CSE 331

Arrays

Kevin Zatloukal

Indexing

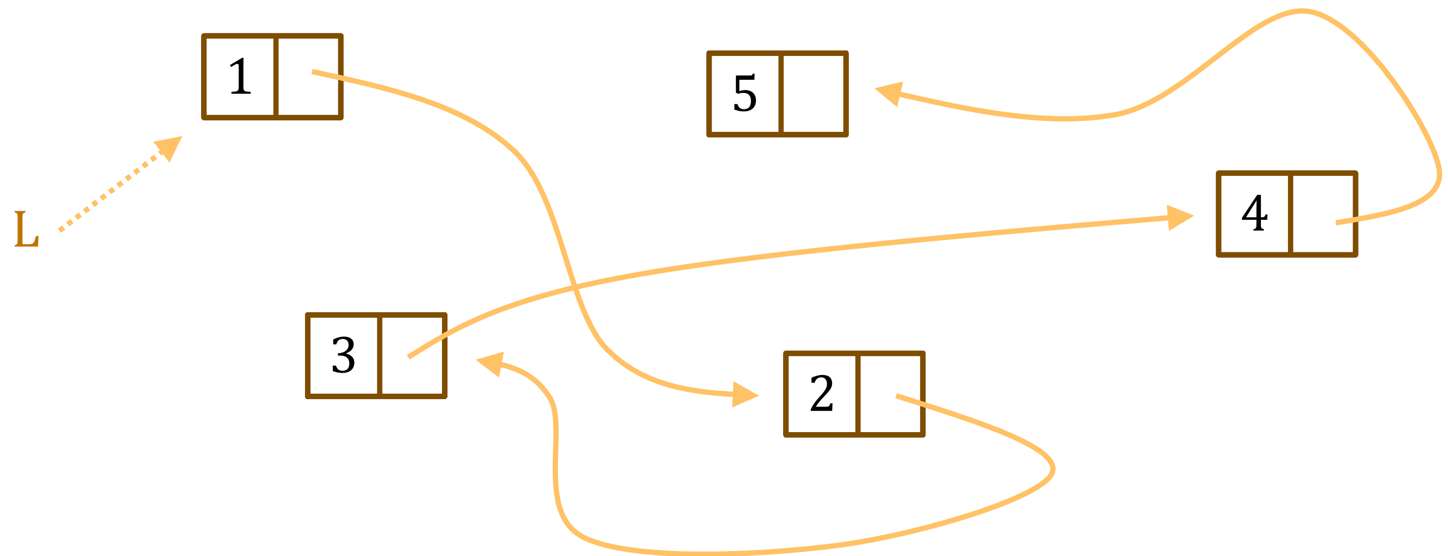
$$\text{at} : (\text{List}, \mathbb{N}) \to \mathbb{Z}$$

$$\begin{aligned}
&\text{at}(\text{nil}\,,\, \text{n}) &&:= \text{ undefined} \\
&\text{at}(\text{x} :: \text{L}\,,\, 0) &&:= \text{ x} \\
&\text{at}(\text{x} :: \text{L}\,,\, \text{n}+1) &&:= \text{ at}(\text{L}, \text{n})
\end{aligned}$$

- **Retrieve an element of the list by <u>index</u>**
  - **use "$\text{L}[\text{j}]$" as an abbreviation for $\text{at}(\text{j}, \text{L})$**

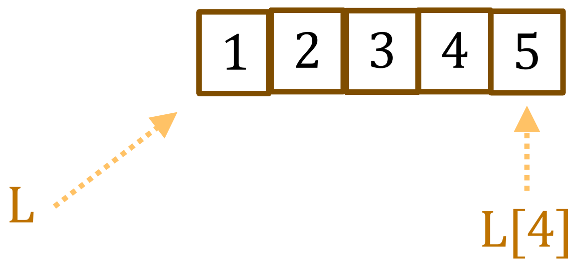
- **Not an efficient operation on lists…**

Linked Lists in Memory



- **Must follow the "next" pointers to find elements**
  - $\mathrm{at}(\mathrm{L}, \mathrm{n})$ **is an** $\mathrm{O}(\mathrm{n})$ **operation**
  - **no faster way to do this**

Faster Implementation of at

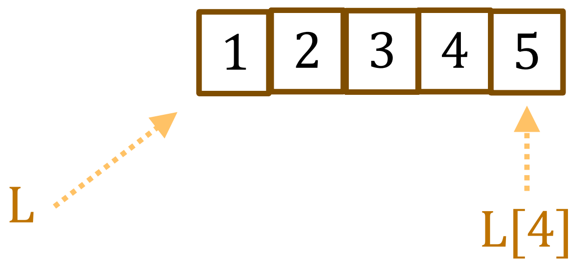


- Alternative: store the elements next to each other
  - can find the n-th entry by arithmetic:

location of L[4] = (location of L) + 4 * sizeof(node)

- Resulting data structure is an **array**

Faster Implementation of at



- Resulting data structure is an **array**
- Efficient to read L[i]
- Inefficient to...
  - insert elements anywhere but the end
  - write operations with an immutable ADT
  - trees can do <u>all of this</u> in O(log n) time

Access By Index

- **Easily access both $\mathrm{L}[0]$ and $\mathrm{L}[\mathrm{n}\text{-}1]$, where $\mathrm{n} = \mathrm{len}(\mathrm{L})$**
  - **can process a list in either direction**

- **“With great power, comes great responsibility”**

  — the Peter Parker Principle

- **Whenever we write “$\mathrm{A}[\mathrm{j}]$”, we must check $0 \leq \mathrm{j} < \mathrm{n}$**
  - **new bug just dropped!**

    with list, we only need to worry about nil and non-nil

    once we know L is non-nil, we know L.hd exists
  - **TypeScript will not help us with this!**

    type checker does catch “could be nil” bugs, but not this

Recall: Sum List With a Loop

$$\begin{aligned} \text{sum-acc}(\text{nil}, r) &:= r \\ \text{sum-acc}(x :: L, r) &:= \text{sum-acc}(L, x + r) \end{aligned}$$

- Tail recursive version is a loop

```
const sum = (S: List<bigint>): bigint => {
  let r = 0;
  // Inv: sum(S₀) = r + sum(S)
  while (S.kind !== "nil") {
    r = S.hd + r;
    S = S.tl;
  }
  return r;
};
```

Change to a version that uses indexes...

Sum List by Index

- Change to using an array and accessing by index

```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = 0;
  // Inv: …
  while (j !== S.length) { // … S.kind !== "nil"
    r = S[j] + r;          // … r = S.hd + r
    j = j + 1;             // … S = S.tl
  }
  return r;
};
```

Note that **S** is no longer changing

Sum List by Index

$$\text{sum-acc} : (\mathbb{N}, \text{List}, \mathbb{Z}) \to \mathbb{Z}$$

$$\begin{aligned}
&\text{sum-acc}(S, j, r) && := r && \textbf{if } j = \text{len}(S) \\
&\text{sum-acc}(S, j, r) && := \text{sum-acc}(S, j{+}1, S[j] + r) && \textbf{if } j < \text{len}(S)
\end{aligned}$$

- **Change to using an array and accessing by index**

```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = 0;
  // Inv: …
  while (j !== S.length) {
    r = S[j] + r;
    j = j + 1;
  }
  return r;
};
```

Sublists

- Use indexes to refer to a section of a list (a "sublist"):

$$\text{sublist} : (\text{List}, \mathbb{Z}, \mathbb{Z}) \to \mathbb{Z}$$

$$\begin{array}{lll} \text{sublist}(L, i, j) & := \text{nil} & \textbf{if } j < i \\ \text{sublist}(L, i, j) & := L[i] :: \text{sublist}(L, i + 1, j) & \textbf{if } i \leq j \end{array}$$

- Useful for *reasoning* about lists and indexes

- This includes <u>both</u> $L[i]$ and $L[j]$

$$\begin{array}{lll} \text{sublist}(L, 0, 2) & = L[0] :: \text{sublist}(L, 1, 2) & \textbf{def of } \text{sublist } (\textbf{since } 0 \leq 2) \\ & = L[0] :: L[1] :: \text{sublist}(L, 2, 2) & \textbf{def of } \text{sublist } (\textbf{since } 1 \leq 2) \\ & = L[0] :: L[1] :: L[2] :: \text{sublist}(L, 3, 2) & \textbf{def of } \text{sublist } (\textbf{since } 2 \leq 2) \\ & = L[0] :: L[1] :: L[2] :: \text{nil} & \textbf{def of } \text{sublist } (\textbf{since } 3 < 2) \\ & = [L[0], L[1], L[2]] & \end{array}$$

Sublists

- Use indexes to refer to a section of a list (a "sublist"):

$$\text{sublist} : (\text{List}, \mathbb{Z}, \mathbb{Z}) \rightarrow \mathbb{Z}$$

$$\begin{aligned} \text{sublist}(L, i, j) &:= \text{nil} && \textbf{if } j < i \\ \text{sublist}(L, i, j) &:= L[i] :: \text{sublist}(L, i+1, j) && \textbf{if } i \leq j \end{aligned}$$

- The sublist is empty when the **range** is empty

$$\text{sublist}(L, 3, 2) = \text{nil}$$

  – weird-looking example that comes up a lot:

$$\text{sublist}(L, 0, \text{-}1) = \text{nil}$$

  – not an array out of bonds error! (this is math, not Java)

Sublists

$$\text{sublist} : (\text{List}, \mathbb{Z}, \mathbb{Z}) \to \mathbb{Z}$$

$$\begin{aligned} &\text{sublist}(L, i, j) &&:= \text{nil} && \textbf{if } j < i \\ &\text{sublist}(L, i, j) &&:= L[i] :: \text{sublist}(L, i+1, j) && \textbf{if } i \leq j \end{aligned}$$

- **Will use "$L[i .. j]$" as shorthand for "$\text{sublist}(L, i, j)$"**
  - **again, using an operator for most common operations**

- **Some useful facts about sublists:**

$$L = L[0 .. \text{len}(L)\text{-}1]$$

$$L[i .. j] = L[i .. k] + L[k+1 .. j]$$ **for any** k **with** $i - 1 \leq k \leq j$ (and $0 \leq i \leq j < n$)

Sum List by Index

$$\begin{array}{lll} \text{sum-acc}(S, j, r) & := r & \textbf{if } j = \text{len}(S) \\ \text{sum-acc}(S, j, r) & := \text{sum-acc}(S, j+1, S[j] + r) & \textbf{if } j < \text{len}(S) \end{array}$$

- **Change to using an array and accessing by index**

```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = 0;
  // Inv: … ?? …
  while (j != S.length) {
    r = S[j] + r;
    j = j + 1;
  }
  return r;
};
```

Still need to fill in Inv…

Need a version using indexes.

Recall: Sum List With a Loop

$$\text{sum-acc}(S, j, r) := r \qquad \textbf{if } j = \text{len}(S)$$
$$\text{sum-acc}(S, j, r) := \text{sum-acc}(S, j+1, S[j] + r) \qquad \textbf{if } j < \text{len}(S)$$

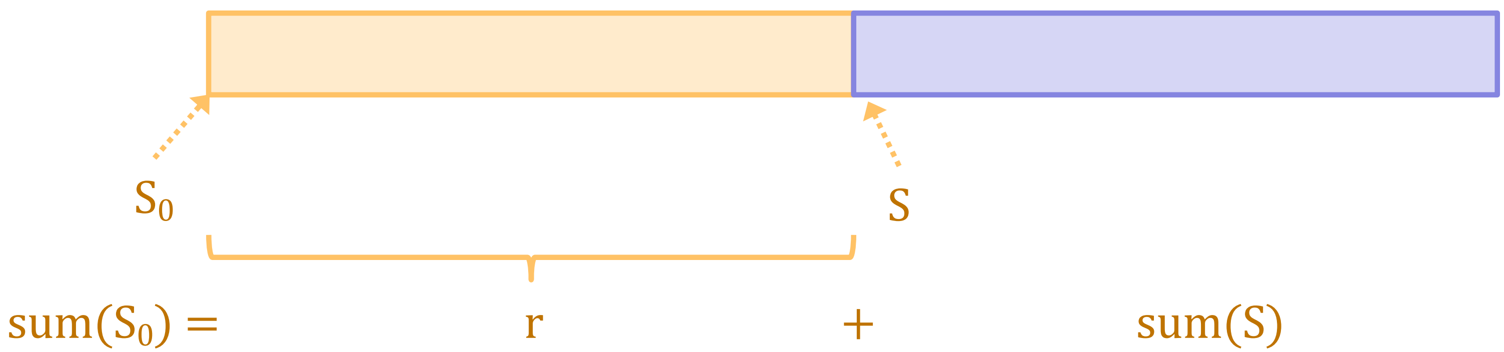
- Tail recursive version is a loop

```
const sum = (S: List<bigint>): bigint => {
  let r = 0;
  // Inv: sum(S0) = r + sum(S)
  while (S.kind !== "nil") {
    r = S.hd + r;
    S = S.tl;
  }
  return r;
};
```

Inv says $\text{sum}(S_0)$ is r plus sum of rest (S)

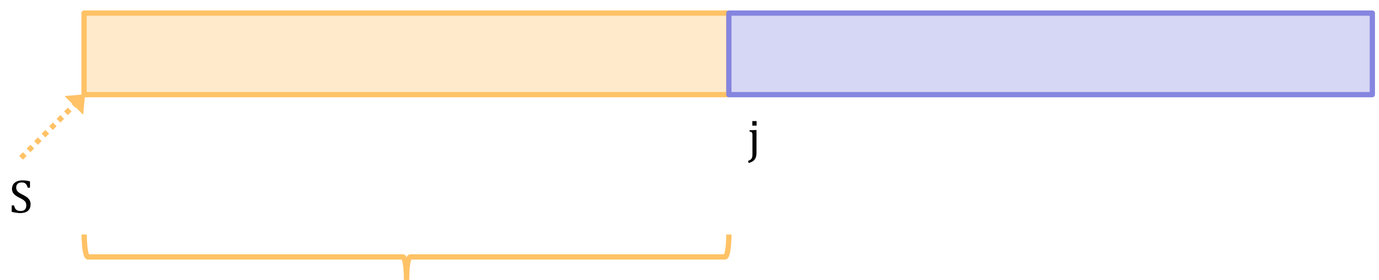
Not the most explicit way of explaining "r"...

Recall: Sum List With a Loop



- "r" contains sum of the part of the list *seen so far*
- Can explain this more simply with indexes…
  - no longer need to move S

Using Sublists With Loops



sum(S) = r + sum(S[j .. n-1])

- **Sum is the part in "r" plus the part left in S[j .. n-1]**

- **What sum is in "r"?**

  r = sum(S[0 .. j-1])

  – **we can use just this as our invariant! (it's all we need)**

Using Sublists With Loops

- Array version uses access by index

```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = 0;
  // Inv: r = sum(S[0 .. j-1])
  while (j != S.length) {
    r = S[j] + r;
    j = j + 1;
  }
  return r;
};
```

Are we sure this is right?

Let's think it through...

Sum of an Array

```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = 0;
  {{ r = 0 and j = 0 }}
  {{ Inv: r = sum(S[0 .. j-1]) }}
  while (j != S.length) {
    r = S[j] + r;
    j = j + 1;
  }
  return r;
};
```

Does Inv hold initially?

sum(S[0 .. j-1])
 = sum(S[0 .. -1])   since j = 0
 = sum([])
 = 0   def of sum
 = r

Sum of an Array

```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = 0;
  {{ Inv: r = sum(S[0 .. j-1]) }}
  while (j != S.length) {
    r = S[j] + r;
    j = j + 1;
  }
  {{ r = sum(S[0 .. j-1]) and j = len(S) }}
  {{ r = sum(S) }}
  return r;
};
```

Does the postcondition hold?

$r = \text{sum}(S[0 .. j\text{-}1])$
$\quad = \text{sum}(S[0 .. \text{len}(S)\text{-}1])$ **since** $j = \text{len}(S)$
$\quad = \text{sum}(S)$

Sum of an Array

```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = 0;
  {{ Inv: r = sum(S[0 .. j-1]) }}
  while (j != S.length) {
    {{ r = sum(S[0 .. j-1]) and j ≠ len(S) }}
    r = S[j] + r;
    j = j + 1;
    {{ r = sum(S[0 .. j-1]) }}
  }
  return r;
};
```

Sum of an Array

```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = 0;
  {{ Inv: r = sum(S[0 .. j-1]) }}
  while (j != S.length) {
    {{ r = sum(S[0 .. j-1]) and j ≠ len(S) }}
    r = S[j] + r;
    {{ r = sum(S[0 .. j]) }}
    j = j + 1;
    {{ r = sum(S[0 .. j-1]) }}
  }
  return r;
};
```

Sum of an Array

```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = 0;
  {{ Inv: r = sum(S[0 .. j-1]) }}
  while (j != S.length) {
    {{ r = sum(S[0 .. j-1]) and j ≠ len(S) }}
    {{ S[j] + r = sum(S[0 .. j]) }}
    r = S[j] + r;
    {{ r = sum(S[0 .. j]) }}
    j = j + 1;
    {{ r = sum(S[0 .. j-1]) }}
  }
  return r;
};
```

Sum of an Array

```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = 0;
  {{ Inv: r = sum(S[0 .. j-1]) }}
  while (j != S.length) {
    {{ r = sum(S[0 .. j-1]) and j ≠ len(S) }}
    {{ S[j] + r = sum(S[0 .. j]) }}
    r = S[j] + r;
    {{ r = sum(S[0 .. j]) }}
    j = j + 1;
    {{ r = sum(S[0 .. j-1]) }}
  }
  return r;
};
```

Is this valid?

Sum of an Array

{{ r = sum(S[0 .. j-1]) and j ≠ len(S) }}
{{ S[j] + r = sum(S[0 .. j]) }}

S[j] + r
= S[j] + sum(S[0 .. j-1]) — **since** r = sum(S[0 .. j-1])
= sum(S[0 .. j-1]) + S[j]
= sum(S[0 .. j-1]) + sum([S[j]]) — **def of** sum
= sum(S[0 .. j-1]) + sum(S[j .. j])

Sum of an Array

{{ r = sum(S[0 .. j-1]) and j ≠ len(S) }}
{{ S[j] + r = sum(S[0 .. j]) }}

S[j] + r
= S[j] + sum(S[0 .. j-1]) **since** r = sum(S[0 .. j-1])
= sum(S[0 .. j-1]) + S[j]
= sum(S[0 .. j-1]) + sum([S[j]]) **def of** sum
= sum(S[0 .. j-1]) + sum(S[j .. j])
= ...
= sum(S[0 .. j])

Sum of an Array

$\{\{\ r = \mathrm{sum}(S[0 .. j\text{-}1]) \text{ and } j \neq \mathrm{len}(S)\ \}\}$

$\{\{\ S[j] + r = \mathrm{sum}(S[0 .. j])\ \}\}$

$$
\begin{aligned}
& S[j] + r \\
&\quad = S[j] + \mathrm{sum}(S[0 .. j\text{-}1]) && \textbf{since } r = \mathrm{sum}(S[0 .. j\text{-}1]) \\
&\quad = \mathrm{sum}(S[0 .. j\text{-}1]) + S[j] \\
&\quad = \mathrm{sum}(S[0 .. j\text{-}1]) + \mathrm{sum}([S[j]]) && \textbf{def of } \mathrm{sum} \\
&\quad = \mathrm{sum}(S[0 .. j\text{-}1]) + \mathrm{sum}(S[j .. j]) \\
&\quad = \ldots \\
&\quad = \mathrm{sum}(S[0 .. j\text{-}1] \mathbin{+\!\!\!+} S[j .. j]) \\
&\quad = \mathrm{sum}(S[0 .. j])
\end{aligned}
$$

- **We saw that** $\mathrm{len}(L \mathbin{+\!\!\!+} R) = \mathrm{len}(L) + \mathrm{len}(R)$
- **Does** $\mathrm{sum}(L \mathbin{+\!\!\!+} R) = \mathrm{sum}(L) + \mathrm{sum}(R)$?
  - **Yes! Very similar proof by structural induction. (Call this Lemma 3)**

Sum of an Array

$\{\{\ r = \mathrm{sum}(S[0 .. j\text{-}1]) \text{ and } j \neq \mathrm{len}(S)\ \}\}$

$\{\{\ S[j] + r = \mathrm{sum}(S[0 .. j])\ \}\}$

$S[j] + r$

$= S[j] + \mathrm{sum}(S[0 .. j\text{-}1])$ **since** $r = \mathrm{sum}(S[0 .. j\text{-}1])$

$= \mathrm{sum}(S[0 .. j\text{-}1]) + S[j]$

$= \mathrm{sum}(S[0 .. j\text{-}1]) + \mathrm{sum}([S[j]])$ **def of** sum

$= \mathrm{sum}(S[0 .. j\text{-}1]) + \mathrm{sum}(S[j .. j])$

$= \mathrm{sum}(S[0 .. j\text{-}1] \mathbin{+\!\!\!+} S[j .. j])$ **by Lemma 3**

$= \mathrm{sum}(S[0 .. j])$

(The need to reason by induction comes up all the time.)

Sum of an Array

$\{\{\ r - S[j\text{-}1] = \mathrm{sum}(S[0\ ..\ j\text{-}2])$ **and** $j\text{-}1 \neq \mathrm{len}(S)\ \}\}$

$\{\{\ r = \mathrm{sum}(S[0\ ..\ j\text{-}1])\ \}\}$

$$
\begin{aligned}
r &= S[j\text{-}1] + \mathrm{sum}(S[0\ ..\ j\text{-}2]) && \textbf{since } r - S[j\text{-}1] = \mathrm{sum}(S[0\ ..\ j\text{-}2]) \\
&= \mathrm{sum}(S[0\ ..\ j\text{-}2]) + S[j\text{-}1] \\
&= \mathrm{sum}(S[0\ ..\ j\text{-}2]) + \mathrm{sum}([S[j\text{-}1]]) && \textbf{def of } \mathrm{sum} \\
&= \mathrm{sum}(S[0\ ..\ j\text{-}2]) + \mathrm{sum}(S[j\text{-}1\ ..\ j\text{-}1]) \\
&= \ldots \\
&= \mathrm{sum}(S[0\ ..\ j\text{-}2] \mathbin{+\!\!+} S[j\text{-}1\ ..\ j\text{-}1]) \\
&= \mathrm{sum}(S[0\ ..\ j\text{-}1])
\end{aligned}
$$

- **We saw that** $\mathrm{len}(L \mathbin{+\!\!+} R) = \mathrm{len}(L) + \mathrm{len}(R)$
- **Does** $\mathrm{sum}(L \mathbin{+\!\!+} R) = \mathrm{sum}(L) + \mathrm{sum}(R)$?
  - Yes! Very similar proof by structural induction. (Call this Lemma 3)

Sum of an Array

{{ $r - S[j\text{-}1] = \mathrm{sum}(S[0 \,..\, j\text{-}2])$ and $j\text{-}1 \neq \mathrm{len}(S)$ }}
{{ $r = \mathrm{sum}(S[0 \,..\, j\text{-}1])$ }}

$r = S[j\text{-}1] + \mathrm{sum}(S[0 \,..\, j\text{-}2])$	**since** $r - S[j\text{-}1] = \mathrm{sum}(S[0 \,..\, j\text{-}2])$
$= \mathrm{sum}(S[0 \,..\, j\text{-}2]) + S[j\text{-}1]$	
$= \mathrm{sum}(S[0 \,..\, j\text{-}2]) + \mathrm{sum}([S[j\text{-}1]])$	**def of** sum
$= \mathrm{sum}(S[0 \,..\, j\text{-}2]) + \mathrm{sum}(S[j\text{-}1 \,..\, j\text{-}1])$	
$= \mathrm{sum}(S[0 \,..\, j\text{-}2] \mathbin{+\!\!+} S[j\text{-}1 \,..\, j\text{-}1])$	**by Lemma 3**
$= \mathrm{sum}(S[0 \,..\, j\text{-}1])$	

(The need to reason by induction comes up all the time.)

Linear Search of a List

$$
\begin{array}{lll}
\text{contains}(\text{nil}, y) & := \text{false} & \\
\text{contains}(x :: L, y) & := \text{true} & \textbf{if } x = y \\
\text{contains}(x :: L, y) & := \text{contains}(L, y) & \textbf{if } x \neq y
\end{array}
$$

- **Tail-recursive definition from HW5**

```
const contains =
    (S: List<bigint>, y: bigint): bigint => {
  // Inv: contains(S0, y) = contains(S, y)
  while (S.kind !== "nil" && S.hd !== y) {
    S = S.tl;
  }
  return S.kind !== "nil";  // implies S.hd === y
};
```

Change to a version that uses indexes…

Linear Search of an Array

$$\begin{aligned}
&\text{contains}(\text{nil}, y) && := \text{false} && \\
&\text{contains}(x :: L, y) && := \text{true} && \textbf{if } x = y \\
&\text{contains}(x :: L, y) && := \text{contains}(L, y) && \textbf{if } x \neq y
\end{aligned}$$

- **Change to using an array and accessing by index**

```
const contains =
    (S: Array<bigint>, y: bigint): bigint => {
  let j = 0;
  // Inv: …
  while (j !== S.length && S[j] !== y) {
    j = j + 1;
  }
  return j !== S.length;
};
```

`S.hd` with `S` changing becomes
`S[j]` with `j` changing

What is the invariant now?

Linear Search of an Array

$$
\begin{array}{lll}
\text{contains}(\text{nil}, y) & := \text{false} & \\
\text{contains}(x :: L, y) & := \text{true} & \textbf{if } x = y \\
\text{contains}(x :: L, y) & := \text{contains}(L, y) & \textbf{if } x \neq y
\end{array}
$$

- **Change to using an array and accessing by index**

```
const contains =
    (S: Array<bigint>, y: bigint): bigint => {
  let j = 0;
  // Inv: contains(S, y) = contains(S[j .. n-1], y)
  while (j !== S.length && S[j] !== y) {
    j = j + 1;
  }
  return j !== S.length;
};
```

Can we explain this better?

Linear Search of an Array

contains(S, y) = contains(S[j .. n-1], y)

- **What do we know about the left segment?**
  - it does not contain "y"
  - that's why we kept searching

Linear Search of an Array

- **Update the invariant to be more informative**

```
const contains =
    (S: Array<bigint>, y: bigint): bigint => {
  let j = 0;
  // Inv: S[i] /= y for any i = 0 .. j-1
  while (j !== S.length && S[j] !== y) {
    j = j + 1;
  }
  return j !== S.length;
};
```

Facts About Sublists

- "With great power, comes great responsibility"
- **Since we can easily access any $L[j]$, may need to keep track of facts about it**
  - may need facts about *every* element in the list

    applies to preconditions, postconditions, and intermediate assertions
- **We can write facts about several elements at once:**
  - this says that elements at indexes 0 .. $j\text{-}1$ are not y

    $S[i] \neq y$ for any $0 \leq i < j$
  - shorthand for j facts: $S[0] \neq y, \ldots, S[j\text{-}1] \neq y$

Reasoning Toolkit

Description	Testing	Tools	Reasoning
no mutation	full coverage	type checker	calculation induction
local variable mutation	“	“	Floyd logic
heap state	“	“	rep invariants
arrays	“	“	for-any facts

Facts About Sublists

- "With great power, comes great responsibility"
  - since we can easily access any $L[j]$, may need facts about it

- We can write facts about several elements at once:
  - this says that elements at indexes 0 .. $j\text{-}1$ are not y

$$S[i] \neq y \qquad \text{for any } 0 \leq i < j$$

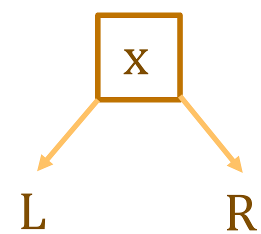
- These facts get hard to write down!
  - we will need to find ways to make this <u>easier</u>
  - a common trick is to draw pictures instead…

Visual Presentation of Facts

- **Just saw this example**

- **But we have seen "for any" facts with BSTs...**

contains-key(y, L) $\rightarrow$ (y < x)
contains-key(z, R) $\rightarrow$ (x < z)



  - "for any" facts are common in more complex code
  - drawing pictures is a typical coping mechanism

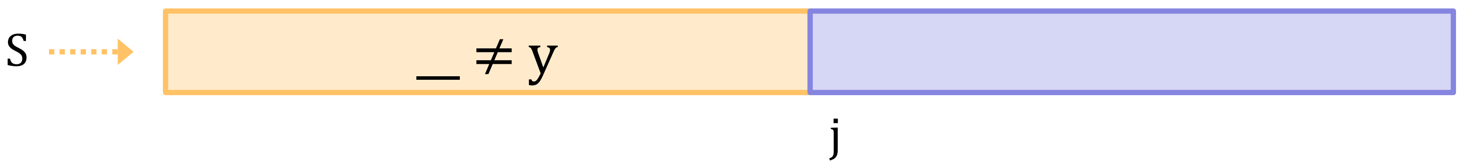
Recall: Linear Search of an Array

- **Let's check the correctness of this loop (w/ pictures)**

```
const contains =
    (S: Array<bigint>, y: bigint): boolean => {
  let j = 0;
  // Inv: S[k] /= y for any k = 0 .. j-1
  while (j !== S.length && S[j] !== y) {
    j = j + 1;
  }
  return j !== S.length;
};
```

Inv: gold part contains no y

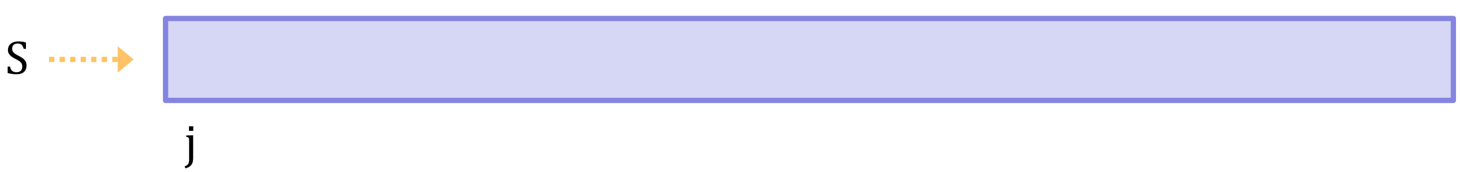
Linear Search of an Array




```
const contains =
    (S: Array<bigint>, y: bigint): boolean => {
  let j = 0;
  {{ j = 0 }}
  {{ Inv: S[i] ≠ y  for any 0 ≤ i ≤ j – 1 }}
  while (j !== S.length && S[j] !== y) {
    j = j + 1;
  }
  return j !== S.length;
};
```

What is the picture when $j = 0$?

Inv holds because there is no gold part.



Linear Search of an Array


```
const contains =
    (S: Array<bigint>, y: bigint): boolean => {
  let j = 0;
  {{ Inv: S[i] ≠ y  for any 0 ≤ i ≤ j – 1 }}
  while (j !== S.length && S[j] !== y) {
    {{ (S[i] ≠ y  for any 0 ≤ i ≤ j – 1) and j ≠ len(S) and S[j] ≠ y }}
    j = j + 1;
    {{ S[i] ≠ y  for any 0 ≤ i ≤ j – 1 }}
  }
  return j !== S.length;
};
```

Linear Search of an Array


```
const contains =
    (S: Array<bigint>, y: bigint): boolean => {
  let j = 0;
  {{ Inv: S[i] ≠ y  for any 0 ≤ i ≤ j – 1 }}
  while (j !== S.length && S[j] !== y) {
    {{ (S[i] ≠ y  for any 0 ≤ i ≤ j – 1) and j ≠ len(S) and S[j] ≠ y }}
    {{ S[i] ≠ y  for any 0 ≤ i ≤ j }}
    j = j + 1;
    {{ S[i] ≠ y  for any 0 ≤ i ≤ j – 1 }}
  }
  return j !== S.length;
};
```

Is this valid?

Linear Search of an Array

{{ (S[i] $\neq$ y for any $0 \leq i \leq j - 1$) and j $\neq$ len(S) and S[j] $\neq$ y }}

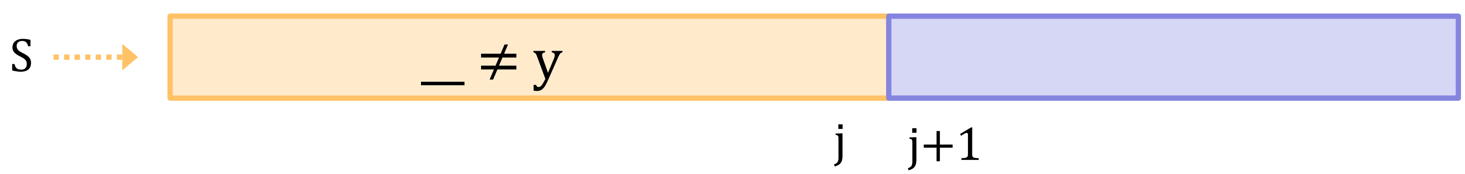
{{ S[i] $\neq$ y for any $0 \leq i \leq j$ }}

- **What does the top assertion say about $S[j]$?**
  - **it is not y**

Linear Search of an Array

{{ ($S[i] \neq y$ for any $0 \leq i \leq j - 1$) and $j \neq len(S)$ and $S[j] \neq y$ }}
{{ $S[i] \neq y$ for any $0 \leq i \leq j$ }}

- **What is the picture for the bottom assertion?**

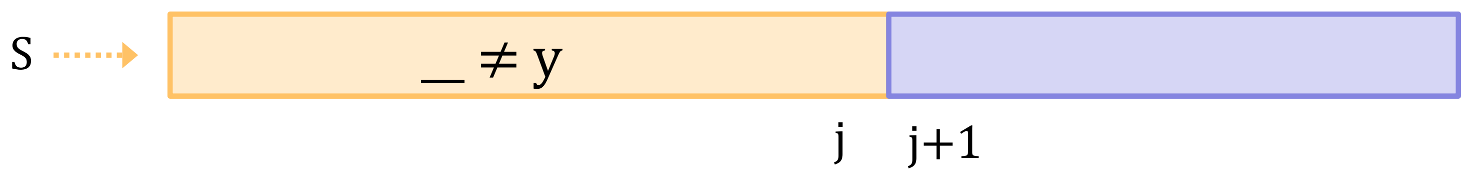


- **Do the facts above imply this holds?**
  – **Yes! It's the same picture**

Linear Search of an Array

{{ (S[i] ≠ y for any 0 ≤ i ≤ j – 1) and j ≠ len(S) and S[j] ≠ y }}
{{ S[i] ≠ y for any 0 ≤ i ≤ j }}

- **What is the picture for the bottom assertion?**



- **Most likely bug is an off-by-one error**
  - must check S[j], not S[j-1] or S[j+1]

Linear Search of an Array


```
const contains =
    (S: Array<bigint>, y: bigint): boolean => {
  let j = 0;
  {{ Inv: S[i] ≠ y  for any 0 ≤ i ≤ j – 1 }}
  while (j !== S.length && S[j] !== y) {
    j = j + 1;
  }
  {{ Inv and (j = len(S) or S[j] = y) }}
  {{ contains(S, y) = (j ≠ len(S)) }}
  return j !== S.length;
};
```

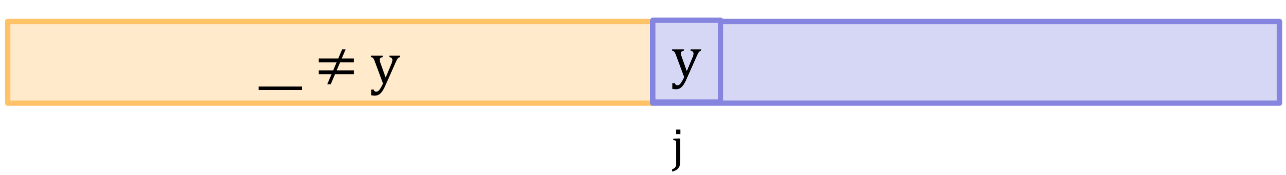
"or" means cases...

Case $j \neq \text{len}(S)$:

Must have $S[j] = y$.

What does Inv say now?

Code should and does return true.



Linear Search of an Array


```
const contains =
    (S: Array<bigint>, y: bigint): boolean => {
  let j = 0;
  {{ Inv: S[i] ≠ y  for any 0 ≤ i ≤ j – 1 }}
  while (j !== S.length && S[j] !== y) {
    j = j + 1;
  }
  {{ Inv and (j = len(S) or S[j] = y) }}
  {{ contains(S, y) = (j ≠ len(S)) }}
  return j !== S.length;
};
```

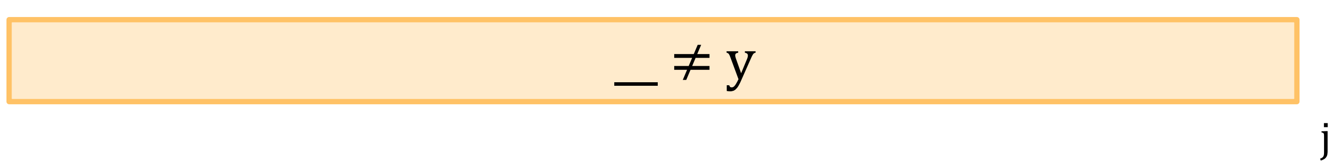
"or" means cases…

Case $j = \text{len}(S)$:

What does Inv say now?

Says y is not in the array!

Code should and does return false.



Finding an Element in an Array

- Can search for an element in an array as follows

$$
\begin{array}{lll}
\text{contains}(\text{nil}, y) & := \text{false} & \\
\text{contains}(x :: L, y) & := \text{true} & \textbf{if } x = y \\
\text{contains}(x :: L, y) & := \text{contains}(L, y) & \textbf{if } x \neq y
\end{array}
$$

- Searches through the array in linear time
  - did the same on lists

- Can be done more quickly if the list is sorted
  - binary search!

Finding an Element in a Sorted Array

- Can search more quickly if the list is sorted
  - precondition is A[0] ≤ A[1] ≤ … ≤ A[n-1]   (informal)
  - write this formally as

$$A[j] \leq A[j+1] \text{ for any } 0 \leq j \leq n-2$$

- Not easy to describe this visually…
  - how about a gradient?

Binary Search of an Array


```
const bsearch = (S: …, y: …): boolean => {
  let j = 0, k = S.length;
  {{ Inv: (S[i] < y  for any 0 ≤ i < j) and (y ≤ S[i] for any k ≤ i < n) }}
  while (j !== k) {
    const m = (j + k) / 2n;
    if (S[m] < y) {
      j = m + 1;
    } else {
      k = m;
    }
  }
  return (S[k] === y);
};
```

Inv includes facts about two regions.

Let's check that this is right...

Binary Search of an Array


```
const bsearch = (S: …, y: …): boolean => {
  let j = 0, k = S.length;
  {{ j = 0 and k = n }}
  {{ Inv: (S[i] < y  for any 0 ≤ i < j) and (y ≤ S[i] for any k ≤ i < n) }}
```

- What does the picture look like with $j = 0$ **and** $k = n$?

- Does this hold?
  - Yes! It's vacuously true

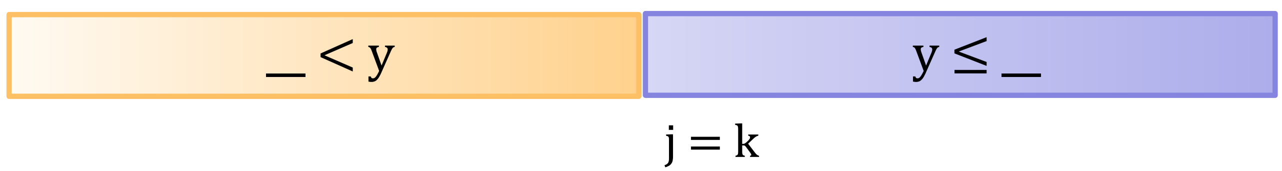
Binary Search of an Array


```
const bsearch = (S: …, y: …): boolean => {
  let j = 0, k = S.length;
  {{ Inv: (S[i] < y  for any 0 ≤ i < j) and (y ≤ S[i] for any k ≤ i < n) }}
  while (j !== k) {
   …
  }
  {{ Inv and (j = k) }}
  {{ contains(S, y) = (S[y] = y) }}
  return (S[k] === y);
};
```

Binary Search of an Array


```
    {{ Inv and (j = k) }}
    {{ contains(S, y) = (S[y] = y) }}
    return (S[k] === y);
};
```

- What does the picture look like with $j = k$?



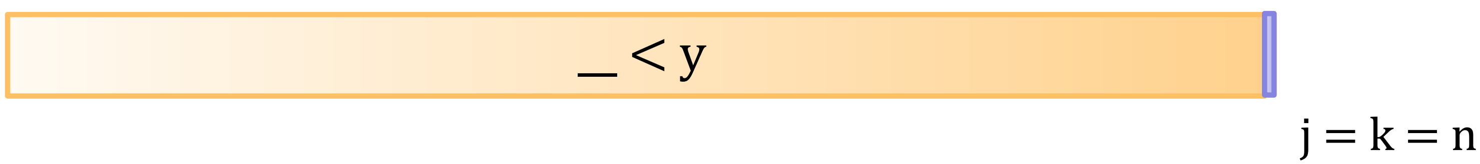
- Does S contain y iff $S[k] = y$?
  - If $S[k] = y$, then $\text{contains}(S, y) = \text{true}$
  - If $S[k] \ne y$, then $S[k] < y$ and $S[i] < y$ for every $k < i$, so $\text{contains}(S, y) = \text{false}$

What case are we missing?

Binary Search of an Array


```
    {{ Inv and (j = k) }}
    {{ contains(S, y) = (S[y] = y) }}
    return (S[k] === y);
};
```

- What does the picture look like with $j = k = n$?



- In this case...
  - we see that $\text{contains}(S, y) = \text{false}$
  - and the code returns false because "`undefined === y`" is false

    (Okay, but yuck.)

Binary Search of an Array


```
{{ Inv: (S[i] < y  for any 0 ≤ i < j) and (y ≤ S[i] for any k ≤ i < n) }}
while (j !== k) {
  {{ Inv and (j < k) }}
  const m = (j + k) / 2n;
  if (S[m] < y) {
    j = m + 1;
  } else {
    k = m;
  }
  {{ (S[i] < y  for any 0 ≤ i < j) and (y ≤ S[i] for any k ≤ i < n) }}
}
```

Reason through both paths...

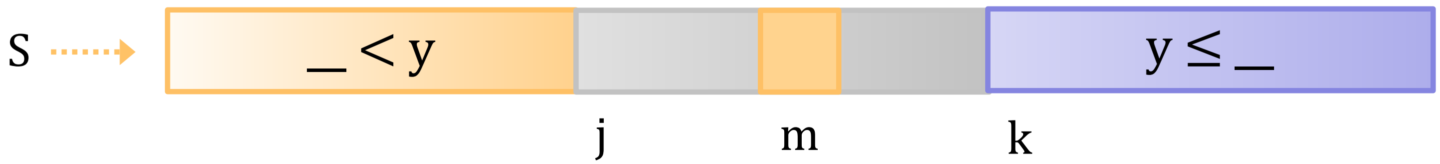
Binary Search of an Array


```
{{ Inv and (j < k) }}
const m = (j + k) / 2n;
if (S[m] < y) {
  {{ Inv and (j < k) and (S[m] < y) }}
  j = m + 1;
} else {
  {{ Inv and (j < k) and (S[m] ≥ y) }}
  k = m;
}
{{ (S[i] < y  for any 0 ≤ i < j) and (y ≤ S[i] for any k ≤ i < n) }}
}
```

Binary Search of an Array


```
const m = (j + k) / 2n;
if (S[m] < y) {
    {{ Inv and (j < k) and (S[m] < y) }}
    {{ (S[i] < y  for any 0 ≤ i < m+1) and (y ≤ S[i] for any k ≤ i < n) }}
    j = m + 1;
} else {
    {{ Inv and (j < k) and (S[m] ≥ y) }}
    {{ (S[i] < y  for any 0 ≤ i < j) and (y ≤ S[i] for any m ≤ i < n) }}
    k = m;
}
{{ (S[i] < y  for any 0 ≤ i < j) and (y ≤ S[i] for any k ≤ i < n) }}
```

Binary Search of an Array



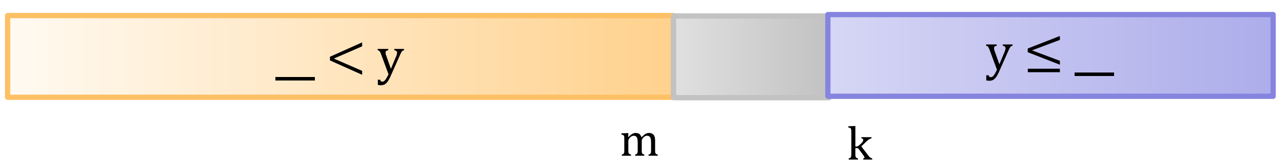

```
const m = (j + k) / 2n;
if (S[m] < y) {
```

    {{ Inv and (j < k) and (S[m] < y) }}

    {{ (S[i] < y for any $0 \le i < m+1$) **and** (y ≤ S[i] for any $k \le i < n$) }}

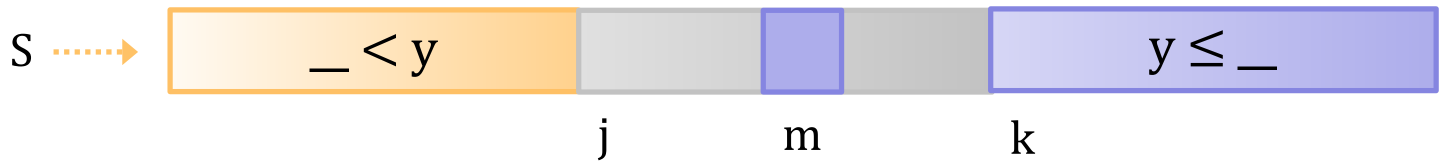
```
  j = m + 1;
} …
```

- **What does the picture look like in the bottom assertion?**



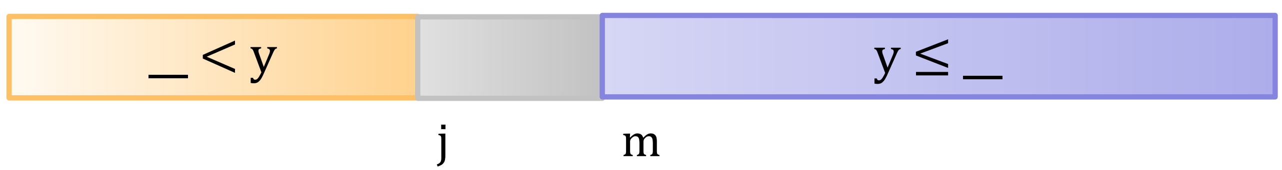
- **Does this hold?**
  - **Yes! Because the array is sorted (everything before S[m] is even smaller)**

Binary Search of an Array




```
const m = (j + k) / 2n;
… else {
    {{ Inv and (j < k) and (S[m] ≥ y) }}
    {{ (S[i] < y  for any 0 ≤ i < j) and (y ≤ S[i] for any m ≤ i < n) }}
    k = m;
}
```

- **What does the picture look like in the bottom assertion?**



- **Does this hold?**
  - **Yes! Because the array is sorted (everything after $S[m]$ is even larger)**

Binary Search of an Array


```
const bsearch = (S: …, y: …): boolean => {
  let j = 0, k = S.length;
  {{ Inv: (S[i] < y  for any 0 ≤ i < j) and (y ≤ S[i] for any k ≤ i < n) }}
  while (j !== k) {
    const m = (j + k) / 2n;
    if (S[m] < y) {
      j = m + 1;
    } else {
      k = m;
    }
  }
  return (S[k] === y);
};
```

Does this terminate?

Need to check that $k - j$ *decreases*

Can see that $j \leq m \leq k$, so the "then" branch is fine.

Can see that $j < k$ implies $m < k$ (integer division rounds <u>down</u>), so the "else" branch is also fine

Loop Invariants

Loop Invariants with Arrays

- Previous example:

  $\{\{$ **Inv**: $s = \mathrm{sum}(S[0 \,..\, j-1])$ … $\}\}$ **sum of array**

  $\{\{$ **Post**: $s = \mathrm{sum}(S[0 \,..\, n-1])$ $\}\}$

  – **in this case,** Post **is a special case of** Inv **(where** $j = n$**)**

  – **in other words,** Inv **is a weakening of** Post

- Heuristic for loop invariants: weaken the postcondition

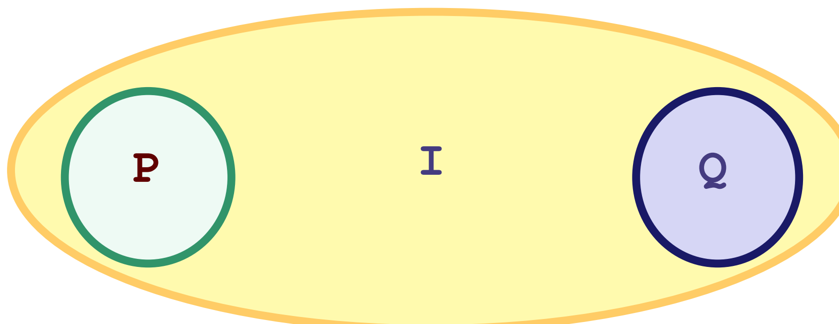
  – assertion that allows postcondition as a special case

  – must also allow states that are easy to prepare

Heuristic for Loop Invariants

- Loop Invariant allows both start and stop states
  - describing more states = weakening

```
{{ P }}
{{ Inv: I }}
while (cond) {
  S
}
{{ Q }}
```



  - usually are many ways to weaken it...

Loop Invariants with Arrays

- Previous example

{{ **Inv**: $s = \text{sum}(S[0 .. j-1])$ … }}   **sum of array**

{{ **Post**: $s = \text{sum}(S[0 .. n-1])$ }}

- Linear search also fits this pattern:

{{ **Inv**: $S[i] \neq y$ for any $0 \leq i < j$ }}   **search an array**

{{ **Post**: $(S[i] = y)$ or $(S[i] \neq y$ for any $0 \leq i < n)$ }}

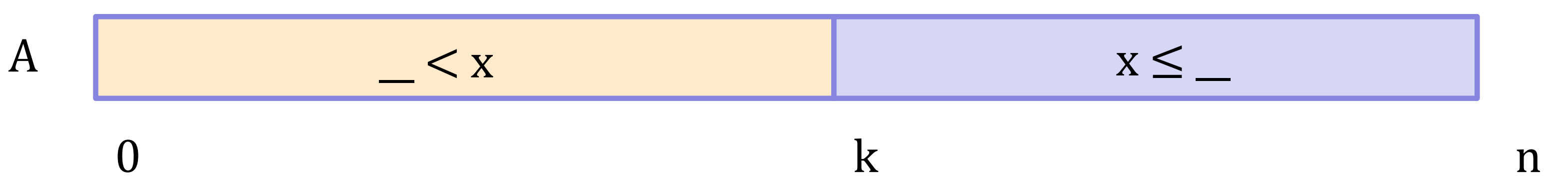
  – less obvious but still a **weakening**

Searching a Sorted Array

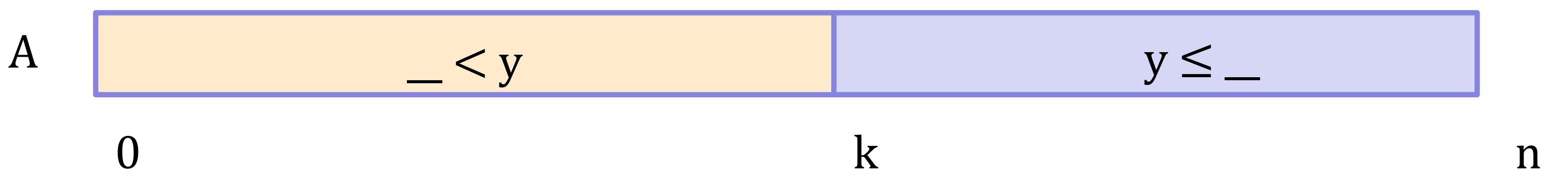
- Suppose we require A to be sorted:
  - precondition includes

$A[j-1] \leq A[j]$ for any $1 \leq j < n$ (where $n := A.\text{length}$)

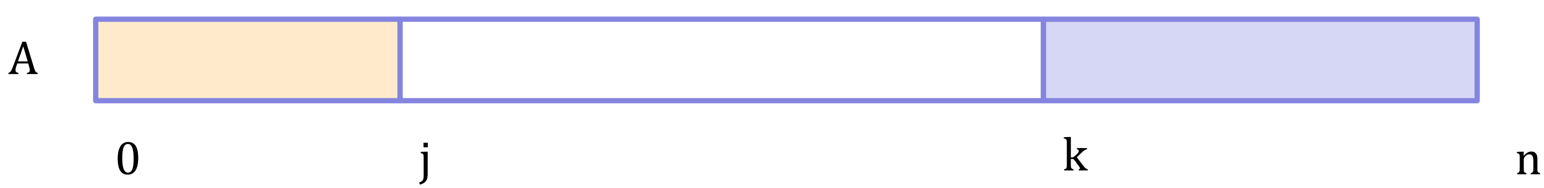
- Want to find the index k where "x" would be...
  - picture would look like this:



Searching a Sorted Array



- **End with complete knowledge of $A[i]$ vs x**
  - **how can we describe *partial* knowledge?**
  - **know some elements are smaller and some larger**



$A[i] < y$ for any $0 \leq i < j$

$y \leq A[i]$ for any $k \leq i < n$

Loop Invariants with Arrays

- **Previous example**

{{ **Inv**: $s = sum(S[0 .. j - 1])$ … }}
{{ **Post**: $s = sum(S[0 .. n - 1])$ }}

sum of array

- **Linear search also fits this pattern:**

{{ **Inv**: $S[i] \neq y$ for any $0 \leq i < j$ }}
{{ **Post**: $(S[i] = y)$ or $(S[i] \neq y$ for any $0 \leq i < n)$ }}

search an array

- **Binary search also still fits this pattern**

{{ **Inv**: $(S[i] < y$ for any $0 \leq i < j)$ **and** $(y \leq S[i]$ for any $k \leq i < n)$ }}
{{ **Post**: $(S[i] < y$ for any $0 \leq i <$ **k**$)$ **and** $(y \leq S[i]$ for any $k \leq i < n)$ }}

Loop Invariants

- Heuristic for loop invariants: weaken the postcondition
  - assertion that allows postcondition as a special case
  - must also allow states that are easy to prepare

- 421 covers complex heuristics for finding invariants…
  - for 331, this heuristic is enough
  - (will give you the invariant for anything more complex)

Writing Loops

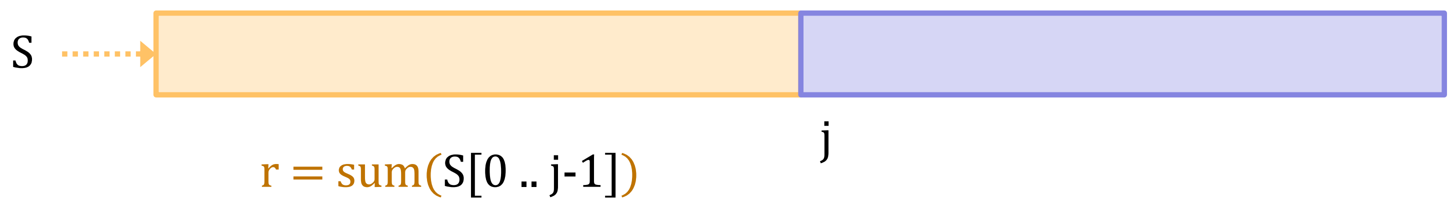
Writing Loops

- Examples so far have been code reviews
  - checking correctness of given code

- Steps to write a loop to solve a problem:
  1. Come up with an idea for the loop
  2. Formalize the idea in the invariant
  3. Write the code so that it is correct with that invariant

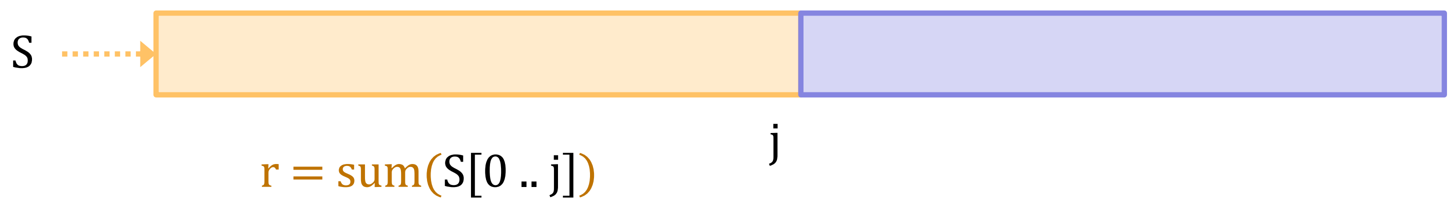
- Let's see some examples…

Recall: Sum of an Array




```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = 0;
  // Inv: r = sum(S[0 .. j-1])
  while (j != S.length) {
    r = S[j] + r;
    j = j + 1;
  }
  return r;
};
```

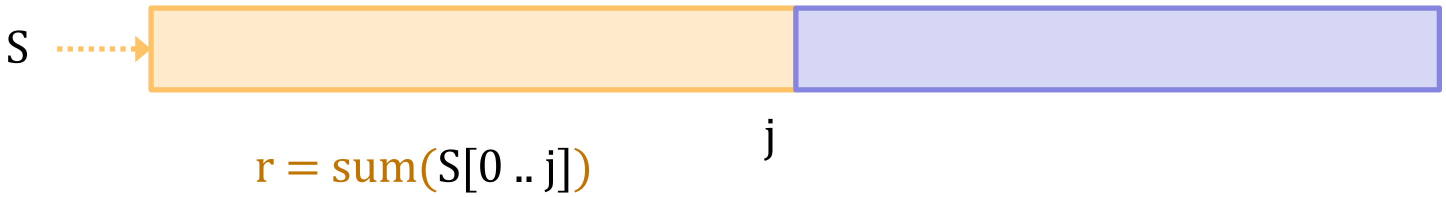
Sum of an Array (version 2)




```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = ??
  // Inv: r = sum(S[0 .. j])
  while (??) {
    r = ??
    j = j + 1;
  }
  return r;
};
```

How do we fill in the blanks to make this code correct?

Sum of an Array (version 2)

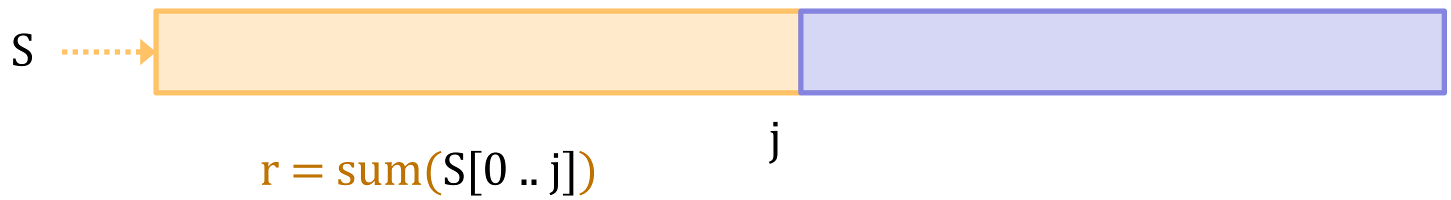



```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = ??
  // Inv: r = sum(S[0 .. j])
```

- **What do we set j to so that $\text{sum}(S[0 .. j]) = 0$?**
  - **must set it to -1:**

$$\text{sum}(S[0 .. -1]) = \text{sum}([]) = 0$$

Sum of an Array (version 2)

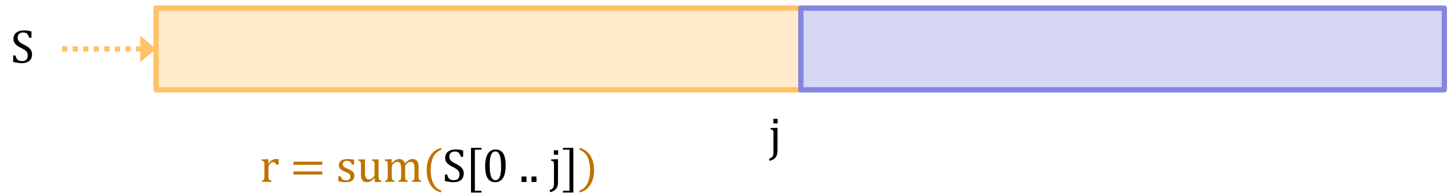



```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = -1;
  // Inv: r = sum(S[0 .. j])
  while (??) {
    ...
  }
  {{ Post: r = sum(S[0 .. n-1]) }}
  return r;
};
```

When do we exit to ensure that
sum([0 .. j]) = sum(S[0 .. n-1])**?**

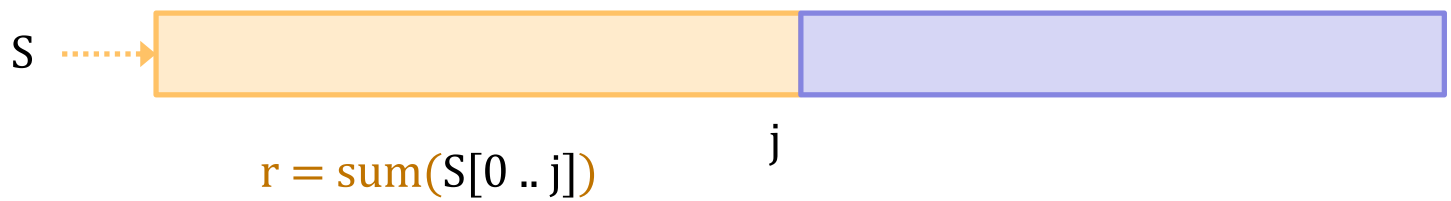
Exit when $j = n - 1$

Sum of an Array (version 2)




```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = -1;
  // Inv: r = sum(S[0 .. j])
  while (j !== S.length - 1) {
    {{ r = sum(S[0 .. j]) and j ≠ n – 1 }}
    r = ??
    j = j + 1;
    {{ r = sum(S[0 .. j]) }}
  }
  return r;
};
```

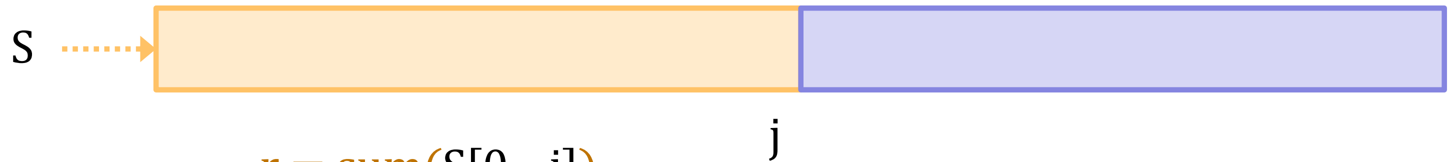
Sum of an Array (version 2)




```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = -1;
  // Inv: r = sum(S[0 .. j])
  while (j !== S.length - 1) {
    {{ r = sum(S[0 .. j]) and j ≠ n – 1 }}
    r = ??
    {{ r = sum(S[0 .. j+1]) }}
    j = j + 1;
    {{ r = sum(S[0 .. j]) }}
  }
```

Let's draw the second picture...

Sum of an Array (version 2)



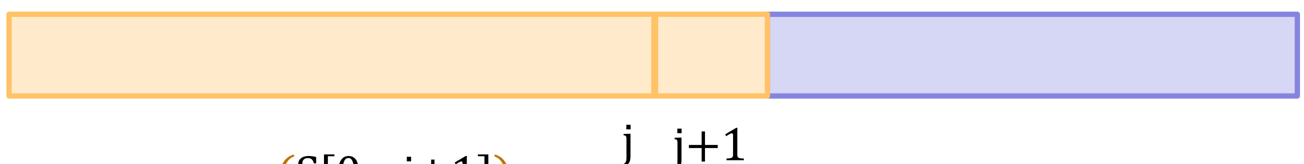
$r = \mathrm{sum}(S[0 .. j])$

$\{\{\ r = \mathrm{sum}(S[0 .. j]) \text{ and } j \neq n - 1\ \}\}$

```
r = ??
```

$\{\{\ r = \mathrm{sum}(S[0 .. j+1])\ \}\}$

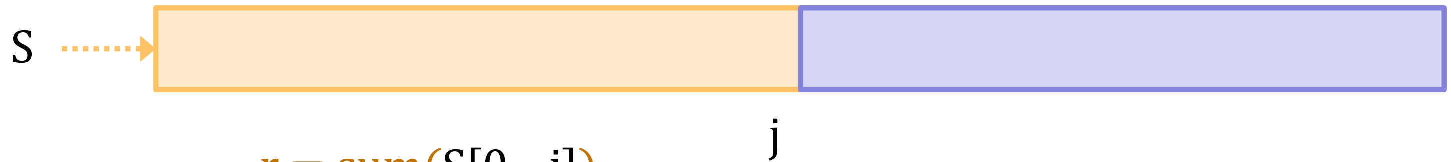
- **What is the picture in the second case?**



$r = \mathrm{sum}(S[0 .. j+1])$

- **What do we add to r to make this hold?**
  - **must add in** $S[j+1]$

Sum of an Array (version 2)



$r = \mathrm{sum}(S[0\ ..\ j])$

```
const sum = (S: Array<bigint>): bigint => {
  let r = 0;
  let j = -1;
  // Inv: r = sum(S[0 .. j])
  while (j !== S.length - 1) {
    r = S[j+1] + r;
    j = j + 1;
  }
  return r;
};
```

This code is correct by construction.

Different from $r = \mathrm{sum}(S[0\ ..\ j\text{-}1])$ **but does the same thing.**