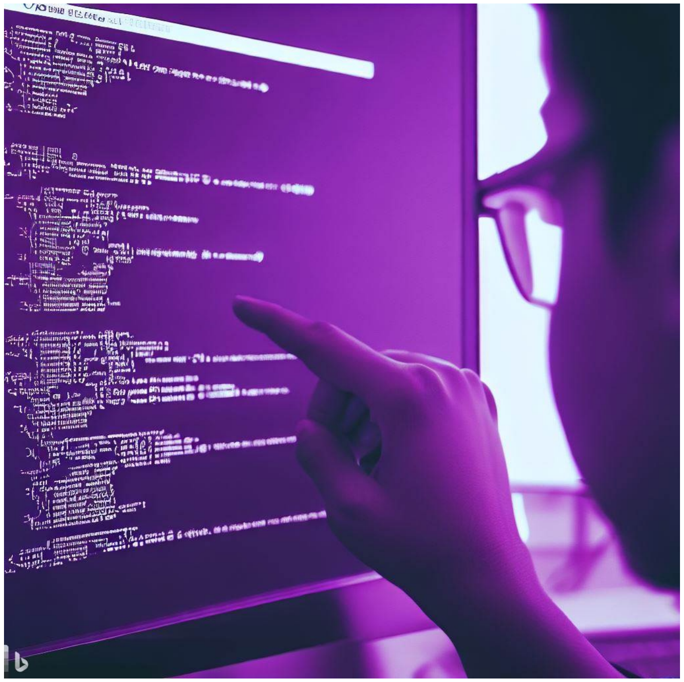


CSE 331

Floyd Logic

Kevin Zatloukal

Reasoning So Far

- Code so far made up of three elements
  - straight-line code
  - conditionals
  - recursion

- All code without mutation looks like this

Recall: Finding Facts at a Return Statement

- Consider this code

```
// Inputs a and b must be integers.
// Returns a non-negative integer.
const f = (a: bigint, b: bigint): bigint => {
  if (a >= 0n && b >= 0n) {
    const L: List = cons(a, cons(b, nil));
    return sum(L);
  }
  …
```

find facts by reading along <u>path</u> from top to return statement

- Known facts include "$\mathrm{a} \geq 0$", "$\mathrm{b} \geq 0$", and "$\mathrm{L} = \mathrm{cons}(\ldots)$"

- Prove that postcondition holds: "$\mathrm{sum}(\mathrm{L}) \geq 0$"

Recall: Finding Facts at a Return Statement

- Consider this code

```
// Inputs a and b must be integers.
// Returns a non-negative integer.
const f = (a: bigint, b: bigint): bigint => {
  if (a >= 0n && b >= 0n) {
    a = a - 1n;
    const L: List = cons(a, cons(b, nil));
    return sum(L);
  }
  …
```

$a \geq 0$

$a \geq 0$? No!

- Facts no longer hold throughout the function
- When we state a fact, we have to say <u>where</u> it holds

Correctness Levels

Description	Testing	Tools	Reasoning
no mutation	coverage	type checking	calculation induction
local variable mutation	“	“	Floyd logic
array mutation	“	“	for-any facts
heap state mutation	“	“	rep invariants

Recall: Finding Facts at a Return Statement

```
// Inputs a and b must be integers.
// Returns a non-negative integer.
const f = (a: bigint, b: bigint): bigint => {
  if (a >= 0n && b >= 0n) {
    {{ a ≥ 0 }}
    a = a - 1n;
    {{ a ≥ -1 }}
    const L: List = cons(a, cons(b, nil));
    return sum(L);
  }
```

- When we state a fact, we have to say <u>where</u> it holds
- {{ .. }} notation indicates facts true at that point
  - cannot assume those are true anywhere else

Recall: Finding Facts at a Return Statement

```
// Inputs a and b must be integers.
// Returns a non-negative integer.
const f = (a: bigint, b: bigint): bigint => {
  if (a >= 0n && b >= 0n) {
    {{ a ≥ 0 }}
    a = a - 1n;
    {{ a ≥ -1 }}
    const L: List = cons(a, cons(b, nil));
    return sum(L);
  }
```

- There are <u>mechanical</u> tools for moving facts around
  - "forward reasoning" says how they change as we move down
  - "backward reasoning" says how they change as we move up

Recall: Finding Facts at a Return Statement

```
// Inputs a and b must be integers.
// Returns a non-negative integer.
const f = (a: bigint, b: bigint): bigint => {
  if (a >= 0n && b >= 0n) {
    {{ a ≥ 0 }}
    a = a - 1n;
    {{ a ≥ -1 }}
    const L: List = cons(a, cons(b, nil));
    return sum(L);
  }
```

- Professionals are *insanely* good at forward reasoning
  - "programmers are the Olympic athletes of forward reasoning"
  - you'll have an edge by learning backward reasoning too

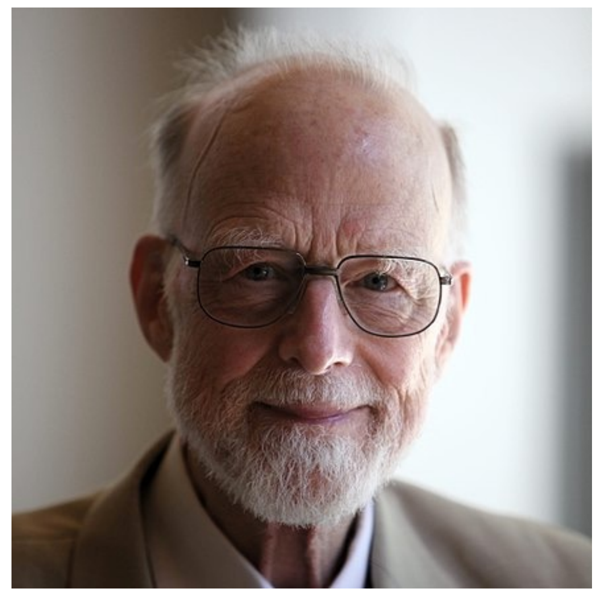
Floyd Logic

Floyd Logic

- Invented by Robert Floyd and Sir Anthony Hoare
  - Floyd won the Turing award in 1978
  - Hoare won the Turing award in 1980

Robert Floyd

picture from Wikipedia



Tony Hoare

Floyd Logic Terminology

- The **program state** is the values of the variables

- An **assertion** (in {{ .. }}) is a T/F claim about the state
  - an assertion "holds" if the claim is true
  - assertions are *math* not code

    (we do our reasoning in math)

- Most important assertions:
  - **precondition**: claim about the state when the function starts
  - **postcondition**: claim about the state when the function ends

Hoare Triples

- A **Hoare triple** has two assertions and some code

$$\{\{\ P\ \}\}$$

$$\mathsf{S}$$

$$\{\{\ Q\ \}\}$$

  - P is the precondition, Q is the postcondition
  - S is the code

- Triple is "**valid**" if the code is correct:
  - S takes *any* state satisfying P into a state satisfying Q

    does not matter what the code does if P does not hold initially
  - otherwise, the triple is invalid

Correctness Example

```
/**
 * @param n an integer with n >= 1
 * @returns an integer m with m >= 10
 */
const f = (n: bigint): bigint => {
  n = n + 3n;
  return n * n;
};
```

- **Check that value returned,** $m = n^2$**, satisfies** $m \geq 10$

Correctness Example

```
/**
 * @param n an integer with n >= 1
 * @returns an integer m with m >= 10
 */
const f = (n: bigint): bigint => {
  {{ n ≥ 1 }}
  n = n + 3n;
  {{ n² ≥ 10 }}
  return n * n;
};
```

- Precondition and postcondition come from spec
- Remains to check that the triple is valid

Hoare Triples with No Code

- Code could be empty:

  {{ P }}
  {{ Q }}

- When is such a triple valid?
  - valid iff P implies Q
  - we already know how to check validity in this case: prove each fact in Q by calculation, using facts from P

Hoare Triples with No Code

- **Code could be empty:**

$$\{\{\ a \geq 0,\ \ b \geq 0,\ \ L = \mathrm{cons}(a, \mathrm{cons}(b, \mathrm{nil}))\ \}\}$$
$$\{\{\ \mathrm{sum}(L) \geq 0\ \}\}$$

- **Check that P implies Q by calculation**

$$\begin{aligned}
\mathrm{sum}(L) \quad &= \mathrm{sum}(\mathrm{cons}(a, \mathrm{cons}(b, \mathrm{nil}))) && \text{since } L = \ldots \\
&= a + \mathrm{sum}(\mathrm{cons}(b, \mathrm{nil})) && \text{def of sum} \\
&= a + b + \mathrm{sum}(\mathrm{nil}) && \text{def of sum} \\
&= a + b && \text{def of sum} \\
&\geq 0 + b && \text{since } a \geq 0 \\
&\geq 0 + 0 && \text{since } b \geq 0 \\
&= 0
\end{aligned}$$

Hoare Triples with Multiple Lines of Code

- Code with multiple lines:

{{ P }}
S
T
{{ Q }}

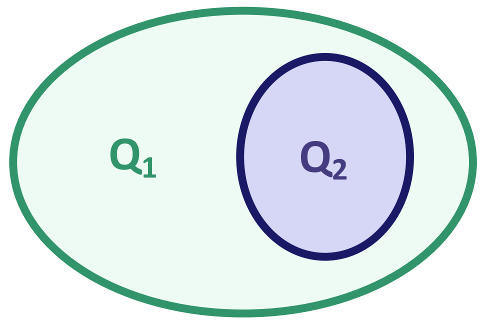
{{ P }}
S
{{ R }}
T
{{ Q }}

- Valid iff there exists an R making both triples valid
  - i.e., {{ P }} **S** {{ R }} is valid and {{ R }} **T** {{ Q }} is valid

- Will see next how to put these to good use…

Stronger Assertions vs Specifications

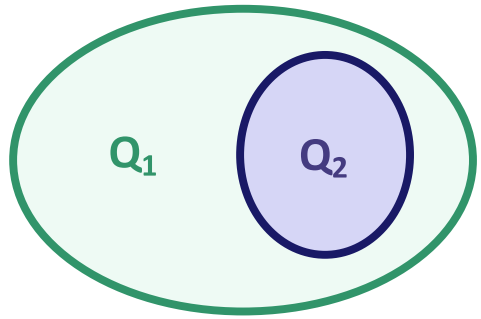
- **Assertion** is **stronger** iff it holds in a subset of states



- **Stronger** assertion <u>implies</u> the **weaker** one
  - stronger is a synonym for "implies"
  - weaker is a synonym for "is implied by"

Stronger Assertions vs Specifications

- Assertion is stronger iff it holds in a subset of states



- Weakest possible assertion is "true" (all states)
  – an empty assertion ("") also means "true"

- Strongest possible assertion is "false" (no states!)

Mechanical Reasoning Tools

- Forward / backward reasoning fill in assertions
  - mechanically create valid triples

- **Forward** reasoning fills in postcondition

$$\{\{\ P\ \}\}\ \ \mathtt{S}\ \ \{\{\ \_\_\ \}\}$$

  - gives *strongest* postcondition making the triple valid

- **Backward** reasoning fills in precondition

$$\{\{\ \_\_\ \}\}\ \ \mathtt{S}\ \ \{\{\ Q\ \}\}$$

  - gives *weakest* precondition making the triple valid

Correctness via Forward Reasoning

- Apply forward reasoning

{{ P }}
S
{{ Q }}

{{ P }}
S 1
{{ $\color{blue}{R}$ }}
2
{{ Q }}

  - first triple is always valid
  - only need to check second triple

    just requires proving an implication (since no code is present)

- If second triple is invalid, the code is **incorrect**
  - true because $\color{blue}{R}$ is the strongest assertion possible here

Correctness via Backward Reasoning

- Apply backward reasoning

$$\begin{array}{c} \{\{\ P\ \}\} \\ S \\ \{\{\ Q\ \}\} \end{array} \qquad\qquad \begin{array}{c} \{\{\ P\ \}\} \\ \{\{\ R\ \}\} \\ S \\ \{\{\ Q\ \}\} \end{array}$$

1

2

  - second triple is always valid
  - only need to check first triple

    just requires proving an implication (since no code is present)

- If first triple is invalid, the code is **incorrect**
  - true because R is the weakest assertion possible here

Mechanical Reasoning Tools

- Forward / backward reasoning fill in assertions
  - mechanically create valid triples
- Reduce correctness to proving implications
  - this was already true for functional code
  - will soon have the same for imperative code
- Implication will be false if the code is **incorrect**
  - reasoning can verify correct code
  - reasoning will never accept incorrect code

Correctness via Forward & Backward

- Can use both types of reasoning on longer code

$\{\{\ P\ \}\}$

S 1

$\{\{\ R_1\ \}\}$

2

$\{\{\ R_2\ \}\}$

T 3

$\{\{\ Q\ \}\}$

  - first and third triples is always valid
  - only need to check second triple

    verify that R_1 implies R_2

Forward & Backward Reasoning

Forward and Backward Reasoning

- Imperative code made up of
  - assignments (mutation)
  - conditionals
  - loops

- Anything can be rewritten with just these

- We will learn forward / backward rules to handle them
  - will also learn a rule for function calls
  - once we have those, we are done

Example Forward Reasoning through Assignments

```
{{ w > 0 }}
 x = 17n;
{{ ____________ }}
 y = 42n;
{{ ____________ }}
 z = w + x + y;
{{ ____________ }}
```

- **What do we know is true after `x = 17` ?**
  - want the strongest postcondition (most precise)

Example Forward Reasoning through Assignments

$\{\{\ w > 0\ \}\}$

```
x = 17n;
```

$\{\{\ w > 0 \text{ and } x = 17\ \}\}$

```
y = 42n;
```

$\{\{$ ____________________ $\}\}$

```
z = w + x + y;
```

$\{\{$ ____________________ $\}\}$

- **What do we know is true after `x = 17`?**
  - w **was not changed, so** $w > 0$ **is still true**
  - x **is now** 17

- **What do we know is true after `y = 42`?**

Example Forward Reasoning through Assignments

{{ $w > 0$ }}
```
x = 17n;
```
{{ $w > 0$ and $x = 17$ }}
```
y = 42n;
```
{{ $w > 0$ and $x = 17$ and $y = 42$ }}
```
z = w + x + y;
```
{{ ________________ }}

- **What do we know is true after `y = 42`?**
  - w **and** x **were not changed, so previous facts still true**
  - y **is now** 42

- **What do we know is true after `z = w + x + y`?**

Example Forward Reasoning through Assignments

$\{\{\ w > 0\ \}\}$

```
x = 17n;
```

$\{\{\ w > 0 \text{ and } x = 17\ \}\}$

```
y = 42n;
```

$\{\{\ w > 0 \text{ and } x = 17 \text{ and } y = 42\ \}\}$

```
z = w + x + y;
```

$\{\{\ w > 0 \text{ and } x = 17 \text{ and } y = 42 \text{ and } z = w + x + y\ \}\}$

- **What do we know is true after** `z = w + x + y`**?**
  - w, x, **and** y **were not changed, so previous facts still true**
  - z **is now** $w + x + y$
- **Could also write** $z = w + 59$ **(since** $x = 17$ **and** $y = 42$**)**

Example Forward Reasoning through Assignments

{{ $w > 0$ }}

```
x = 17n;
```

{{ $w > 0$ and $x = 17$ }}

```
y = 42n;
```

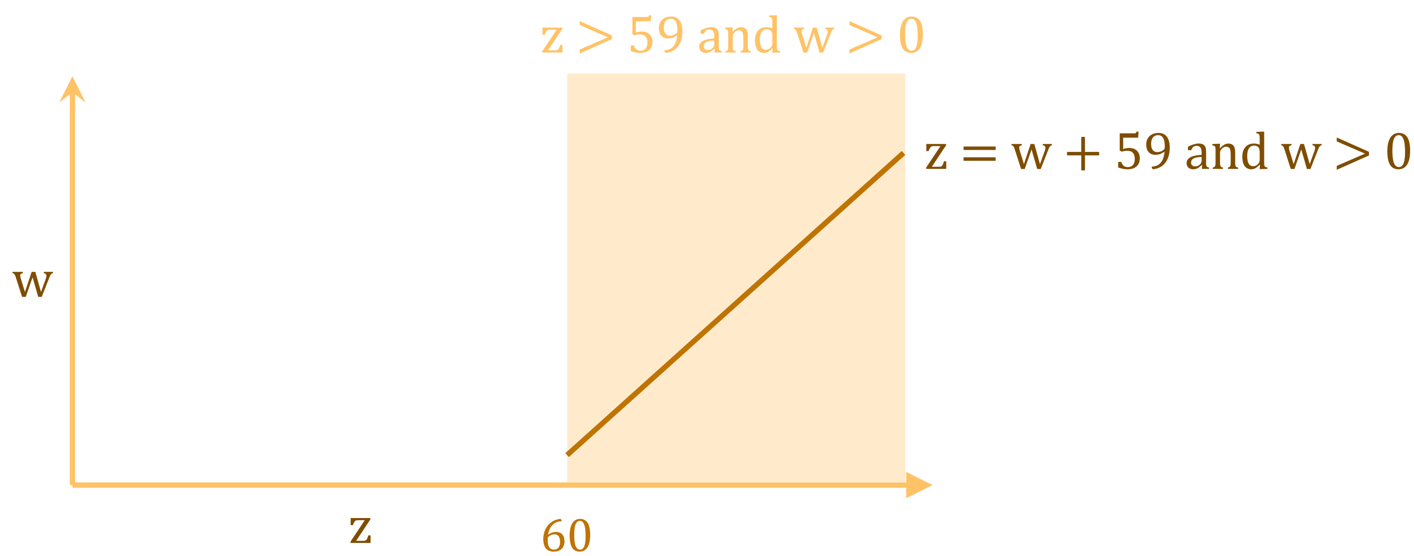
{{ $w > 0$ and $x = 17$ and $y = 42$ }}

```
z = w + x + y;
```

{{ $w > 0$ and $x = 17$ and $y = 42$ and $z = w + x + y$ }}

- **Could write $z = w + 59$, but <u>do not</u> write $z > 59$!**
  - **that is true since $w > 0$, but…**

Example Forward Reasoning through Assignments



- **Could write** $z = w + 59$**, but <u>do not</u> write** $z > 59$ **!**
  - **that is true since** $w > 0$**, but…**

Example Forward Reasoning through Assignments

$\{\{\ w > 0\ \}\}$
```
x = 17n;
```
$\{\{\ w > 0 \text{ and } x = 17\ \}\}$
```
y = 42n;
```
$\{\{\ w > 0 \text{ and } x = 17 \text{ and } y = 42\ \}\}$
```
z = w + x + y;
```
$\{\{w > 0 \text{ and } x = 17 \text{ and } y = 42 \text{ and } z = w + x + y\ \}\}$

- **Could write $z = w + 59$, but do not write $z > 59$!**
  - **that is true since $w > 0$, but…**
  - **that is not the strongest postcondition**

    correctness check could now fail even if the code is right

Code Example of Forward Reasoning

```
// @param w an integer > 0
// @returns an integer z > 59
const f = (w: bigint): bigint => {
  const x = 17n;
  const y = 42n;
  const z = w + x + y;
  return z;
};
```

- **Let's check correctness using Floyd logic…**

Code Example of Forward Reasoning

```
// @param w an integer > 0
// @returns an integer z > 59
const f = (w: bigint): bigint => {
  {{ w > 0 }}
  const x = 17n;
  const y = 42n;
  const z = w + x + y;
  {{ z > 59 }}
  return z;
};
```

- Reason forward…

Code Example of Forward Reasoning

```
// @param w an integer > 0
// @returns an integer z > 59
const f = (w: bigint): bigint => {
  {{ w > 0 }}
  const x = 17n;
  const y = 42n;
  const z = w + x + y;
  {{ w > 0 and x = 17 and y = 42 and z = w + x + y }}
  {{ z > 59 }}
  return z;
};
```

- **Check implication:**

$$
\begin{aligned}
z &= w + x + y \\
&= w + 17 + y && \text{since } x = 17 \\
&= w + 59 && \text{since } y = 42 \\
&> 59 && \text{since } w > 0
\end{aligned}
$$

Code Example of Forward Reasoning

```
// @param w an integer > 0
// @returns an integer z > 59
const f = (w: bigint): bigint => {
  const x = 17n;
  const y = 42n;
  const z = w + x + y;
  return z;
};
```

find facts by reading along path
from top to return statement

- **How about if we use our old approach?**
- **Known facts:** $w > 0$, $x = 17$, $y = 42$, **and** $z = w + x + y$
- **Prove that postcondition holds:** $z > 59$

Code Example of Forward Reasoning

```
// @param w an integer > 0
// @returns an integer z > 59
const f = (w: bigint): bigint => {
  const x = 17n;
  const y = 42n;
  const z = w + x + y;
  return z;
};
```

- We've been doing forward reasoning already!
  - forward reasoning is (only) "and" with *no mutation*

- Line-by-line facts are for "`let`" (not "`const`")

Forward Reasoning through Assignments

- **Forward reasoning is trickier with mutation**
  - **gets harder if we mutate a variable**

```
 w = x + y;
```
$\{\{\ w = x + y\ \}\}$
```
 x = 4n;
```
$\{\{\ w = x + y \text{ and } x = 4\ \}\}$
```
 y = 3n;
```
$\{\{\ w = x + y \text{ and } x = 4 \text{ and } y = 3\ \}\}$

- **Final assertion is not necessarily true**
  - $w = x + y$ **is true with their old values, not the new ones**
  - **changing the value of "**x**" can invalidate facts about** x

    facts refer to the old value, not the new value
  - **avoid this by using different names for old and new values**

Forward Reasoning through Assignments

- Can use subscripts to refer to values at different times

```
… (x: bigint) => …
  …
  x = …
  …
  x = …
  …
  x = …
  …
  x = …
  …
```

x_0	$x = x_0$	"x" means current value
x_1	$x = x_1$	
x_2	$x = x_2$	
x_3	$x = x_3$	
x_4	$x = x_4$	

Forward Reasoning through Assignments

- **<u>Rewrite</u> existing facts to use names of earlier values**
  - will use “x” and “y” to refer to <u>current</u> values
  - can use “x_0” and “y_0” (or other subscripts) for earlier values

$\{\{\ \mathrm{w} = \mathrm{x} + \mathrm{y}\ \}\}$

```
x = 4n;
```

$\{\{\ \mathrm{w} = \mathbf{x_0} + \mathrm{y} \text{ and } \mathbf{x} = 4\ \}\}$

```
y = 3n;
```

$\{\{\ \mathrm{w} = \mathrm{x}_0 + \mathbf{y_0} \text{ and } \mathrm{x} = 4 \text{ and } \mathbf{y} = 3\ \}\}$

- **Final assertion is now accurate**
  - w is equal to the sum of the initial values of x and y

Forward Reasoning through Assignments

- For assignments, general forward reasoning rule is

$\{\{ P \}\}$

```
  x = y;
```

$\{\{ P[x \mapsto x_k] \text{ and } x = y[x \mapsto x_k] \}\}$

  - replace all "x"s in P and y with "x_k"s

- This process can be simplified in many cases
  - no need for x_0 if we can write it in terms of new value
  - e.g., if "$x = x_0 + 1$", then "$x_0 = x - 1$"
  - assertions will be easier to read without old values

(Technically, this is weakening, but it's usually fine

Postconditions usually do not refer to old values of variables.)

Forward Reasoning through Assignments

- For assignments, general forward reasoning rule is

$\{\{\ P\ \}\}$

```
  x = y;
```

$\{\{\ P[x \mapsto x_k] \text{ and } x = y[x \mapsto x_k]\ \}\}$

x_k is name of previous value

- If $x_0 = f(x)$, then we can simplify this to

$\{\{\ P\ \}\}$

```
  x = … x …;
```

$\{\{\ P[x \mapsto f(x)]\ \}\}$

no need for, e.g., "and $x = x_0 + 1$"

  - if assignment is "$x = x_0 + 1$", then "$x_0 = x - 1$"
  - if assignment is "$x = 2x_0$", then "$x_0 = x/2$"
  - does not work for integer division (an un-invertible operation)

Correctness Example by Forward Reasoning

```
/**
 * @param n an integer with n >= 1
 * @returns an integer m with m >= 10
 */
const f = (n: bigint): bigint => {
  {{ n ≥ 1 }}
  n = n + 3n;
  {{ n – 3 ≥ 1 }}
  {{ n² ≥ 10 }}
  return n * n;
};
```

$n = n_0 + 3$ **means** $n - 3 = n_0$

check this implication

$$\begin{aligned} n^2 &\geq 4^2 && \text{since } n - 3 \geq 1 \text{ (i.e., } n \geq 4) \\ &= 16 \\ &> 10 \end{aligned}$$

This is the preferred approach.
Avoid subscripts when possible.

Recall: Forward Reasoning through Assignments

- **Forward reasoning is trickier with mutation**
  - gets harder if we mutate a variable

{{ $w > 0$ }}

```
 x = 4n;
```

{{ $w > 0$ and $x = 4$ }}

```
 y = 3n;
```

{{ $w > 0$ and $x = 4$ and $y = 3$ }}

- **Each assignment just adds one new fact ("and")**

Recall: Code Example of Forward Reasoning

```
// @param w an integer > 0
// @returns an integer z > 59
const f = (w: bigint): bigint => {
  {{ w > 0 }}
  const x = 17n;
  const y = 42n;
  const z = w + x + y;
  {{ w > 0 and x = 17 and y = 42 and z = w + x + y }}
  {{ z > 59 }}
  return z;
};
```

- "Collecting the facts" was forward reasoning
  - only this simple because there was *no mutation*

Recall: Forward Reasoning through Assignments

- **Forward reasoning is trickier with mutation**
  - gets harder if we mutate a variable

```
 w = x + y;
```
$\{\{\ w = x + y\ \}\}$
```
 x = 4n;
```
$\{\{\ w = x + y \text{ and } x = 4\ \}\}$
```
 y = 3n;
```
$\{\{\ w = x + y \text{ and } x = 4 \text{ and } y = 3\ \}\}$

- **Final assertion is not necessarily true!**
  - fact $w = x + y$ was about the *old values* of x and y
  - still true if we clarify which value of x and y we mean

Unique Names for Different Values

- Can use subscripts to refer to values at different times

```
… (x: bigint) => …
  …
  x = …
  …
  x = …
  …
  x = …
  …
  x = …
  …
```

x_0 $x = x_0$ "x" means current value

x_1 $x = x_1$

x_2 $x = x_2$

x_3 $x = x_3$

x_4 $x = x_4$

Recall: Forward Reasoning through Assignments

- <u>Rewrite</u> existing facts to use names of earlier values
  - will use “x” and “y” to refer to <u>current</u> values
  - can use “x_0” and “y_0” (or other subscripts) for earlier values

$\{\{\ w = x + y\ \}\}$

```
x = 4n;
```

$\{\{\ w = \mathbf{x_0} + y \text{ and } \mathbf{x} = 4\ \}\}$

```
y = 3n;
```

$\{\{\ w = x_0 + \mathbf{y_0} \text{ and } x = 4 \text{ and } \mathbf{y} = 3\ \}\}$

- Final assertion is now accurate
  - w is equal to the sum of the initial values of x and y

Recall: Forward Reasoning through Assignments

- For assignments, general forward reasoning rule is

$\{\{\ P\ \}\}$
```
  x = y;
```
$\{\{\ P[x \mapsto x_k] \text{ and } x = y[x \mapsto x_k]\ \}\}$

  – replace all "x"s in P and y with "x_k"s

- This process can be simplified in many cases…

Recall: Forward Reasoning through Assignments

- For assignments, general forward reasoning rule is

{{ P }}
```
  x = x + 1;
```
{{ P and $x = x_0 + 1$ }}

- Can express the old value x_0 in terms of new value
  - if assignment is "$x = x_0 + 1$", then "$x_0 = x - 1$"
  - if assignment is "$x = 2x_0$", then "$x_0 = x/2$"

Recall: Forward Reasoning through Assignments

- **For assignments, general forward reasoning rule is**

$\{\{\ P\ \}\}$

```
x = y;
```

$\{\{\ P[x \mapsto x_k] \text{ and } x = y[x \mapsto x_k]\ \}\}$

x_k **is name of previous value**

- **If $x_0 = f(x)$, then we can simplify this to**

$\{\{\ P\ \}\}$

```
x = … x …;
```

$\{\{\ P[x \mapsto f(x)]\ \}\}$

no need for, e.g., "$\text{and } x = x_0 + 1$"

  - **easier to read without subscripts, so this is preferred**

Recall: Correctness by Forward Reasoning

```
/**
 * @param n an integer with n >= 1
 * @returns an integer m with m >= 10
 */
const f = (n: bigint): bigint => {
  {{ n ≥ 1 }}
  n = n + 3n;
  {{ n – 3 ≥ 1 }}
  {{ n² ≥ 10 }}
  return n * n;
};
```

$n = n_0 + 3$ **means** $n - 3 = n_0$

check this implication

$$\begin{aligned} n^2 &\geq 4^2 && \text{since } n - 3 \geq 1 \text{ (i.e., } n \geq 4) \\ &= 16 \\ &> 10 \end{aligned}$$

This is the preferred approach.
Avoid subscripts when possible.

Mutation in Straight-Line Code

- Alternative ways of writing this code:

```
n = n + 3n;
return n * n;
```

```
const n1 = n + 3n;
return n1 * n1;
```

- Mutation in *straight-line* code is unnecessary
  - can always use different names for each value

- Why would we prefer the former?
  - seems like it might save memory…
  - but it doesn't!

    most compilers will turn the left into the right on their own (SSA form)

    it's better at saving memory than you are, so it does it itself

Example Backward Reasoning with Assignments

$\{\{$ ____________________ $\}\}$

```
x = 17n;
```

$\{\{$ ____________________ $\}\}$

```
y = 42n;
```

$\{\{$ ____________________ $\}\}$

```
z = w + x + y;
```

$\{\{\ z < 0\ \}\}$

- **What must be true before** $\mathtt{z = w + x + y}$ **so** $\mathtt{z} < 0$ **?**
  - want the weakest postcondition (most allowed states)

Example Backward Reasoning with Assignments

$\{\{$ ____________________ $\}\}$

```
 x = 17n;
```

$\{\{$ ____________________ $\}\}$

```
 y = 42n;
```

$\{\{\, w + x + y < 0 \,\}\}$

```
 z = w + x + y;
```

$\{\{\, z < 0 \,\}\}$

- **What must be true before** `z = w + x + y` **so** $z < 0$ **?**
  - **must have** $w + x + y < 0$ **beforehand**

- **What must be true before** `y = 42` **for** $w + x + y < 0$ **?**

Example Backward Reasoning with Assignments

$\{\{$ ________________ $\}\}$

```
x = 17n;
```

$\{\{\, w + x + 42 < 0 \,\}\}$

```
y = 42n;
```

$\{\{\, w + x + y < 0 \,\}\}$

```
z = w + x + y;
```

$\{\{\, z < 0 \,\}\}$

- **What must be true before** `y = 42` **for** $w + x + y < 0$ **?**
  - **must have** $w + x + 42 < 0$ **beforehand**

- **What must be true before** `x = 17` **for** $w + x + 42 < 0$ **?**

Example Backward Reasoning with Assignments

$\{\{\, w + 17 + 42 < 0 \,\}\}$

```
x = 17n;
```

$\{\{\, w + x + 42 < 0 \,\}\}$

```
y = 42n;
```

$\{\{\, w + x + y < 0 \,\}\}$

```
z = w + x + y;
```

$\{\{\, z < 0 \,\}\}$

- **What must be true before** `x = 17` **for** $w + x + 42 < 0$ **?**
  - **must have** $w + 59 < 0$ **beforehand**

- **All we did was <u>substitute</u> right side for the left side**
  - **e.g., substitute** "$w + x + y$" **for** "z" **in** "$z < 0$"
  - **e.g., substitute** "42" **for** "y" **in** "$w + x + y < 0$"
  - **e.g., substitute** "17" **for** "x" **in** "$w + x + 42 < 0$"

Backward Reasoning through Assignments

- For assignments, backward reasoning is substitution

$$\{\{\ Q[x \mapsto y]\ \}\}$$

```
x = y;
```

$$\{\{\ Q\ \}\}$$

  - just replace all the "x"s with "y"s
  - we will denote this substitution by $Q[x \mapsto y]$

- Mechanically simpler than forward reasoning
  - no need for subscripts

Correctness Example by Forward Reasoning

```
/**
 * @param n an integer with n >= 1
 * @returns an integer m with m >= 10
 */
const f = (n: bigint): bigint => {
  {{ n ≥ 1 }}
  n = n + 3n;
  {{ n^2 ≥ 10 }}
  return n * n;
};
```

- Code is correct if this triple is valid…

Correctness Example by Backward Reasoning

```
/**
 * @param n an integer with n >= 1
 * @returns an integer m with m >= 10
 */
const f = (n: bigint): bigint => {
```

$\{\{\ n \geq 1\ \}\}$

$\{\{\ (n+3)^2 \geq 10\ \}\}$

check this implication

```
  n = n + 3n;
```

$\{\{\ n^2 \geq 10\ \}\}$

```
  return n * n;
};
```

$$
\begin{aligned}
(n+3)^2 &\geq (1+3)^2 && \text{since } n \geq 1 \\
&= 16 \\
&> 10
\end{aligned}
$$

Correctness Example by Forward Reasoning

```
/**
 * @param n an integer with n >= 1
 * @returns an integer m with m >= 10
 */
const f = (n: bigint): bigint => {
  {{ n ≥ 1 }}
  n = n + 3n;
  {{ n – 3 ≥ 1 }}
  {{ n² ≥ 10 }}
  return n * n;
};
```

check this implication

$$
\begin{aligned}
n^2 &\geq 4^2 && \text{since } n - 3 \geq 1 \text{ (i.e., } n \geq 4) \\
&= 16 \\
&> 10
\end{aligned}
$$

Forward reasoning produces known facts.
Backward reasoning produces fact to prove.

Conditionals

Conditionals in Functional Programming

```
// Inputs a and b must be integers.
// Returns a non-negative integer.
const f = (a: bigint, b: bigint): bigint => {
  if (a >= 0n && b >= 0n) {
    const L: List = cons(a, cons(b, nil));
    return sum(L);
  }
  …
```

- Prior reasoning also included *conditionals*
  - what does that look like in Floyd logic?

Conditionals in Floyd Logic

```
// Inputs a and b must be integers.
// Returns a non-negative integer.
const f = (a: bigint, b: bigint): bigint => {
  {{ }}
  if (a >= 0n && b >= 0n) {
    {{ a ≥ 0 and b ≥ 0 }}
    const L: List = cons(a, cons(b, nil));
    return sum(L);
  }
  …
```

- Conditionals introduce extra facts in forward reasoning
  - simple “and” since nothing is mutated

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
const g = (n: bigint): bigint => {
  let m;
  if (n >= 0n) {
    m = 2n * n + 1n;
  } else {
    m = 0n;
  }
  return m;
}
```

- Code like this was impossible without mutation
  - cannot write to a "`const`" after its declaration

- How do we handle it now?

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
const g = (n: bigint): bigint => {
  let m;
  if (n >= 0n) {
    m = 2n * n + 1n;
  } else {
    m = 0n;
  }
  return m;
}
```

- Reason *separately* about each path to a `return`
  - handle each path the same as before
  - but now there can be multiple paths to one `return`

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
const g = (n: bigint): bigint => {
  {{ }}
  let m;
  if (n >= 0n) {
    m = 2n * n + 1n;
  } else {
    m = 0n;
  }
  {{ m > n }}
  return m;
}
```

- **Check correctness path through "then" branch**

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
const g = (n: bigint): bigint => {
  {{ }}
  let m;
  if (n >= 0n) {
    {{ n ≥ 0 }}
    m = 2n * n + 1n;
  } else {
    m = 0n;
  }
  {{ m > n }}
  return m;
}
```

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
const g = (n: bigint): bigint => {
  {{ }}
  let m;
  if (n >= 0n) {
    {{ n ≥ 0 }}
    m = 2n * n + 1n;
    {{ n ≥ 0 and m = 2n + 1}}
  } else {
    m = 0n;
  }
  {{ m > n }}
  return m;
}
```

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
const g = (n: bigint): bigint => {
  {{ }}
  let m;
  if (n >= 0n) {
    {{ n ≥ 0 }}
    m = 2n * n + 1n;
    {{ n ≥ 0 and m = 2n + 1}}
  } else {
    m = 0n;
  }
  {{ n ≥ 0 and m = 2n + 1 }}
  {{ m > n }}
  return m;
}
```

$$\begin{aligned} m &= 2n+1 && \\ &> 2n && \text{since } 1 > 0 \\ &\geq n && \text{since } n \geq 0 \end{aligned}$$

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
const g = (n: bigint): bigint => {
  {{ }}
  let m;
  if (n >= 0n) {
    m = 2n * n + 1n;
  } else {
    m = 0n;
  }
  {{ n ≥ 0 and m = 2n + 1 }}
  {{ m > n }}
  return m;
}
```

- Note: no mutation, so we can do this in our head
  - read along the path, and collect all the facts

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
const g = (n: bigint): bigint => {
  {{ }}
  let m;
  if (n >= 0n) {
    m = 2n * n + 1n;
  } else {
    m = 0n;
  }
  {{ n < 0 and m = 0 }}
  {{ m > n }}
  return m;
}
```

$m = 0$

$> n$ since $0 > n$

- Check correctness path through "else" branch
  - note: no mutation, so we can do this in our head

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
const g = (n: bigint): bigint => {
  {{ }}
  let m;
  if (n >= 0n) {
    m = 2n * n + 1n;
    {{ n ≥ 0 and m = 2n + 1 }}
  } else {
    m = 0n;
    {{ n < 0 and m = 0 }}
  }
  {{ ________________________________ }}
  {{ m > n }}
  return m;
}
```

What do we know is true even if we don't know which branch was taken?

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
const g = (n: bigint): bigint => {
  {{ }}
  let m;
  if (n >= 0n) {
    m = 2n * n + 1n;
  } else {
    m = 0n;
  }
  {{ (n ≥ 0 and m = 2n + 1) or (n < 0 and m = 0) }}
  {{ m > n }}
  return m;
}
```

- The “or” means we must reason by cases anyway!

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
const g = (n: bigint): bigint => {
  {{ }}
  let m;
  if (n >= 0n) {
    m = 2n * n + 1n;
  } else {
    return 0n;
  }
  {{ (n ≥ 0 and m = 2n + 1) or (n < 0 and ??) }}
  {{ m > n }}
  return m;
}
```

- What is the state after a "`return`"?

Conditionals in Floyd Logic

```
// Returns an integer m with m > n
const g = (n: bigint): bigint => {
  {{ }}
  let m;
  if (n >= 0n) {
    m = 2n * n + 1n;
  } else {
    return 0n;
  }
  {{ (n ≥ 0 and m = 2n + 1) or (n < 0 and false) }}
  {{ m > n }}
  return m;
}
```

simplifies to just $n \geq 0$ and $m = 2n + 1$

- State after a "`return`" is false (no states)

Conditionals With Returns

- Latter rule for "`if` .. `return`" is useful:

```
{{ P }}
if (cond)
  return something;
{{ P and not cond }}
…
return something else;
```

- Only reach the line after the "`if`" if `cond` was false

- Only one path to each "`return`" statement
  - forward reason to the "`return`" inside the "`if`"
  - forward reason to the "`return`" after the "`if`"

Function Calls

Reasoning about Function Calls

- **Causes no extra difficulties if...**
  1. **defined for all inputs**
  2. **no inputs are mutated** (much, much harder with mutation)

- **Forward reasoning rule is**

$\{\{\ P\ \}\}$

```
x = Math.sin(a);
```

$\{\{\ P[x \mapsto x_0] \text{ and } x = \sin(a)\ \}\}$

- **Backward reasoning rule is**

$\{\{\ Q[x \mapsto \sin(a)]\ \}\}$

```
x = Math.sin(a);
```

$\{\{\ Q\ \}\}$

Reasoning about Function Calls

- **Preconditions must be checked**
  - not valid to call the function on disallowed inputs

- **Forward reasoning rule is**

$\{\{\ P\ \}\}$

```
x = Math.log(a);
```

$\{\{\ P[x \mapsto x_0] \text{ and } x = \log(a)\ \}\}$

Must also check $a \geq 0$

- **Backward reasoning rule is**

$\{\{\ Q[x \mapsto \log(a)] \text{ and } \mathbf{a \geq 0}\ \}\}$

```
x = Math.log(a);
```

$\{\{\ Q\ \}\}$

Function Calls with Imperative Specs

- Applies to functions we define with imperative specs

```
// @param n a non-negative integer
// @returns square(n), where
//         square(0)  := 0
//       square(n+1)  := square(n) + 2n + 1
const square = (n: bigint): bigint => {..}
```

- Reasoning is the same. E.g., forward rule is

{{ P }}

```
x = square(n);
```

{{ $P[x \mapsto x_0]$ and $x = \text{square}(n)$ }}

Must also check that n is non-negative

Function Calls with Declarative Specs

```
// @requires P₂            -- preconditions a, b
// @returns x such that R -- conditions on a, b, x
const f = (a: bigint, b: bigint): bigint => {..}
```

- Forward reasoning rule is

$\{\{\ P\ \}\}$

```
x = f(a, b);
```

$\{\{\ P[x \mapsto x_0] \text{ and } R\ \}\}$

Must also check that P implies P_2

- Backward reasoning rule is

$\{\{\ Q_1 \text{ and } P_2\ \}\}$

```
x = f(a, b);
```

$\{\{\ Q_1 \text{ and } Q_2\ \}\}$

Must also check that R implies Q_2

Q_2 is the part of postcondition using "x"

Loops

Correctness of Loops

- **Assignment and condition reasoning is mechanical**
- **Loop reasoning <u>cannot</u> be made mechanical**
  - **no way around this**

    (**311 alert**: this follows from Rice's Theorem)
- **Thankfully, one *extra* bit of information fixes this**
  - **need to provide a "loop invariant"**
  - **with the invariant, reasoning is again mechanical**

Loop Invariants

- Loop invariant is true **<u>every time</u>** at the top of the loop

```
{{ Inv: I }}
while (cond) {
  S
}
```

  - must be true when we get to the top the first time
  - must remain true each time execute `S` and loop back up

- Use "**Inv:**" to indicate a loop invariant

  otherwise, this only claims to be true the first time at the loop

Loop Invariants

- Loop invariant is true <u>**every time**</u> at the top of the loop

```
{{ Inv: I }}
while (cond) {
  S
}
```

  - must be true 0 times through the loop (at top the first time)
  - if true n times through, must be true $n+1$ times through

- Why do these imply it is always true?
  - follows by structural induction (on $\mathbb{N}$)

Checking Correctness with Loop Invariants

```
{{ P }}
{{ Inv: I }}
while (cond) {
  S
}
{{ Q }}
```

- **How do we check validity with a loop invariant?**
  - intermediate assertion splits into *three* triples to check

Checking Correctness with Loop Invariants

```
{{ P }}
{{ Inv: I }}
while (cond) {
  s
}
{{ Q }}
```

1. I holds initially

Splits correctness into three parts

1. I holds initially
2. s preserves I
3. Q holds when loop exits

Checking Correctness with Loop Invariants

```
{{ P }}
{{ Inv: I }}
while (cond) {
  {{ I and cond }}
    S
  {{ I }}
}
{{ Q }}
```

1. I holds initially

2. S preserves I

Splits correctness into three parts

1. I holds initially
2. S preserves I
3. Q holds when loop exits

Checking Correctness with Loop Invariants

```
{{ P }}
{{ Inv: I }}
while (cond) {
  {{ I and cond }}
    S
  {{ I }}
}
{{ I and not cond }}
{{ Q }}
```

1. I holds initially
2. S preserves I
3. Q holds when loop exits

Splits correctness into three parts

1. I holds initially — implication
2. S preserves I — forward/back then implication
3. Q holds when loop exits — implication

Checking Correctness with Loop Invariants

```
{{ P }}
{{ Inv: I }}
while (cond) {
  S
}
{{ Q }}
```

Formally, invariant split this into three Hoare triples:

1. {{ P }} {{ I }} — I holds initially
2. {{ I and cond }} **S** {{ I }} — **S** preserves I
3. {{ I and not cond }} {{ Q }} — Q holds when loop exits

Loop Correctness Example 1

- This loop claims to n^2

```
{{ }}
let j: bigint = 0n;
let s: bigint = 0n;
{{ Inv: s = j² }}
while (j !== n) {
  j = j + 1n;
  s = s + j + j - 1;
}
{{ s = n² }}
```

Easy to get this wrong!

- might be initializing "j" wrong ($j = 1$?)
- might be exiting at the wrong time ($j \neq n-1$?)
- might have the assignments in wrong order
- ...

Fact that we need to check 3 implications is a strong indication that more bugs are possible.

Loop Correctness Example 1

- This loop claims to n^2

```
{{ }}
let j: bigint = 0n;
let s: bigint = 0n;
{{ Inv: s = j² }}
while (j !== n) {
  j = j + 1n;
  s = s + j + j - 1;
}
{{ s = n² }}
```

Loop Idea

- move j from 0 to n
- keep track of j^2 in s

j	s
0	0
1	1
2	4
3	9
4	16
...	...

Loop Invariant formalizes the **Loop Idea**

Loop Correctness Example 1

- **This loop claims to n^2**

{{ }}

```
let j: number = 0n;
let s: number = 0n;
```

{{ $\mathrm{j} = 0$ and $\mathrm{s} = 0$ }}

{{ Inv: $\mathrm{s} = \mathrm{j}^2$ }}

```
while (j !== n) {
  j = j + 1n;
  s = s + j + j - 1;
}
```

{{ $\mathrm{s} = \mathrm{n}^2$ }}

$$
\begin{aligned}
\mathrm{s} &= 0 \\
&= 0^2 \\
&= \mathrm{j}^2
\end{aligned}
$$

since $\mathrm{j} = 2$

Loop Correctness Example 1

- This loop claims to n^2

{{ **Inv**: $s = j^2$ }}

```
while (j !== n) {
  j = j + 1n;
  s = s + j + j - 1;
}
```

{{ $s = j^2$ and $j = n$ }}

{{ $s = n^2$ }}

$s = j^2$

$\quad = n^2$ **since** $j = n$

Loop Correctness Example 1

- This loop claims to n^2

$\{\{$ **Inv**: $s = j^2$ $\}\}$

```
while (j !== n) {
```

$\{\{\ s = j^2 \text{ and } j \neq n\ \}\}$

```
  j = j + 1n;
  s = s + j + j - 1;
```

$\{\{\ s = j^2\ \}\}$

```
}
```

$\{\{\ s = n^2\ \}\}$

Loop Correctness Example 1

- **This loop claims to n^2**

{{ **Inv**: $s = j^2$ }}

```
while (j !== n) {
```

{{ $s = j^2$ and $j \neq n$ }}

```
  j = j + 1n;
```

{{ $s = (j - 1)^2$ and $j - 1 \neq n$ }}

```
  s = s + j + j - 1;
```

{{ $s = j^2$ }}

```
}
```

{{ $s = n^2$ }}

$j = j_0 + 1$ **means** $j_0 = j - 1$

Loop Correctness Example 1

- This loop claims to n^2

{{ **Inv**: $s = j^2$ }}

```
while (j !== n) {
```

{{ $s = j^2$ and $j \neq n$ }}

```
j = j + 1n;
```

{{ $s = (j - 1)^2$ and $j - 1 \neq n$ }}

```
s = s + j + j - 1;
```

$s = s_0 + 2j - 1$ **means** $s_0 = s - 2j + 1$

{{ $s - 2j + 1 = (j - 1)^2$ and $j - 1 \neq n$ }}

{{ $s = j^2$ }}

```
}
```

{{ $s = n^2$ }}

Loop Correctness Example 1

- **This loop claims to n^2**

$\{\{$ **Inv**: $s = j^2 \}\}$

```
while (j !== n) {
```

$\{\{ s = j^2$ and $j \neq n \}\}$

```
  j = j + 1n;
```

$\{\{ s = (j - 1)^2$ and $j - 1 \neq n \}\}$

```
  s = s + j + j - 1;
```

$\{\{ s - 2j + 1 = (j - 1)^2$ and $j - 1 \neq n \}\}$

$\{\{ s = j^2 \}\}$

```
}
```

$\{\{ s = n^2 \}\}$

$s = 2j - 1 + (j - 1)^2$
$= 2j - 1 + j^2 - 2j + 1$
$= j^2$

since $s - 2j + 1 = (j - 1)^2$

Loop Correctness Example 2

- **Recursive function to calculate sum of list**

$$\begin{aligned} \text{sum}(\text{nil}) &:= 0 \\ \text{sum}(x :: L) &:= x + \text{sum}(L) \end{aligned}$$

- **This loop claims to calculate it as well:**

$\{\{\ L = L_0\ \}\}$

```
let s: bigint = 0n;
```

$\{\{$ **Inv**: $\text{sum}(L_0) = s + \text{sum}(L)\ \}\}$

```
while (L.kind !== "nil") {
  s = s + L.hd;
  L = L.tl;
}
```

$\{\{\ s = \text{sum}(L_0)\ \}\}$

Loop Idea

- move through L front-to-back
- keep sum of *prior* part in s

Loop Correctness Example 2

- **Recursive function to calculate sum of list**

$$\begin{aligned} \mathrm{sum}(\mathrm{nil}) &:= 0 \\ \mathrm{sum}(x :: L) &:= x + \mathrm{sum}(L) \end{aligned}$$

- **Check that the invariant holds initially**

{{ $L = L_0$ }}

```
let s: number = 0n;
```

{{ $L = L_0$ **and** $s = 0$ }}

{{ **Inv**: $\mathrm{sum}(L_0) = s + \mathrm{sum}(L)$ }}

```
while (L.kind !== "nil") {
  ...
```

$\mathrm{sum}(L_0)$

$= \mathrm{sum}(L)$ **since** $L = L_0$

$= 0 + \mathrm{sum}(L)$

$= s + \mathrm{sum}(L)$ **since** $s = 0$

Loop Correctness Example 2

- **Recursive function to calculate sum of list**

$$\begin{aligned} \mathrm{sum}(\mathrm{nil}) &:= 0 \\ \mathrm{sum}(x :: L) &:= x + \mathrm{sum}(L) \end{aligned}$$

- **Check that the postcondition holds at loop exit**

{{ **Inv**: $\mathrm{sum}(L_0) = s + \mathrm{sum}(L)$ }}

```
while (L.kind !== "nil") {
  s = s + L.hd;
  L = L.tl;
}
```

{{ $\mathrm{sum}(L_0) = s + \mathrm{sum}(L)$ and $L = \mathrm{nil}$ }}

{{ $s = \mathrm{sum}(L_0)$ }}

$$\begin{aligned} &\mathrm{sum}(L_0) \\ &\quad = s + \mathrm{sum}(L) \\ &\quad = s + \mathrm{sum}(\mathrm{nil}) && \textbf{since } L = \mathrm{nil} \\ &\quad = s && \textbf{def of } \mathrm{sum} \end{aligned}$$

Loop Correctness Example 2

- **Recursive function to calculate sum of list**

$$\begin{aligned} \text{sum}(\text{nil}) &:= 0 \\ \text{sum}(x :: L) &:= x + \text{sum}(L) \end{aligned}$$

- **Check that the loop body preserves the invariant**

{{ **Inv**: $\text{sum}(L_0) = s + \text{sum}(L)$ }}

```
while (L.kind !== "nil") {
```

{{ $\text{sum}(L_0) = s + \text{sum}(L)$ and $L \neq \text{nil}$ }}

```
  s = s + L.hd;
  L = L.tl;
```

{{ $\text{sum}(L_0) = s + \text{sum}(L)$ }}

```
}
```

$L \neq \text{nil}$ **means** $L = L.\text{hd} :: L.\text{tl}$

Loop Correctness Example 2

- **Recursive function to calculate sum of list**

$$\begin{aligned} \text{sum}(\text{nil}) &:= 0 \\ \text{sum}(x :: L) &:= x + \text{sum}(L) \end{aligned}$$

- **Check that the loop body preserves the invariant**

```
{{ Inv: sum(L_0) = s + sum(L) }}
while (L.kind !== "nil") {
  {{ sum(L_0) = s + sum(L) and L = L.hd :: L.tl }}
  s = s + L.hd;
  L = L.tl;
  {{ sum(L_0) = s + sum(L) }}
}
```

Loop Correctness Example 2

- **Recursive function to calculate sum of list**

$$\begin{aligned} \mathrm{sum}(\mathrm{nil}) &:= 0 \\ \mathrm{sum}(x :: L) &:= x + \mathrm{sum}(L) \end{aligned}$$

- **Check that the loop body preserves the invariant**

{{ **Inv**: $\mathrm{sum}(L_0) = s + \mathrm{sum}(L)$ }}

```
while (L.kind !== "nil") {
```

{{ $\mathrm{sum}(L_0) = s + \mathrm{sum}(L)$ and $L = L.hd :: L.tl$ }}

```
  s = s + L.hd;
```

{{ $\mathrm{sum}(L_0) = s + \mathrm{sum}(L.tl)$ }}

```
  L = L.tl;
```

{{ $\mathrm{sum}(L_0) = s + \mathrm{sum}(L)$ }}

```
}
```

Loop Correctness Example 2

- **Recursive function to calculate sum of list**

$$\begin{aligned} \mathrm{sum}(\mathrm{nil}) &:= 0 \\ \mathrm{sum}(x :: L) &:= x + \mathrm{sum}(L) \end{aligned}$$

- **Check that the loop body preserves the invariant**

{{ **Inv**: $\mathrm{sum}(L_0) = s + \mathrm{sum}(L)$ }}

```
while (L.kind !== "nil") {
```

{{ $\mathrm{sum}(L_0) = s + \mathrm{sum}(L)$ and $L = L.hd :: L.tl$ }}

{{ $\mathrm{sum}(L_0) = s + L.hd + \mathrm{sum}(L.tl)$ }}

```
  s = s + L.hd;
```

{{ $\mathrm{sum}(L_0) = s + \mathrm{sum}(L.tl)$ }}

```
  L = L.tl;
```

{{ $\mathrm{sum}(L_0) = s + \mathrm{sum}(L)$ }}

```
}
```

Loop Correctness Example 2

- **Recursive function to calculate sum of list**

$$\begin{aligned} \text{sum}(\text{nil}) &:= 0 \\ \text{sum}(x :: L) &:= x + \text{sum}(L) \end{aligned}$$

- **Check that the loop body preserves the invariant**

{{ **Inv**: $\text{sum}(L_0) = s + \text{sum}(L)$ }}

```
while (L.kind !== "nil") {
```

{{ $\text{sum}(L_0) = s + \text{sum}(L)$ and $L = L.hd :: L.tl$ }}

{{ $\text{sum}(L_0) = s + L.hd + \text{sum}(L.tl)$ }}

```
  s = s + L.hd;
```

{{ $\text{sum}(L_0) = s + \text{sum}(L.tl)$ }}

```
  L = L.tl;
```

{{ $\text{sum}(L_0) = s + \text{sum}(L)$ }}

```
}
```

$$\begin{aligned} &\text{sum}(L_0) \\ &= s + \text{sum}(L) \\ &= s + \text{sum}(L.hd :: L.tl) && \textbf{since } L = L.hd :: L.tl \\ &= s + L.hd + \text{sum}(L.tl) && \textbf{def of } \text{sum} \end{aligned}$$

Loop Correctness Example 3

- Recursive function to check if y appears in list L

contains(y, nil)	:= false	
contains(y, x :: L)	:= true	if $x = y$
contains(y, x :: L)	:= contains(y, L)	if $x \neq y$

- This loop claims to calculate it as well:

{{ **Inv**: contains(y, L_0) = contains(y, L) }}

```
while (L.kind !== "nil") {
  if (L.hd === y)
    return true;
  L = L.tl;
}
return false;
```

Loop Idea

- move through L front-to-back
- answer remains the same as on the original list L_0
- can only do that if y is *not* found

Loop Correctness Example 3

- **Check that the invariant holds initially**

$\{\{\ L_0 = L\ \}\}$

$\{\{\ \textbf{Inv}\text{: contains}(y, L_0) = \text{contains}(y, L)\ \}\}$

```
while (L.kind !== "nil") {
  if (L.hd === y)
    return true;
  L = L.tl;
}
return false;
```

$\text{contains}(y, L_0)$

$= \text{contains}(y, L)$ **since** $L_0 = L$

$\text{contains}(y, \text{nil})$	$:= \text{false}$	
$\text{contains}(y, x :: L)$	$:= \text{true}$	**if** $x = y$
$\text{contains}(y, x :: L)$	$:= \text{contains}(y, L)$	**if** $x \neq y$

Loop Correctness Example 3

- **Check that the invariant implies the postcondition**

{{ **Inv**: contains(y, L_0) = contains(y, L) }}

```
while (L.kind !== "nil") {
  if (L.hd === y)
    return true;
  L = L.tl;
}
```

{{ contains(y, L_0) = contains(y, L) and L = nil }}
{{ contains(y, L_0) = false }}

```
return false;
```

contains(y, L_0)
= contains(y, L)
= contains(y, nil) **since** L = nil
= false **def of** contains

contains(y, nil) := false
contains(y, x :: L) := true **if** x = y
contains(y, x :: L) := contains(y, L) **if** x ≠ y

Loop Correctness Example 3

- **Check that the body preserves the invariant**

{{ **Inv**: $\text{contains}(y, L_0) = \text{contains}(y, L)$ }}

```
while (L.kind !== "nil") {
```

{{ $\text{contains}(y, L_0) = \text{contains}(y, L)$ and $L \neq \text{nil}$ }}

```
  if (L.hd === y)
    return true;
  L = L.tl;
```

{{ $\text{contains}(y, L_0) = \text{contains}(y, L)$ }}

```
}
return false;
```

$L \neq \text{nil}$ **means** $L = L.\text{hd} :: L.\text{tl}$

$\text{contains}(y, \text{nil})$	$:= \text{false}$	
$\text{contains}(y, x :: L)$	$:= \text{true}$	**if** $x = y$
$\text{contains}(y, x :: L)$	$:= \text{contains}(y, L)$	**if** $x \neq y$

Loop Correctness Example 3

- **Check that the body preserves the invariant**

$\{\{$ **Inv**: $\text{contains}(y, L_0) = \text{contains}(y, L)$ $\}\}$

```
while (L.kind !== "nil") {
```

$\{\{ \text{contains}(y, L_0) = \text{contains}(y, L) \text{ and } L = L.hd :: L.tl \}\}$

```
  if (L.hd === y)
```

$\{\{ \text{contains}(y, L_0) = \text{contains}(y, L) \text{ and } L = L.hd :: L.tl \text{ and } L.hd = y \}\}$

$\{\{ \text{contains}(y, L_0) = \text{true} \}\}$

```
    return true;
  L = L.tl;
```

$\{\{ \text{contains}(y, L_0) = \text{contains}(y, L) \}\}$

```
}
return false;
```

$\text{contains}(y, L_0)$
$= \text{contains}(y, L)$
$= \text{contains}(y, L.hd :: L.tl)$ **since** $L = L.hd :: L.tl$
$= \text{true}$ **since** $y = L.hd$

$\text{contains}(y, \text{nil})$	$:= \text{false}$	
$\text{contains}(y, x :: L)$	$:= \text{true}$	**if** $x = y$
$\text{contains}(y, x :: L)$	$:= \text{contains}(y, L)$	**if** $x \neq y$

Loop Correctness Example 3

- **Check that the body preserves the invariant**

{{ **Inv**: $\text{contains}(y, L_0) = \text{contains}(y, L)$ }}

```
while (L.kind !== "nil") {
```

{{ $\text{contains}(y, L_0) = \text{contains}(y, L)$ and $L = L.hd :: L.tl$ }}

```
  if (L.hd === y)
```

{{ $\text{contains}(y, L_0) = \text{true}$ }}

```
    return true;
```

{{ $\text{contains}(y, L_0) = \text{contains}(y, L)$ and $L = L.hd :: L.tl$ and $L.hd \neq y$ }}

```
  L = L.tl;
```

{{ $\text{contains}(y, L_0) = \text{contains}(y, L)$ }}

```
}
return false;
```

contains(y, nil)	:= false	
contains(y, x :: L)	:= true	if $x = y$
contains(y, x :: L)	:= contains(y, L)	if $x \neq y$

Loop Correctness Example 3

- **Check that the body preserves the invariant**

{{ **Inv**: $\text{contains}(y, L_0) = \text{contains}(y, L)$ }}

```
while (L.kind !== "nil") {
```

{{ $\text{contains}(y, L_0) = \text{contains}(y, L)$ and $L = L.hd :: L.tl$ }}

```
  if (L.hd === y)
```

{{ $\text{contains}(y, L_0) = \text{true}$ }}

```
    return true;
```

{{ $\text{contains}(y, L_0) = \text{contains}(y, L)$ and $L = L.hd :: L.tl$ and $L.hd \neq y$ }}

{{ $\text{contains}(y, L_0) = \text{contains}(y, L.tl)$ }}

```
  L = L.tl;
```

{{ $\text{contains}(y, L_0) = \text{contains}(y, L)$ }}

```
}
return false;
```

$\text{contains}(y, L_0)$
$= \text{contains}(y, L)$
$= \text{contains}(y, L.hd :: L.tl)$ **since** $L = L.hd :: L.tl$
$= \text{contains}(y, L.tl)$ **since** $y \neq L.hd$

$\text{contains}(y, \text{nil}) := \text{false}$
$\text{contains}(y, x :: L) := \text{true}$ **if** $x = y$
$\text{contains}(y, x :: L) := \text{contains}(y, L)$ **if** $x \neq y$

Loop Correctness Example 4

- **Declarative spec of** $\mathrm{sqrt}(\mathrm{x})$

  return $\mathrm{y} \in \mathbb{Z}$ **such that** $(\mathrm{y} - 1)^2 < \mathrm{x} \leq \mathrm{y}^2$

  - **precondition that** x **is positive:** $0 < \mathrm{x}$
  - **precondition that x is not too large:** $\mathrm{x} < 10^{12} = (10^6)^2$

Loop Correctness Example 4

return $y \in \mathbb{Z}$ such that $(y - 1)^2 < x \leq y^2$

- This loop claims to calculate it:

```
let a: bigint = 0;
let b: bigint = 1000000;
{{ Inv: a² < x ≤ b² }}
while (a !== b - 1) {
  const m = (a + b) / 2n;
  if (m*m < x) {
    a = m;
  } else {
    b = m;
  }
}
return b;
```

Loop Idea

- maintain a range $a \dots b$ with x in the range $a^2 \dots b^2$

Loop Correctness Example 4

return $y \in \mathbb{Z}$ **such that** $(y - 1)^2 < x \leq y^2$

- **Check that the invariant holds initially:**

{{ **Pre**: $0 < x \leq 10^{12}$ }}
```
let a: bigint = 0;
let b: bigint = 1000000;
```
{{ **Inv**: $a^2 < x \leq b^2$ }}
```
while (a !== b - 1) {
  …
}
return b;
```

Loop Correctness Example 4

return $y \in \mathbb{Z}$ **such that** $(y - 1)^2 < x \le y^2$

- **Check that the invariant holds initially:**

{{ **Pre**: $0 < x \le 10^{12}$ }}

```
let a: bigint = 0;
let b: bigint = 1000000;
```

{{ $0 < x \le 10^{12}$ and $a = 0$ and $b = 10^6$ }}

{{ **Inv**: $a^2 < x \le b^2$ }}

```
while (a !== b - 1) {
  ...
}
return b;
```

$a^2 = 0^2$ **since** $a = 0$
$= 0$
$< x$

$x < 10^{12}$
$= (10^6)^2$
$= b^2$ **since** $b = 10^6$

Loop Correctness Example 4

return $y \in \mathbb{Z}$ **such that** $(y - 1)^2 < x \leq y^2$

- **Check that the postcondition hold after exit**

{{ **Inv**: $a^2 < x \leq b^2$ }}

```
while (a !== b - 1) {
  …
}
```

{{ **Inv**: $a^2 < x \leq b^2$ and $a = b - 1$ }}

{{ $(b - 1)^2 < x \leq b^2$ }}

```
return b;
```

Does $(y - 1)^2 < x < y^2$ **holds with** $y = b$**?**

$(b - 1)^2$

$= a^2$ since $a = b - 1$

$< x$

Loop Correctness Example 4

return $y \in \mathbb{Z}$ **such that** $(y - 1)^2 < x \leq y^2$

- **Check that the body preserves the invariant:**

$\{\{$ **Inv**: $a^2 < x \leq b^2$ $\}\}$

```
while (a !== b - 1) {
  {{ a² < x ≤ b²  and a ≠ b - 1 }}
  const m = (a + b) / 2n;
  if (m*m < x) {
    a = m;
  } else {
    b = m;
  }
  {{ a² < x ≤ b² }}
}
```

Loop Correctness Example 4

return $y \in \mathbb{Z}$ **such that** $(y - 1)^2 < x \leq y^2$

- **Check that the body preserves the invariant:**

$\{\{$ **Inv**: $a^2 < x \leq b^2$ $\}\}$

```
while (a !== b - 1) {
```

$\{\{$ $a^2 < x \leq b^2$ and $a \neq b - 1$ $\}\}$

```
  const m = (a + b) / 2n;
  if (m*m < x) {
```

$\{\{$ $a^2 < x \leq b^2$ and $a \neq b - 1$ and $m^2 < x$ $\}\}$

```
    a = m;
  } else {
```

$\{\{$ $a^2 < x \leq b^2$ and $a \neq b - 1$ and $x \leq m^2$ $\}\}$

```
    b = m;
  }
```

$\{\{$ $a^2 < x \leq b^2$ $\}\}$

```
}
```

Loop Correctness Example 4

return $y \in \mathbb{Z}$ such that $(y-1)^2 < x \le y^2$

- **Check that the body preserves the invariant:**

```
{{ Inv: a² < x ≤ b² }}
while (a !== b - 1) {
  const m = (a + b) / 2n;
  if (m*m < x) {
    {{ a² < x ≤ b²  and a ≠ b – 1 and m² < x }}
    {{ m² < x ≤ b² }}
    a = m;
  } else {
    {{ a² < x ≤ b²  and a ≠ b – 1 and x ≤ m² }}
    b = m;
  }
  {{ a² < x ≤ b² }}
}
```

Immediate!

Loop Correctness Example 4

return $y \in \mathbb{Z}$ **such that** $(y-1)^2 < x \le y^2$

- **Check that the body preserves the invariant:**

```
{{ Inv: a² < x ≤ b² }}
while (a !== b - 1) {
  const m = (a + b) / 2n;
  if (m*m < x) {
    a = m;
  } else {
    {{ a² < x ≤ b²  and a ≠ b – 1 and x ≤ m² }}
    {{ a² < x ≤ m² }}
    b = m;
  }
  {{ a² < x ≤ b² }}
}
```

Immediate!

Correctness of binary search is pretty easy *once* you have the invariant clear!

Termination

- This analysis does not check that the code **terminates**
  - it shows that the postcondition holds if the loop exits
  - but we never showed that the loop does exit
- Termination follows from the running time analysis
  - e.g., if the code runs in $O(n^2)$ time, then it terminates
  - an infinite loop would be O(infinity)
  - any finite bound on the running time proves it terminates
- Normal to also analyze the running time of our code, and we get termination already from that analysis

Correctness of Loops

- **With straight-line code and conditionals, if the triple is not valid…**
  - **the code is wrong**
  - **there is *some* test case that will prove it**

    (doesn't mean we found that case in our tests, but it exists)

- **With loops, if the triples are not valid…**
  - **the code is wrong *with that invariant***
  - **there may <u>not</u> be any test case that proves it**

    the code may behave correctly on all inputs
  - **the code could be right but with a *different* invariant**

- **Loops are inherently more complicated**