CSE 331
Software Design & Implementation

Autumn 2025

Section 2 - Reasoning

Administrivia

- HW2 will be released later tonight and is due **Friday @ 6pm!**

Proof By Calculation - Review

- Proving implications is the **core step** of reasoning
- Uses known facts and definitions (ex: len(nil) = 0)
  - Written in our math notation!
- Start from the left side of the inequality to be proved
- Chain of "=" shows first = last
- Chain of "=" and "≤" shows first ≤ last
- Directly cite the definition of a function

Proof By Calculation Reminders

- The goal of proof by calculation is to *show* that an assertion is true *given* facts that you already know
- You should ***start*** the proof with either the left or the right side of the assertion and ***end*** the proof with the other side of the assertion.
- Every symbol (=, >, <, etc.) connecting each line of the proof is the current line's relationship to the previous line in the proof (not any other lines)
- Only modify one side
- **Every** line requires justification (except for algebraic manipulations)

Proof By Calculation - Example

```
// Inputs x and y are positive integers
// Returns a positive integer.
public static int f(int x, int y) {
  return x * y;
};
```

- Known facts "$x \geq 1$" and "$y \geq 1$"
- Correct if the return value is a positive integer

$$
\begin{aligned}
x * y &\geq x * 1 \quad \text{since } y \geq 1 \\
&\geq 1 * 1 \quad \text{since } x \geq 1 \\
&= 1
\end{aligned}
$$

- Calculation shows that $x * y \geq 1$

Proof By Calculation - Citing Functions

$$\begin{aligned} \text{sum}(\text{nil}) &:= 0 \\ \text{sum}(x :: L) &:= x + \text{sum}(L) \end{aligned}$$

- Know "$a \geq 0$", "$b \geq 0$", and "$L = a :: b :: \text{nil}$"

- Prove the "sum(L)" is non-negative

$$\begin{aligned} \text{sum}(L) &= \text{sum}(a :: b :: \text{nil}) && \text{since } L = a :: b :: \text{nil} \\ &= a + \text{sum}(b :: \text{nil}) && \text{def of sum} \\ &= a + b + \text{sum}(\text{nil}) && \text{def of sum} \\ &= a + b && \text{def of sum} \\ &\geq 0 + b && \text{since } a \geq 0 \\ &\geq 0 && \text{since } b \geq 0 \end{aligned}$$

Proof By Calculation Bad Example

Suppose we have the facts: $x = 3$, $y = 4$, $z > 5$ and we want to use proof by calculation to prove $x^2 + y^2 < z^2$. Our proof by calculation would look like this:

$x^2 + y^2$	$< z^2$	beginning of a backwards proof
0	$< z^2 - x^2 - y^2$	
0	$< z^2 - 3^2 - y^2$	since $x = 3$
0	$< z^2 - 3^2 - 4^2$	since $y = 4$
0	$< z^2 - 25$	
25	$< z^2$	
5	$< \|z\|$	

Since $z > 5$, we know $x^2+y^2<z^2$ by above.

Manipulates both sides of the equation

Not a single chain of equalities

doesn't end with right side of the assertion (z^2)

What is wrong with this proof?

Proof by Calculation Bug: Explanation

The previous proof is an example of *Circular Reasoning*. We begin the proof with the conclusion manipulating both sides until we reach one of the given facts.

Just because we can prove one direction does **not** mean the other direction necessarily holds.

We must always start from what we know and end with what we want to prove.

Facts → Conclusion ✔

Conclusion → Facts ✖

Proof By Calculation Example Correct

Suppose we have the facts: $x = 3$, $y = 4$, $z > 5$ and we want to use proof by calculation to prove $x^2 + y^2 < z^2$. Our proof by calculation would look like this:

$x^2 + y^2$	$= 3^2 + y^2$	since $x = 3$
	$= 3^2 + 4^2$	since $y = 4$
	$\boxed{= 25}$	
	$= 5^2$	
	$< z^2$	$\boxed{\text{since } z > 5}$

note that each line shows the relationship *only* to the previous line

start with left side of assertion

end with right side of assertion

note that every line has justification (except for algebraic manipulations)

Task 1a - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

a) Given that $a = 1$ and $b = 1$, it follows that $a = 2b - 1$.

Task 1a - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

a) Given that $a = 1$ and $b = 1$, it follows that $a = 2b - 1$.

$$a = 1$$

Task 1a - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

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a) Given that $a = 1$ and $b = 1$, it follows that $a = 2b - 1$.

$$\begin{aligned} a &= 1 \\ &= 2 - 1 \end{aligned}$$

Task 1a - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

a) Given that $a = 1$ and $b = 1$, it follows that $a = 2b - 1$.

$$\begin{aligned} a &= 1 \\ &= 2 - 1 \\ &= 2 \cdot 1 - 1 \end{aligned}$$

Task 1a - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

a) Given that $a = 1$ and $b = 1$, it follows that $a = 2b - 1$.

$$\begin{aligned} a &= 1 \\ &= 2 - 1 \\ &= 2 \cdot 1 - 1 \\ &= 2b - 1 && \text{since } b = 1 \end{aligned}$$

Task 1b - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

b) Given that $a = 1$, $b = 2a - 1$, and $c > 0$, it follows that $(b-1)^2 < c$.

Task 1b - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

b) Given that $a = 1$, $b = 2a - 1$, and $c > 0$, it follows that $(b-1)^2 < c$.

$(b-1)^2 = (2a-1-1)^2$ since $b = 2a - 1$

Task 1b - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

b) Given that $a = 1$, $b = 2a - 1$, and $c > 0$, it follows that $(b-1)^2 < c$.

$$\begin{aligned} (b-1)^2 &= (2a-1-1)^2 \quad \text{since } b = 2a-1 \\ &= (2a-2)^2 \end{aligned}$$

Task 1b - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

b) Given that $a = 1$, $b = 2a - 1$, and $c > 0$, it follows that $(b-1)^2 < c$.

$$\begin{aligned}(b-1)^2 &= (2a-1-1)^2 && \text{since } b = 2a-1\\ &= (2a-2)^2 \\ &= (2\cdot 1-2)^2 && \text{since } a = 1\end{aligned}$$

Task 1b - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

b) Given that $a = 1$, $b = 2a - 1$, and $c > 0$, it follows that $(b-1)^2 < c$.

$$\begin{aligned}
(b-1)^2 &= (2a-1-1)^2 && \text{since } b = 2a-1 \\
&= (2a-2)^2 \\
&= (2 \cdot 1 - 2)^2 && \text{since } a = 1 \\
&= 0^2 \\
&= 0
\end{aligned}$$

Task 1b - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

b) Given that $a = 1$, $b = 2a - 1$, and $c > 0$, it follows that $(b-1)^2 < c$.

$$\begin{aligned}
(b-1)^2 &= (2a-1-1)^2 && \text{since } b = 2a-1 \\
&= (2a-2)^2 \\
&= (2 \cdot 1 - 2)^2 && \text{since } a = 1 \\
&= 0^2 \\
&= 0 \\
&< c && \text{since } c > 0
\end{aligned}$$

Task 1c - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

c) Given that $d = b + 1$, $c = a - 8$, and $e = a + 8b$, it follows that $e = c + 8d$.

Task 1c - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

c) Given that $d = b + 1$, $c = a - 8$, and $e = a + 8b$, it follows that $e = c + 8d$.

$$e = a + 8b$$

Task 1c - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

c) Given that $d = b + 1$, $c = a - 8$, and $e = a + 8b$, it follows that $e = c + 8d$.

$$\begin{aligned} e &= a + 8b \\ &= c + 8 + 8b & \text{since } a = c + 8 \end{aligned}$$

Task 1c - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

c) Given that $d = b + 1$, $c = a - 8$, and $e = a + 8b$, it follows that $e = c + 8d$.

$$\begin{aligned} e &= a + 8b \\ &= c + 8 + 8b && \text{since } a = c + 8 \\ &= c + 8 + 8(d - 1) && \text{since } b = d - 1 \end{aligned}$$

Task 1c - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

c) Given that $d = b + 1$, $c = a - 8$, and $e = a + 8b$, it follows that $e = c + 8d$.

$$\begin{aligned} e &= a + 8b \\ &= c + 8 + 8b && \text{since } a = c + 8 \\ &= c + 8 + 8(d - 1) && \text{since } b = d - 1 \\ &= c + 8 + 8d - 8 \\ &= c + 8d \end{aligned}$$

Task 1d - Calc You Later

Let a, b, c, d, e be integers. Complete each of the following proofs **by calculation**.

Include an explanation on each step where a given fact is used. You can skip such an explanation only if the claim written in that step is itself, *literally* a known fact. It is also fine to cite a fact that is equivalent (via simple algebra) to a known fact.

d) Given that $b = 2a - 1$, $d = a^2$, and $d + b + 2 < c$, it follows that $(a+1)^2 < c$.

$$\begin{aligned}
(a+1)^2 &= a^2 + 2a + 1 \\
&= a^2 + 2a - 1 + 2 \\
&= a^2 + b + 2 && \text{since } b = 2a - 1 \\
&= d + b + 2 && \text{since } d = a^2 \\
&< c && \text{since } d + b + 2 < c
\end{aligned}$$

Defining Function By Cases – Review

- Sometimes we want to define functions by cases
  - **Ex:** define $f(\mathrm{n})$ where $\mathrm{n} : \mathbb{Z}$

$$
\begin{aligned}
\mathrm{f}(\mathrm{n}) &:= 2\mathrm{n} + 1 && \textbf{if } \mathrm{n} \geq 0 \\
\mathrm{f}(\mathrm{n}) &:= 0 && \textbf{if } \mathrm{n} < 0
\end{aligned}
$$

  - To use the definition $f(\mathrm{n})$, we need to know if $\mathrm{n} > 0$ or not
  - This new code structure requires a new proof structure

Proof By Cases – Review

- Split a proof into cases:
  - **Ex:** $\mathrm{a} = \mathrm{True}$ and $\mathrm{a} = \mathrm{False}$ or $\mathrm{n} >= 0$ and $\mathrm{n} < 0$
  - These cases needs to be *exhaustive*

- **Ex:** $\mathrm{f(n)} := 2\mathrm{n} + 1$ if $\mathrm{n} \geq 0$
  $\mathrm{f(n)} := 0$ if $\mathrm{n} < 0$

  Prove that $\mathrm{f(n)} \geq \mathrm{n}$ **for any** $\mathrm{n} : \mathbb{Z}$

Case $\mathbf{n} \geq \mathbf{0}$:

$f(\mathrm{n}) = 2\mathrm{n} + 1$	def of f (since $n \geq 0$)
$> \mathrm{n}$	since $n \geq 0$

Case $\mathbf{n} < \mathbf{0}$:

$f(\mathrm{n}) = 0$	def of f (since $n < 0$)
$\geq \mathrm{n}$	since $n < 0$

Since these 2 cases are *exhaustive*, $f(\mathrm{n}) >= \mathrm{n}$ holds in general

Task 2 - Absolutely Positive

Let x be an integer and L a list. Complete the following proof **by cases**.

Let abs : $\mathbb{Z} \to \mathbb{Z}$ be defined as follows:

$$\begin{aligned} \mathsf{abs}(x) &= -x \quad && \textbf{if } x < 0 \\ \mathsf{abs}(x) &= x && \textbf{if } x \geqslant 0 \end{aligned}$$

Prove that $\mathsf{abs}(\mathsf{abs}(x)) = \mathsf{abs}(x)$.

Task 2 - Absolutely Positive

Let abs : $\mathbb{Z} \to \mathbb{Z}$ be defined as follows:

$$\begin{aligned} \text{abs}(x) &= -x \quad &&\textbf{if } x < 0 \\ \text{abs}(x) &= x &&\textbf{if } x \geqslant 0 \end{aligned}$$

Prove that $\text{abs}(\text{abs}(x)) = \text{abs}(x)$.

Suppose that $x < 0$. Then, we can see that

$$\begin{aligned} \text{abs}(\text{abs}(x)) &= \text{abs}(-x) &&\text{def of abs (since } x < 0) \\ &= -x &&\text{def of abs (since } -x > 0) \\ &= \text{abs}(x) &&\text{def of abs (since } x < 0) \end{aligned}$$

Now, suppose that $x \geqslant 0$. Then we can see that

$$\text{abs}(\text{abs}(x)) = \text{abs}(x) \quad \text{def of abs (since } x \geqslant 0)$$

These two cases are exhaustive, so we have proven the claim holds in general.

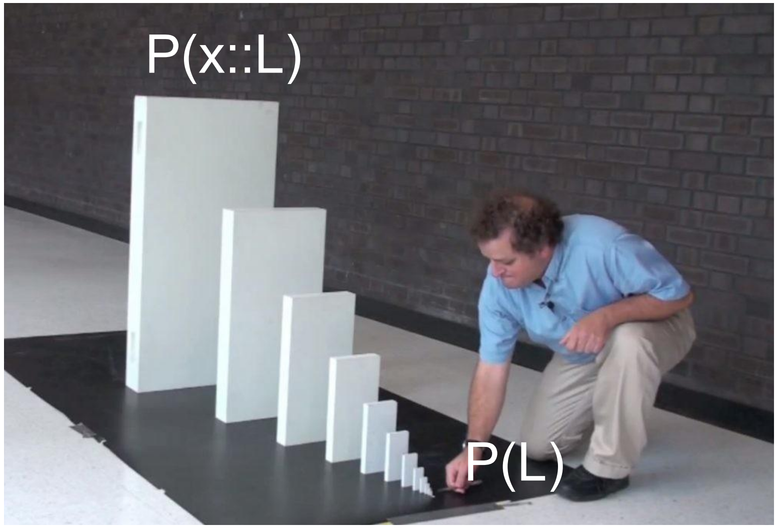
Structural Induction – Review

- Let **P(S)** be the claim
- To Prove P(S) holds for any list S, we need to prove two implications: base case and inductive case
  - **Base Case**: prove P(nil)
    - Use any known facts and definitions
  - **Inductive Hypothesis**: assume **P(L)** is true for a L: List
    - Use this in the inductive step ONLY ⤵️
  - **Inductive Step**: prove **P(x :: L)** for any x : *Z,* L : List
    - Direct proof
    - Use known facts and definitions and Inductive Hypothesis
- Assuming we know P(L), if we prove P(x :: L), we then prove recursively that P(S) holds for any List

Structural Induction - 331 Format

The following is the structural induction format we recommend for using in your homework (the staff solution also follows this format)

1) **Introduction** - define P(S) to be what we are trying to prove
2) **Base Case** - show P(nil) holds
3) **Inductive Hypothesis** - assume P(L) is true for an arbitrary list
4) **Inductive Step** - show P(x :: L) holds
5) **Conclusion** - “We have shown that P(S) holds for any list”



Task 3 - Keeping It Cool

The functions keep, skip : (List) $\rightarrow$ List remove half the elements in a list. These two functions keep and skip every other element of the passed in list. The keep function includes the first element but skips the one after it, while skip drops the first element but keeps the one after that. They are defined formally as follows:

$$\mathsf{keep}(\mathsf{nil}) = \mathsf{nil}$$
$$\mathsf{keep}(x :: L) = x :: \mathsf{skip}(L)$$

$$\mathsf{skip}(\mathsf{nil}) = \mathsf{nil}$$
$$\mathsf{skip}(x :: L) = \mathsf{keep}(L)$$

Also, recall the function echo : (List) $\rightarrow$ List, which was defined in class as follows:

$$\mathsf{echo}(\mathsf{nil}) = \mathsf{nil}$$
$$\mathsf{echo}(x :: L) = x :: x :: \mathsf{echo}(L)$$

and the function sum : (List) $\rightarrow \mathbb{N}$, which was defined in class as follows:

$$\begin{aligned} \mathsf{sum}(\mathsf{nil}) &:= 0 \\ \mathsf{sum}(x :: L) &:= x + \mathsf{sum}(L) \end{aligned}$$

You will use these functions in the problem below.

Prove that $\mathsf{sum}(\mathsf{skip}(\mathsf{echo}(L))) = \mathsf{sum}(L)$ holds by structural induction on L.

Task 3 - Keeping It Cool

Prove that $\mathsf{sum}(\mathsf{skip}(\mathsf{echo}(L))) = \mathsf{sum}(L)$ holds by structural induction on L.

Define $P(L)$ to be the claim $\mathsf{sum}(\mathsf{skip}(\mathsf{echo}(L))) = \mathsf{sum}(L)$. We will prove that this holds for all values of L by structural induction.

Base Case. We can see that $P(\mathsf{nil})$ holds as follows:

$$\begin{aligned} \mathsf{sum}(\mathsf{skip}(\mathsf{echo}(\mathsf{nil}))) &= \mathsf{sum}(\mathsf{skip}(\mathsf{nil})) && \text{def of echo} \\ &= \mathsf{sum}(\mathsf{nil}) && \text{def of skip} \end{aligned}$$

Inductive Hypothesis. Suppose that $P(L)$ holds for some list of integers L.

Inductive Step. Let x be an arbitrary integer. We can see that $P(x :: L)$ holds as follows:

$$\begin{aligned} \mathsf{sum}(\mathsf{skip}(\mathsf{echo}(x :: L))) &= \mathsf{sum}(\mathsf{skip}(x :: x :: \mathsf{echo}(L))) && \text{def of echo} \\ &= \mathsf{sum}(\mathsf{keep}(x :: \mathsf{echo}(L))) && \text{def of skip} \\ &= \mathsf{sum}(x :: \mathsf{skip}(\mathsf{echo}(L))) && \text{def of keep} \\ &= x + \mathsf{sum}(\mathsf{skip}(\mathsf{echo}(L))) && \text{def of sum} \\ &= x + \mathsf{sum}(L) && \text{Ind. Hyp.} \\ &= \mathsf{sum}(x :: L) && \text{def of sum} \end{aligned}$$

Conclusion. $P(L)$ holds for all lists of integers L by structural induction.

Attendance

https://tinyurl.com/25ausec2