
  int m = x;
  if (x < 0) {
    m = m * -1;
  } else if (x == 0) {
    return 1;
  }
  {{ ____________________________________ }}
  m = m + 1;
  {{ m > 0 }}
  return m;
}
```

How many paths can the code take?

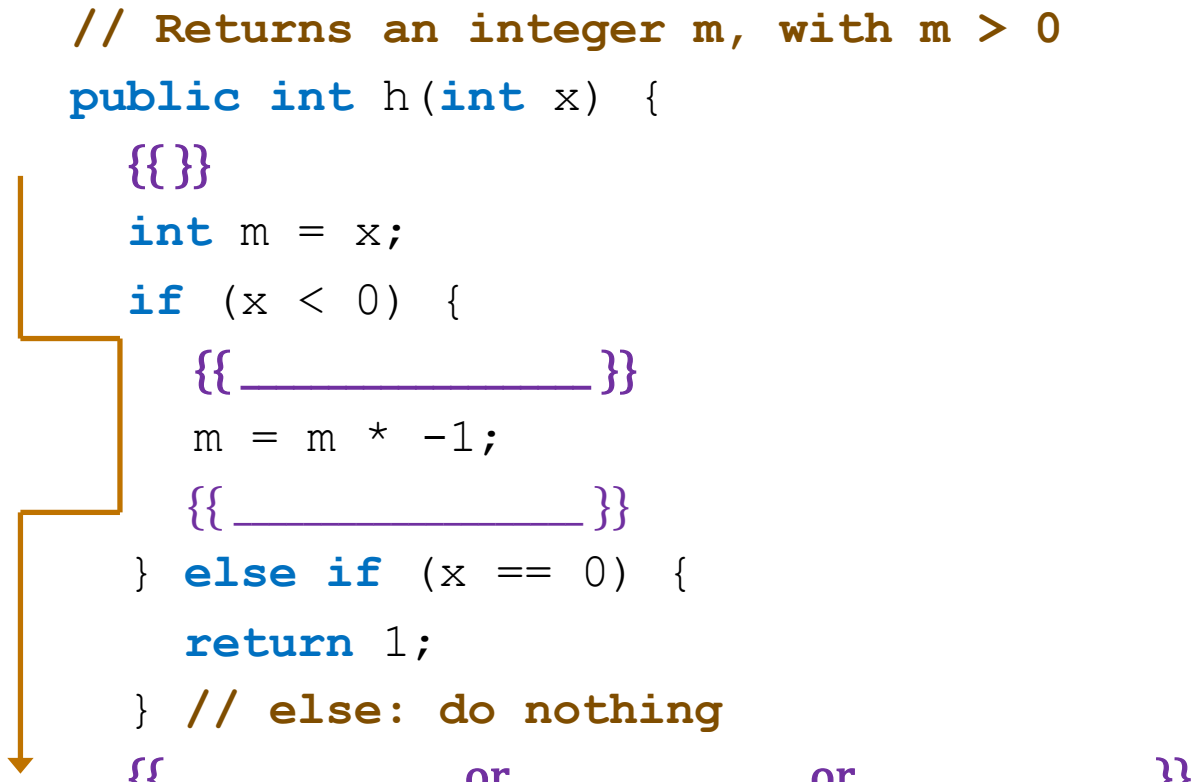
Conditionals in Floyd Logic

```
// Returns an integer m, with m > 0
public int h(int x) {
  {{ }}
  int m = x;
  if (x < 0) {
    m = m * -1;
  } else if (x == 0) {
    return 1;
  } else {
    // do nothing
  }
  {{ _____ or _____ or _____ }}
  m = m + 1;
  {{ m > 0 }}
  return m;
}
```

3 paths! `else` branch is not written out, but it's there implicitly

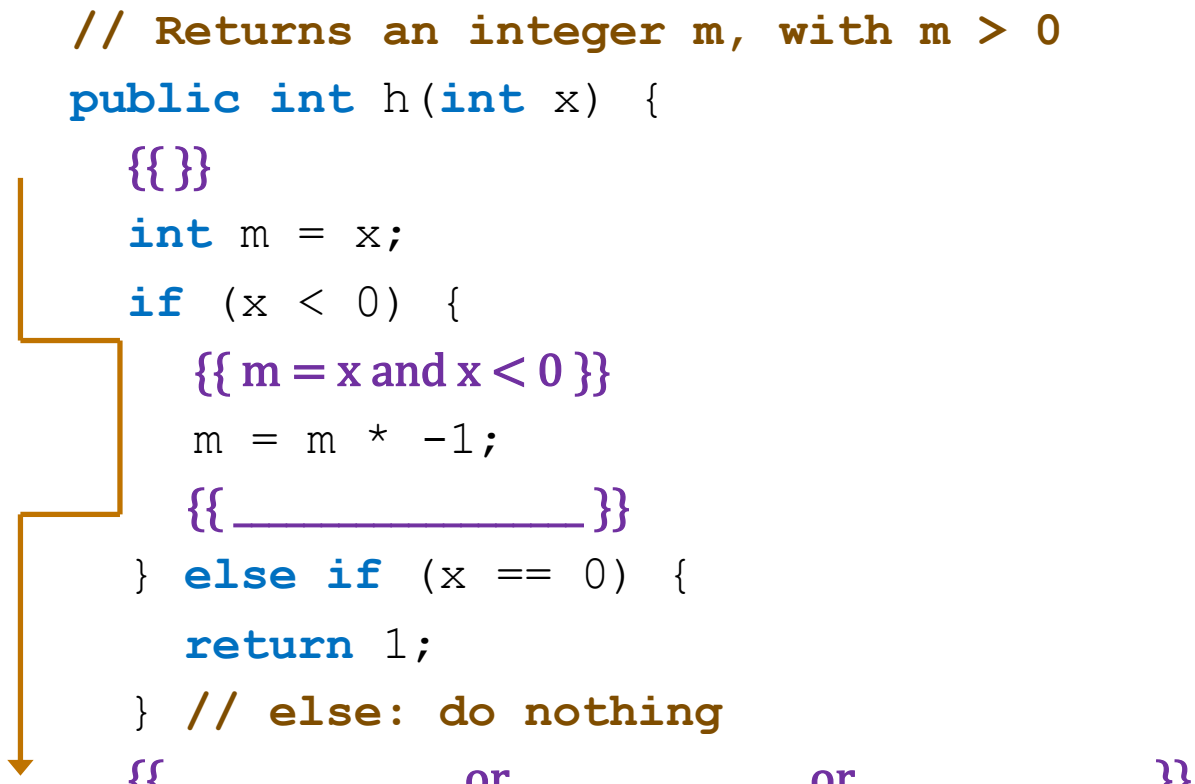
After the conditional, there are 3 sets of facts that could be true

Conditionals in Floyd Logic



```
// Returns an integer m, with m > 0
public int h(int x) {
  {{ }}
  int m = x;
  if (x < 0) {
    {{ _____ }}
    m = m * -1;
    {{ _____ }}
  } else if (x == 0) {
    return 1;
  } // else: do nothing
  {{ _____ or _____ or _____ }}
  m = m + 1;
  {{ m > 0 }}
  return m;
}
```

Conditionals in Floyd Logic



```
// Returns an integer m, with m > 0
public int h(int x) {
  {{ }}
  int m = x;
  if (x < 0) {
    {{ m = x and x < 0 }}
    m = m * -1;
    {{ _____ }}
  } else if (x == 0) {
    return 1;
  } // else: do nothing
  {{ _____ or _____ or _____ }}
  m = m + 1;
  {{ m > 0 }}
  return m;
}
```

Conditionals in Floyd Logic

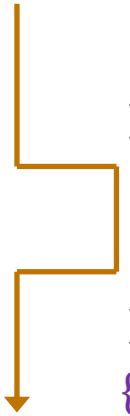
```
// Returns an integer m, with m > 0
public int h(int x) {
  {{ }}
  int m = x;
  if (x < 0) {
    {{ m = x and x < 0 }}
    m = m * -1;
    {{ m = - x and x < 0 }}
  } else if (x == 0) {
    return 1;
  } // else: do nothing
  {{ (m = - x and x < 0) or _____ or _____ }}
  m = m + 1;
  {{ m > 0 }}
  return m;
}
```

Conditionals in Floyd Logic

```
// Returns an integer m, with m > 0
public int h(int x) {
  {{ }}
  int m = x;
  if (x < 0) {
    m = m * -1;
  } else if (x == 0) {
    {{ _____ }}
    return 1;
  } // else: do nothing
  {{ (m = - x and x < 0) or _____ or _____ }}
  m = m + 1;
  {{ m > 0 }}
  return m;
}
```

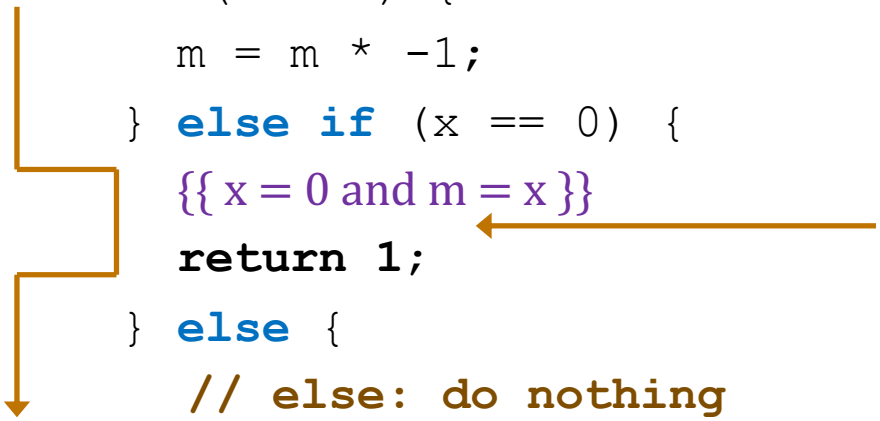
Conditionals in Floyd Logic

```
// Returns an integer m, with m > 0
public int h(int x) {
  {{ }}
  int m = x;
  if (x < 0) {
    m = m * -1;
  } else if (x == 0) {
    {{ x = 0 and m = x }}
    return 1;
  } // else: do nothing
  {{ (m = - x and x < 0) or _____ or _____ }}
  m = m + 1;
  {{ m > 0 }}
  return m;
}
```



Conditionals in Floyd Logic

```
// Returns an integer m, with m > 0
public int h(int x) {
  {{ }}
  int m = x;
  if (x < 0) {
    m = m * -1;
  } else if (x == 0) {
    {{ x = 0 and m = x }}
    return 1;
  } else {
    // else: do nothing
  }
  {{ (m = - x and x < 0) or (x = 0 and m = x and false) or _____ }}
  m = m + 1;
  {{ m > 0 }}
  return m;
}
```



Must prove that post condition holds here

false: no states can reach beyond return

Conditionals in Floyd Logic

```
// Returns an integer m, with m > 0
public int h(int x) {
  {{ }}
  int m = x;
  if (x < 0) {
    m = m * -1;
  } else if (x == 0) {
    return 1;
  } // else: do nothing

  {{ (m = - x and x < 0) or _____ }}
  m = m + 1;
  {{ m > 0 }}
  return m;
}
```

What do we know in implicit else case?
When *neither* of the then cases were entered

Conditionals in Floyd Logic

```
// Returns an integer m, with m > 0
public int h(int x) {
  {{ }}
  int m = x;
  if (x < 0) {
    m = m * -1;
  } else if (x == 0) {
    return 1;
  } // else: do nothing

  {{ (m = - x and x < 0) or (x > 0 and m = x) }}
  m = m + 1;
  {{ m > 0 }}
  return m;
}
```

Conditionals in Floyd Logic

```
// Returns an integer m, with m > 0
public int h(int x) {
  {{ }}
  int m = x;
  if (x < 0) {
    m = m * -1;
  } else if (x == 0) {
    return 1;
  } // else: do nothing
  {{ (m = - x and x < 0) or (x > 0 and m = x) }}
  {{ _____ }}
  m = m + 1;
  {{ m > 0 }}
  return m;
}
```

Can reason backward and forward and meet in the middle

Conditionals in Floyd Logic

```
// Returns an integer m, with m > 0
public int h(int x) {
  {{ }}
  int m = x;
  if (x < 0) {
    m = m * -1;
  } else if (x == 0) {
    return 1;
  } // else: do nothing
  {{ (m = - x and x < 0) or (x > 0 and m = x) }}
  {{ m + 1 > 0 }}
  m = m + 1;
  {{ m > 0 }}
  return m;
}
```

check this implication

Does the set of facts we know at this point in the program satisfy what must be true to reach our post condition

Conditionals in Floyd Logic

- Prove by cases

  {{ (m = - x and x < 0) or (x > 0 and m = x) }}
  {{ m + 1 > 0 }}

  Case 1: m = - x and x < 0

  m + 1 = -x + 1 since m = -x
  > 1 since x < 0
  > 0

  Case 2: x > 0 and m = x

  m + 1 = x + 1 since m = x
  > 1 since x > 0
  > 0

- Already proved for the branch with the return, so proved the postcondition holds, in general

Loops

Correctness of Loops

- Assignment and condition reasoning is mechanical
- Loop reasoning **cannot** be made mechanical
  - no way around this

    (**311 alert**: this follows from Rice's Theorem)
- Thankfully, one *extra* bit of information fixes this
  - need to provide a "loop invariant"
  - with the invariant, reasoning is again mechanical

Loop Invariants

- Loop invariant is true **every time** at the top of the loop

```
{{ Inv: I }}
while (cond) {
  S
}
```

  - must be true when we get to the top the first time
  - must remain true each time execute S and loop back up

- Use "**Inv:**" to indicate a loop invariant

  otherwise, this only claims to be true the first time at the loop

Loop Invariants

- Loop invariant is true **every time** at the top of the loop

```
{{ Inv: I }}
while (cond) {
  s
}
```

  - must be true 0 times through the loop (at top the first time)
  - if true n times through, must be true $n+1$ times through

- Why do these imply it is always true?
  - follows by structural induction (on $\mathbb{N}$)

Checking Correctness with Loop Invariants

```
{{ P }}
{{ Inv: I }}
while (cond) {
  S
}
{{ Q }}
```

- How do we check validity with a loop invariant?
  - intermediate assertion splits into *three* triples to check

Checking Correctness with Loop Invariants

```
{{ P }}
{{ Inv: I }}
while (cond) {
  S
}
{{ Q }}
```

1. I holds initially

Splits correctness into three parts

1. I holds initially
2. S preserves I
3. Q holds when loop exits

Checking Correctness with Loop Invariants

```
{{ P }}
{{ Inv: I }}
while (cond) {
  {{ I and cond }}
    S
  {{ I }}
}
{{ Q }}
```

1. I holds initially

2. S preserves I

Splits correctness into three parts

1. I holds initially
2. S preserves I
3. Q holds when loop exits

Checking Correctness with Loop Invariants

```
{{ P }}
{{ Inv: I }}
while (cond) {
  {{ I and cond }}
    S
  {{ I }}
}
{{ I and not cond }}
{{ Q }}
```

1. I holds initially
2. S preserves I
3. Q holds when loop exits

Splits correctness into three parts

1.	I holds initially	implication
2.	S preserves I	forward/back then implication
3.	Q holds when loop exits	implication

Checking Correctness with Loop Invariants

```
{{ P }}
{{ Inv: I }}
while (cond) {
  S
}
{{ Q }}
```

Formally, invariant split this into three Hoare triples:

1. {{ P }} {{ I }} — **I holds initially**
2. {{ I and cond }} **S** {{ I }} — **S preserves** I
3. {{ I and not cond }} {{ Q }} — Q **holds when loop exits**

Loop Correctness Example 1

- This loop claims to calculate n^2

```
{{ }}
int j = 0;
int s = 0;
{{ Inv: s = j^2 }}
while (j != n) {
  j = j + 1;
  s = s + j + j - 1;
}
{{ s = n^2 }}
```

Easy to get this wrong!

- might be initializing "j" wrong ($j = 1$?)
- might be exiting at the wrong time ($j \neq n-1$?)
- might have the assignments in wrong order
- …

Fact that we need to check 3 implications is a strong indication that more bugs are possible.

Loop Correctness Example 1

- This loop claims to calculate n^2

```
{{ }}
int j = 0;
int s = 0;
{{ Inv: s = j^2 }}
while (j != n) {
  j = j + 1;
  s = s + j + j - 1;
}
{{ s = n^2 }}
```

Loop Idea

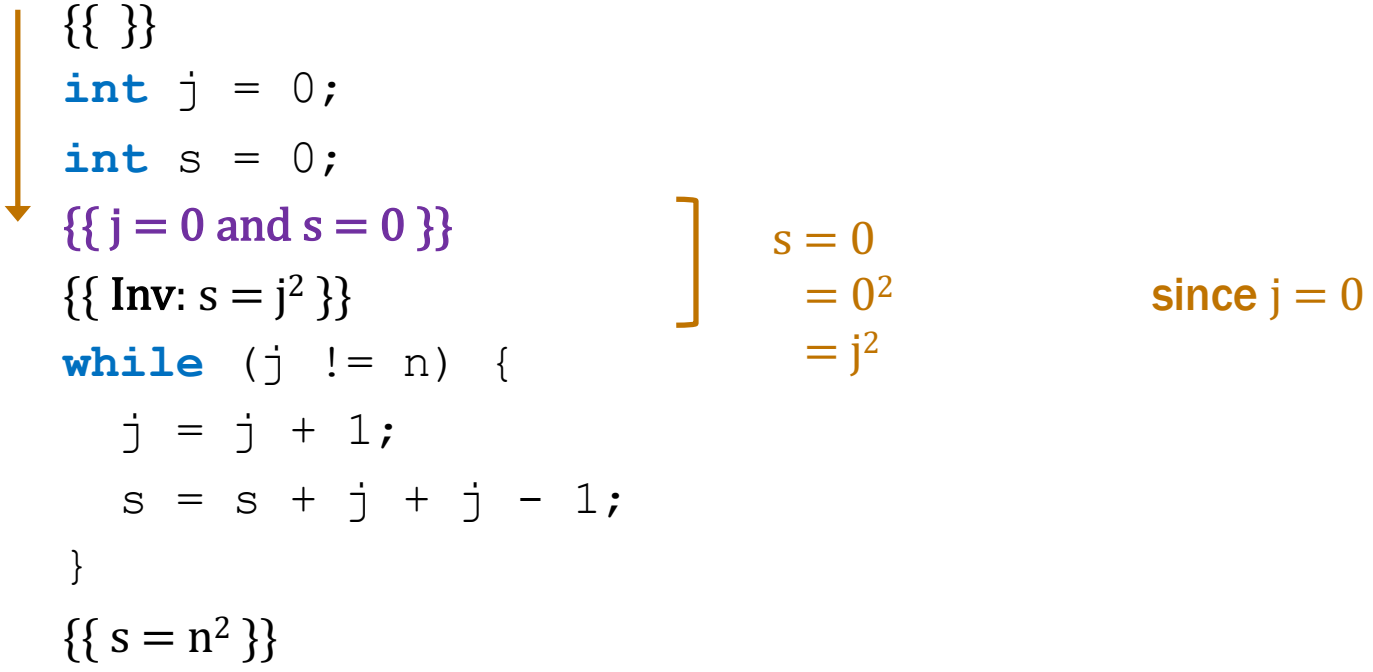
- move j from 0 to n
- keep track of j^2 in s

j	s
0	0
1	1
2	4
3	9
4	16
…	…

Loop Invariant formalizes the Loop Idea

Loop Correctness Example 1

- This loop claims to calculate n^2



Loop Correctness Example 1

- This loop claims to calculate n^2

{{ Inv: $s = j^2$ }}

```
while (j != n) {
  j = j + 1;
  s = s + j + j - 1;
}
```

{{ $s = j^2$ and $j = n$ }}

{{ $s = n^2$ }}

$s = j^2$

$= n^2$ since $j = n$

Loop Correctness Example 1

- This loop claims to calculate n^2

{{ Inv: $s = j^2$ }}

```
while (j != n) {
```

{{ $s = j^2$ and $j \neq n$ }}

```
  j = j + 1;
  s = s + j + j - 1;
```

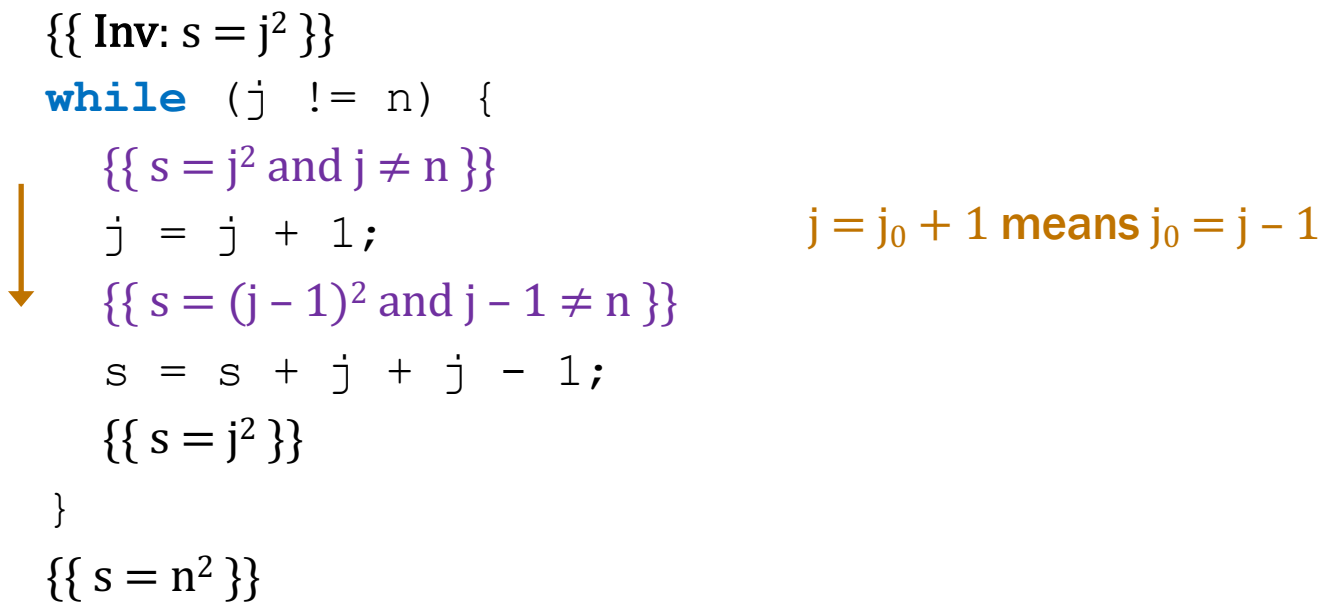
{{ $s = j^2$ }}

```
}
```

{{ $s = n^2$ }}

Loop Correctness Example 1

- This loop claims to calculate n^2



Loop Correctness Example 1

- This loop claims to calculate n^2

{{ Inv: $s = j^2$ }}
```
while (j != n) {
```
{{ $s = j^2$ and $j \neq n$ }}
```
  j = j + 1;
```
{{ $s = (j-1)^2$ and $j - 1 \neq n$ }}
```
  s = s + j + j - 1;
```
{{ $s - 2j + 1 = (j-1)^2$ and $j - 1 \neq n$ }}
{{ $s = j^2$ }}
```
}
```
{{ $s = n^2$ }}

$s = s_0 + 2j - 1$ **means** $s_0 = s - 2j + 1$

Loop Correctness Example 1

- This loop claims to calculate n^2

{{ Inv: $s = j^2$ }}
```
while (j != n) {
```
{{ $s = j^2$ and $j \neq n$ }}
```
  j = j + 1;
```
{{ $s = (j-1)^2$ and $j - 1 \neq n$ }}
```
  s = s + j + j - 1;
```
{{ $s - 2j + 1 = (j-1)^2$ and $j - 1 \neq n$ }}
{{ $s = j^2$ }}
```
}
```
{{ $s = n^2$ }}

$s = 2j - 1 + (j-1)^2$
$= 2j - 1 + j^2 - 2j + 1$
$= j^2$

since $s - 2j + 1 = (j-1)^2$

Loop Correctness Example 2

- Recursive function to calculate sum of list

$$\begin{aligned} \text{sum}(\text{nil}) &:= 0 \\ \text{sum}(x :: L) &:= x + \text{sum}(L) \end{aligned}$$

- This loop claims to calculate it as well:

$\{\{ L = L_0 \}\}$

```
int s = 0;
```

$\{\{ \textbf{Inv}: \text{sum}(L_0) = s + \text{sum}(L) \}\}$

```
while (L != null) {
  s = s + L.hd;
  L = L.tl;
}
```

$\{\{ s = \text{sum}(L_0) \}\}$

Loop Idea

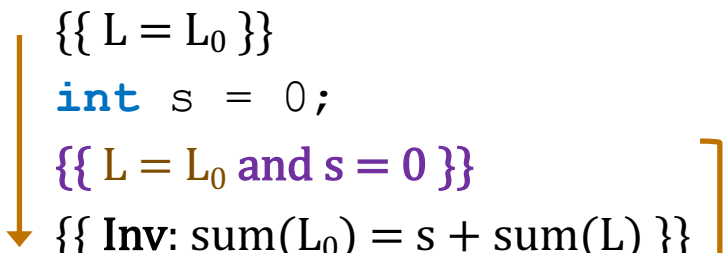
- move through L front-to-back
- keep sum of *prior* part in s

Loop Correctness Example 2

- **Recursive function to calculate sum of list**

$$\begin{aligned} \text{sum}(\text{nil}) &:= 0 \\ \text{sum}(x :: L) &:= x + \text{sum}(L) \end{aligned}$$

- **Check that the invariant holds initially**



$$\begin{aligned} &\text{sum}(L_0) \\ &= \text{sum}(L) && \textbf{since } L = L_0 \\ &= 0 + \text{sum}(L) \\ &= s + \text{sum}(L) && \textbf{since } s = 0 \end{aligned}$$

Loop Correctness Example 2

- **Recursive function to calculate sum of list**

$$\begin{aligned} \text{sum}(\text{nil}) &:= 0 \\ \text{sum}(x :: L) &:= x + \text{sum}(L) \end{aligned}$$

- **Check that the postcondition holds at loop exit**

{{ **Inv**: $\text{sum}(L_0) = s + \text{sum}(L)$ }}

```
while (L != null) {
  s = s + L.hd;
  L = L.tl;
}
```

{{ $\text{sum}(L_0) = s + \text{sum}(L)$ and $L = \text{nil}$ }}

{{ $s = \text{sum}(L_0)$ }}

$$\begin{aligned} &\text{sum}(L_0) \\ &= s + \text{sum}(L) \\ &= s + \text{sum}(\text{nil}) && \textbf{since } L = \text{nil} \\ &= s && \textbf{def of } \text{sum} \end{aligned}$$

Loop Correctness Example 2

- **Recursive function to calculate sum of list**

$$\begin{aligned} \text{sum}(\text{nil}) &:= 0 \\ \text{sum}(x :: L) &:= x + \text{sum}(L) \end{aligned}$$

- **Check that the loop body preserves the invariant**

{{ **Inv**: $\text{sum}(L_0) = s + \text{sum}(L)$ }}

```
while (L != null) {
```

{{ $\text{sum}(L_0) = s + \text{sum}(L)$ and $L \neq \text{nil}$ }}

```
  s = s + L.hd;
  L = L.tl;
```

{{ $\text{sum}(L_0) = s + \text{sum}(L)$ }}

```
}
```

$L \neq \text{nil}$ **means** $L = L.\text{hd} :: L.\text{tl}$

Loop Correctness Example 2

- **Recursive function to calculate sum of list**

$$\begin{aligned} \text{sum}(\text{nil}) &:= 0 \\ \text{sum}(x :: L) &:= x + \text{sum}(L) \end{aligned}$$

- **Check that the loop body preserves the invariant**

{{ **Inv**: $\text{sum}(L_0) = s + \text{sum}(L)$ }}

```
while (L != null) {
```

{{ $\text{sum}(L_0) = s + \text{sum}(L)$ and $L = L.hd :: L.tl$ }}

```
  s = s + L.hd;
  L = L.tl;
```

{{ $\text{sum}(L_0) = s + \text{sum}(L)$ }}

```
}
```

Loop Correctness Example 2

- **Recursive function to calculate sum of list**

$$\begin{aligned} \text{sum}(\text{nil}) &:= 0 \\ \text{sum}(x :: L) &:= x + \text{sum}(L) \end{aligned}$$

- **Check that the loop body preserves the invariant**

{{ **Inv**: $\text{sum}(L_0) = s + \text{sum}(L)$ }}

```
while (L != null) {
```

{{ $\text{sum}(L_0) = s + \text{sum}(L)$ and $L = L.hd :: L.tl$ }}

```
  s = s + L.hd;
```

{{ $\text{sum}(L_0) = s + \text{sum}(L.tl)$ }}

```
  L = L.tl;
```

{{ $\text{sum}(L_0) = s + \text{sum}(L)$ }}

```
}
```

Loop Correctness Example 2

- **Recursive function to calculate sum of list**

$$\begin{aligned} \text{sum}(\text{nil}) &:= 0 \\ \text{sum}(x :: L) &:= x + \text{sum}(L) \end{aligned}$$

- **Check that the loop body preserves the invariant**

{{ **Inv**: $\text{sum}(L_0) = s + \text{sum}(L)$ }}

```
while (L != null) {
```

{{ $\text{sum}(L_0) = s + \text{sum}(L)$ and $L = L.hd :: L.tl$ }}

{{ $\text{sum}(L_0) = s + L.hd + \text{sum}(L.tl)$ }}

```
s = s + L.hd;
```

{{ $\text{sum}(L_0) = s + \text{sum}(L.tl)$ }}

```
L = L.tl;
```

{{ $\text{sum}(L_0) = s + \text{sum}(L)$ }}

```
}
```

Loop Correctness Example 2

- **Recursive function to calculate sum of list**

$$\begin{aligned} \text{sum}(\text{nil}) &:= 0 \\ \text{sum}(x :: L) &:= x + \text{sum}(L) \end{aligned}$$

- **Check that the loop body preserves the invariant**

{{ **Inv**: $\text{sum}(L_0) = s + \text{sum}(L)$ }}
while (L != **null**) {
  {{ $\text{sum}(L_0) = s + \text{sum}(L)$ and $L = L.hd :: L.tl$ }}
  {{ $\text{sum}(L_0) = s + L.hd + \text{sum}(L.tl)$ }}
  s = s + L.hd;
  {{ $\text{sum}(L_0) = s + \text{sum}(L.tl)$ }}
  L = L.tl;
  {{ $\text{sum}(L_0) = s + \text{sum}(L)$ }}
}

$\text{sum}(L_0)$
$= s + \text{sum}(L)$
$= s + \text{sum}(L.hd :: L.tl)$ **since** $L = L.hd :: L.tl$
$= s + L.hd + \text{sum}(L.tl)$ **def of** sum

Loop Correctness Example 3

- **Recursive function to check if y appears in list L**

$$
\begin{array}{lll}
\text{contains}(y, \text{nil}) & := \text{false} & \\
\text{contains}(y, x :: L) & := \text{true} & \textbf{if } x = y \\
\text{contains}(y, x :: L) & := \text{contains}(y, L) & \textbf{if } x \neq y
\end{array}
$$

- **This loop claims to calculate it as well:**

```
{{ Inv: contains(y, L0) = contains(y, L) }}
while (L != null) {
  if (L.hd == y)
    return true;
  L = L.tl;
}
return false;
```

Loop Idea

- move through L front-to-back
- answer remains the same as on the original list L_0
- can only do that if y is *not* found

Loop Correctness Example 3

- **Check that the invariant holds initially**

{{ $L_0 = L$ }}
{{ **Inv**: contains(y, L_0) = contains(y, L) }}

```
while (L != null) {
  if (L.hd == y)
    return true;
  L = L.tl;
}
return false;
```

contains(y, L_0)
= contains(y, L) **since** $L_0 = L$

contains(y, nil) := false
contains(y, x :: L) := true **if** x = y
contains(y, x :: L) := contains(y, L) **if** x ≠ y

Loop Correctness Example 3

- **Check that the invariant implies the postcondition**

{{ **Inv**: contains(y, L_0) = contains(y, L) }}

```
while (L != null) {
  if (L.hd == y)
    return true;
  L = L.tl;
}
```

{{ contains(y, L_0) = contains(y, L) and L = nil }}

{{ contains(y, L_0) = false }}

```
return false;
```

contains(y, L_0)
= contains(y, L)
= contains(y, nil) **since** L = nil
= false **def of** contains

contains(y, nil) := false
contains(y, x :: L) := true **if** x = y
contains(y, x :: L) := contains(y, L) **if** x ≠ y

Loop Correctness Example 3

- **Check that the body preserves the invariant**

{{ **Inv**: contains(y, L_0) = contains(y, L) }}

```
while (L != null) {
```

{{ contains(y, L_0) = contains(y, L) and L ≠ nil }}

```
  if (L.hd == y)
    return true;
  L = L.tl;
```

{{ contains(y, L_0) = contains(y, L) }}

```
}
return false;
```

L ≠ nil **means** L = L.hd :: L.tl

contains(y, nil) := false
contains(y, x :: L) := true **if** x = y
contains(y, x :: L) := contains(y, L) **if** x ≠ y

Loop Correctness Example 3

- **Check that the body preserves the invariant**

{{ **Inv**: contains(y, L_0) = contains(y, L) }}
```
while (L != null) {
```
{{ contains(y, L_0) = contains(y, L) and L = L.hd :: L.tl }}
```
  if (L.hd == y)
```
{{ contains(y, L_0) = contains(y, L) and L = L.hd :: L.tl and L.hd = y }}
{{ contains(y, L_0) = true }}
```
    return true;
  L = L.tl;
```
{{ contains(y, L_0) = contains(y, L) }}
```
}
return false;
```

contains(y, L_0)
= contains(y, L)
= contains(y, L.hd :: L.tl) **since** L = L.hd :: L.tl
= true **since** y = L.hd

contains(y, nil) := false
contains(y, x :: L) := true **if** x = y
contains(y, x :: L) := contains(y, L) **if** x ≠ y

Loop Correctness Example 3

- **Check that the body preserves the invariant**

{{ **Inv**: contains(y, L_0) = contains(y, L) }}
```
while (L != null) {
```
{{ contains(y, L_0) = contains(y, L) and L = L.hd :: L.tl }}
```
  if (L.hd == y)
```
{{ contains(y, L_0) = true }}
```
    return true;
```
{{ contains(y, L_0) = contains(y, L) and L = L.hd :: L.tl and L.hd ≠ y }}
```
  L = L.tl;
```
{{ contains(y, L_0) = contains(y, L) }}
```
}
return false;
```

contains(y, nil) := false
contains(y, x :: L) := true **if** x = y
contains(y, x :: L) := contains(y, L) **if** x ≠ y

Loop Correctness Example 3

- **Check that the body preserves the invariant**

{{ **Inv**: contains(y, L_0) = contains(y, L) }}
```
while (L != null) {
```
{{ contains(y, L_0) = contains(y, L) and L = L.hd :: L.tl }}
```
  if (L.hd == y)
```
{{ contains(y, L_0) = true }}
```
    return true;
```
{{ contains(y, L_0) = contains(y, L) and L = L.hd :: L.tl and L.hd ≠ y }}
{{ contains(y, L_0) = contains(y, L.tl) }}
```
  L = L.tl;
```
{{ contains(y, L_0) = contains(y, L) }}
```
}
return false;
```

contains(y, nil) := false
contains(y, x :: L) := true **if** x = y
contains(y, x :: L) := contains(y, L) **if** x ≠ y

contains(y, L_0)
= contains(y, L)
= contains(y, L.hd :: L.tl) **since** L = L.hd :: L.tl
= contains(y, L.tl) **since** y ≠ L.hd

Loop Correctness Example 4

- **Declarative spec of** $\mathrm{sqrt}(x)$

  return $y \in \mathbb{Z}$ **such that** $(y-1)^2 < x \leq y^2$

  - **precondition that** x **is positive:** $0 < x$
  - **precondition that x is not too large:** $x < 10^{12} = (10^6)^2$

Loop Correctness Example 4

return $y \in \mathbb{Z}$ such that $(y-1)^2 < x \leq y^2$

- This loop claims to calculate it:

```
int a = 0;
int b = 1000000;
{{ Inv: a^2 < x ≤ b^2 }}
while (a != b - 1) {
  int m = (a + b) / 2;
  if (m*m < x) {
    a = m;
  } else {
    b = m;
  }
}
return b;
```

Loop Idea

- maintain a range $a \ldots b$ with x in the range $a^2 \ldots b^2$

Loop Correctness Example 4

return $y \in \mathbb{Z}$ such that $(y-1)^2 < x \leq y^2$

- **Check that the invariant holds initially:**

{{ **Pre:** $0 < x \leq 10^{12}$ }}

```
int a = 0;
int b = 1000000;
```

{{ **Inv:** $a^2 < x \leq b^2$ }}

```
while (a != b - 1) {
  …
}
return b;
```

Loop Correctness Example 4

return $y \in \mathbb{Z}$ **such that** $(y-1)^2 < x \leq y^2$

- **Check that the invariant holds initially:**

{{ **Pre:** $0 < x \leq 10^{12}$ }}

```
int a = 0;
int b = 1000000;
```

{{ $0 < x \leq 10^{12}$ and $a = 0$ and $b = 10^6$ }}

{{ **Inv:** $a^2 < x \leq b^2$ }}

```
while (a != b - 1) {
  …
}
return b;
```

$a^2 = 0^2$ **since** $a = 0$
$= 0$
$< x$

$x < 10^{12}$
$= (10^6)^2$
$= b^2$ **since** $b = 10^6$

Loop Correctness Example 4

return $y \in \mathbb{Z}$ **such that** $(y-1)^2 < x \leq y^2$

- **Check that the postcondition hold after exit**

{{ **Inv:** $a^2 < x \leq b^2$ }}
while (a != b - 1) {
  …
}
{{ $a^2 < x \leq b^2$ **and** $a = b - 1$ }}
{{ $(b-1)^2 < x \leq b^2$ }}
return b;

Does $(y-1)^2 < x < y^2$ **hold with** $y = b$**?**

$(b-1)^2$
$= a^2$ **since** $a = b - 1$
$< x$

Loop Correctness Example 4

return $y \in \mathbb{Z}$ **such that** $(y-1)^2 < x \leq y^2$

- **Check that the body preserves the invariant:**

```
{{ Inv: a² < x ≤ b² }}
while (a != b - 1) {
  {{ a² < x ≤ b² and a ≠ b - 1 }}
  int m = (a + b) / 2;
  if (m*m < x) {
    a = m;
  } else {
    b = m;
  }
  {{ a² < x ≤ b² }}
}
```

Loop Correctness Example 4

return $y \in \mathbb{Z}$ **such that** $(y-1)^2 < x \leq y^2$

- **Check that the body preserves the invariant:**

```
{{ Inv: a² < x ≤ b² }}
while (a != b - 1) {
  {{ a² < x ≤ b² and a ≠ b – 1 }}
  int m = (a + b) / 2;
  if (m*m < x) {
    {{ a² < x ≤ b² and a ≠ b – 1 and m² < x }}
    a = m;
  } else {
    {{ a² < x ≤ b² and a ≠ b – 1 and x ≤ m² }}
    b = m;
  }
  {{ a² < x ≤ b² }}
}
```

Loop Correctness Example 4

return $y \in \mathbb{Z}$ **such that** $(y-1)^2 < x \leq y^2$

- **Check that the body preserves the invariant:**

```
{{ Inv: a² < x ≤ b² }}
while (a != b - 1) {
  int m = (a + b) / 2;
  if (m*m < x) {
    {{ a² < x ≤ b² and a ≠ b – 1 and m² < x }}
    {{ m² < x ≤ b² }}
    a = m;
  } else {
    {{ a² < x ≤ b² and a ≠ b – 1 and x ≤ m² }}
    b = m;
  }
  {{ a² < x ≤ b² }}
}
```

Immediate!

Loop Correctness Example 4

return $y \in \mathbb{Z}$ **such that** $(y-1)^2 < x \leq y^2$

- **Check that the body preserves the invariant:**

```
{{ Inv: a² < x ≤ b² }}
while (a != b - 1) {
  int m = (a + b) / 2;
  if (m*m < x) {
    a = m;
  } else {
    {{ a² < x ≤ b²  and a ≠ b – 1 and x ≤ m² }}
    {{ a² < x ≤ m² }}
    b = m;
  }
  {{ a² < x ≤ b² }}
}
```

Immediate!

Correctness of binary search is pretty easy *once* you have the invariant clear!

Termination

- This analysis does not check that the code **terminates**
  - it shows that the postcondition holds if the loop exits
  - but we never showed that the loop does exit

- Termination follows from the running time analysis
  - e.g., if the code runs in $O(n^2)$ time, then it terminates
  - an infinite loop would be O(infinity)
  - any finite bound on the running time proves it terminates

- Normal to also analyze the running time of our code, and we get termination already from that analysis

Correctness of Loops

- With straight-line code and conditionals, if the triple is not valid…
  - the code is **wrong**
  - there is *some* test case that will prove it

    (doesn't mean we found that case in our tests, but it exists)

- With loops, if the triples are not valid…
  - the code is **wrong** *with that invariant*
  - there may **not** be any test case that proves it

    the code may behave correctly on all inputs
  - the code could be right but with a *different* invariant

- Loops are inherently more complicated