CSE 331
Software Design & Implementation

Kevin Zatloukal

Spring 2022

Lecture 3 – Reasoning about Loops

Floyd Logic

- A Hoare triple is two assertions and one piece of code:

  {*P*} *S* {*Q*}

  - *P* the precondition
  - *S* the code
  - *Q* the postcondition

- A Hoare triple **{*P*} *S* {*Q*}** is called valid if:
  - in any state where P holds,
    executing S produces a state where Q holds
  - i.e., if *P* is true before *S*, then *Q* must be true after it
  - otherwise, the triple is called invalid
  - code is correct iff triple is valid

Reasoning Forward & Backward

- Forward:
  - start with the **given** precondition
  - fill in the **strongest** postcondition

$\{P\}\ S\ \{?\}$ →

- Backward
  - start with the **required** postcondition
  - fill in the **weakest** precondition

$\{?\}\ S\ \{Q\}$ ←

- Finds the “best” assertion that makes the triple valid

Reasoning: Assignments

`x = expr`

- Forward
  - add the fact “x = expr” to what is known
  - BUT you must *fix* any existing references to “x”

- Backward
  - just replace any “x” in the postcondition with expr (substitution)

Reasoning: If Statements

Forward reasoning

{{ P }}
`if (cond)`
{{ P and `cond` }}
`S1`
{{ P1 }}
`else`
{{ P and not `cond` }}
`S2`
{{ P2 }}
{{ P1 or P2 }}

Backward reasoning

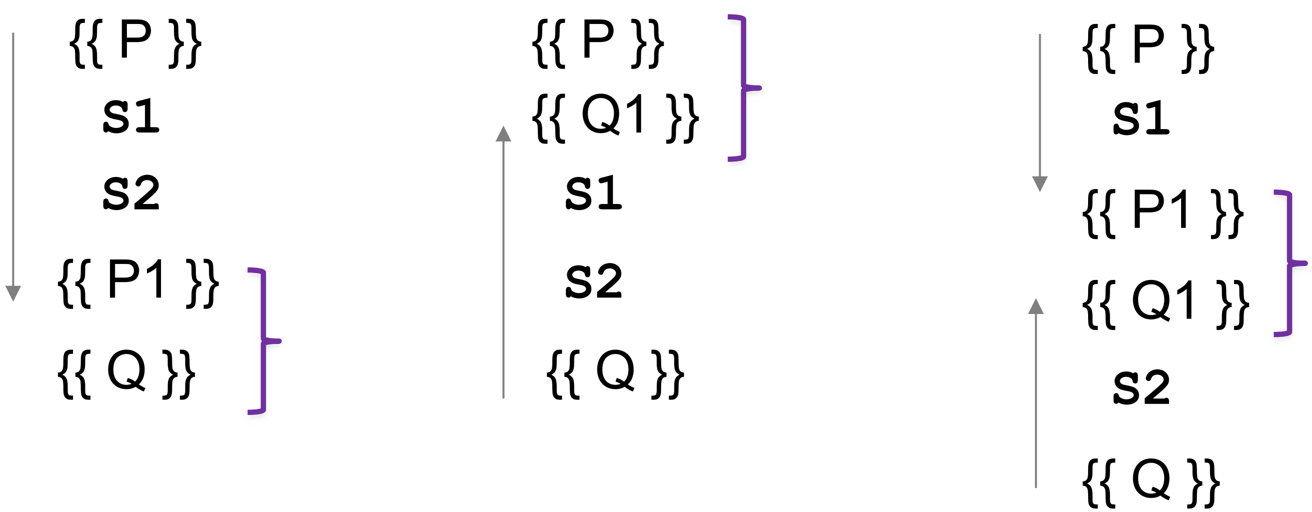
{{ `cond` and Q1 or
not `cond` and Q2 }}
`if (cond)`
{{ Q1 }}
`S1`
{{ Q }}
`else`
{{ Q2 }}
`S2`
{{ Q }}
{{ Q }}

Validity with Fwd & Back Reasoning

Reasoning in either direction gives valid assertions

Just need to check adjacent assertions:

- top assertion must imply bottom one



Reasoning So Far

- “Turn the crank” reasoning for assignment and if statements
- All code (essentially) can be written just using:
  - assignments
  - if statements
  - while loops
- Only part we are missing is **loops**
- (We will also cover function calls later.)

Reasoning About Loops

- Loop reasoning is not as easy as with “=“ and “if”
  - recall Rice’s Theorem (from 311): checking any non-trivial semantic property about programs is **undecidable**
- We need help (more information) before the reasoning again becomes a mechanical process
- That help comes in the form of a “loop invariant”

Loop Invariant

A **loop invariant** is an assertion that holds at the top of the loop:

{{ Inv: I }}
```
while (cond)
  S
```

- It holds when we **first get to** the loop.
- It holds each time we execute `S` and **come back to** the top.

Notation: I'll use "**Inv**:" to indicate a loop invariant.

Lupin variants

Checking Correctness of a Loop

Consider a while-loop (other loop forms not too different) with a loop invariant **`I`**.

Let's try forward reasoning...

{{ P }}
```
S1
```

{{ Inv: **`I`** }}
```
while (cond)
  S2

S3
```
{{ Q }}

Checking Correctness of a Loop

Consider a while-loop (other loop forms not too different) with a loop invariant **`I`**.

Let's try forward reasoning...

```
{{ P }}
 S1
{{ P1 }}

{{ Inv: I }}
 while (cond)
   S2

 S3
{{ Q }}
```

Need to check that P1 implies **`I`** (i.e., that **`I`** is true the first time)

Checking Correctness of a Loop

Consider a while-loop (other loop forms not too different) with a loop invariant **`I`**.

Let's try forward reasoning...

{{ P }}
```
S1
```
{{ Inv: **`I`** }}
```
while (cond)
```
  {{ **`I`** and cond }}
```
  S2
```
  {{ P2 }}

Need to check that P2 implies **`I`** again (i.e., that **`I`** is true each time around)

```
S3
```
{{ Q }}

Checking Correctness of a Loop

Consider a while-loop (other loop forms not too different) with a loop invariant **`I`**.

Let's try forward reasoning...

{{ P }}
```
S1
```

{{ Inv: **`I`** }}
```
while (cond)
  S2
```

{{ **`I`** and not cond }}
```
S3
```
{{ P3 }}
{{ Q }}

Need to check that P3 implies Q
(i.e., Q holds after the loop)

Checking Correctness of a Loop

Consider a while-loop (other loop forms not too different) with a loop invariant **`I`**.

```
{{ P }}
 S1

{{ Inv: I }}
 while (cond)
   S2

 S3
{{ Q }}
```

Informally, we need:

- **`I`** holds initially
- **`I`** holds each time around
- **`Q`** holds after we exit

Formally, we need validity of:

- {{ P }} `S1` {{ **`I`** }}
- {{ **`I`** and cond }} `S2` {{ **`I`** }}
- {{ **`I`** and not cond }} `S3` {{ Q }}

(can check these with backward reasoning instead)

More on Loop Invariants

- Loop invariants are crucial information
  - needs to be provided before reasoning is mechanical
- Pro Tip: always document your invariants for *non-trivial* loops
  - don't make code reviewers guess the invariant
- Pro Tip: with a good loop invariant, the code is easy to write
  - all the creativity can be saved for finding the invariant
  - more on this in later lectures…

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = 0;
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

Equivalent to this “for” loop:

```
s = 0;
for (int i = 0; i != n; i++)
  s = s + b[i];
```

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
i = 0;
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
{{ s = b[0] + ... + b[n-1] }}
```

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
i = 0;
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
{{ s = b[0] + ... + b[n-1] }}
```

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = 0;
```

{{ s = 0 and i = 0 }}

{{ Inv: s = b[0] + ... + b[i-1] }}

```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies
  s = b[0] + … + b[i-1] ?

Less formal

s = sum of first i numbers in b

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
i = 0;
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
{{ s = b[0] + ... + b[n-1] }}
```

- (s = 0 and i = 0) implies
  s = b[0] + … + b[i-1] ?

Less formal

s = sum of first i numbers in b When i = 0, s needs to be the sum of the first 0 numbers, so we need s = 0.

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
i = 0;
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
{{ s = b[0] + ... + b[n-1] }}
```

- (s = 0 and i = 0) implies
  s = b[0] + … + b[i-1] ?

More formal

s = sum of all b[k] with $0 \leq k \leq i-1$

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
i = 0;
```
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```
{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies
  s = b[0] + … + b[i-1] ?

More formal

s = sum of all b[k] with $0 \leq k \leq i-1$

i = 3 ($0 \leq k \leq 2$): s = b[0] + b[1] + b[2]
i = 2 ($0 \leq k \leq 1$): s = b[0] + b[1]
i = 1 ($0 \leq k \leq 0$): s = b[0]
i = 0 ($0 \leq k \leq -1$) s = 0

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = 0;
```

{{ s = 0 and i = 0 }}

{{ Inv: s = b[0] + ... + b[i-1] }}

```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies
  s = b[0] + … + b[i-1] ?

More formal

s = sum of all b[k] with $0 \leq k \leq i-1$

when i = 0, we want to sum over all indexes k satisfying $0 \leq k \leq -1$

There are no such indexes, so we need s = 0

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
i = 0;
```
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```
{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies
  s = b[0] + … + b[i-1] ?

  Yes. (An empty sum is zero.)

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = 0;
```

{{ s = 0 and i = 0 }}

{{ Inv: s = b[0] + ... + b[i-1] }}

```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies **`I`**

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
i = 0;
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
   {{ s = b[0] + ... + b[i-1] and i != n }}
   s = s + b[i];
   i = i + 1;
   {{ s = b[0] + ... + b[i-1] }}
}
{{ s = b[0] + ... + b[n-1] }}
```

- (s = 0 and i = 0) implies **`I`**
- {{ **`I`** and i != n }} `S` {{ **`I`** }} ?

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
i = 0;
```
{{ Inv: s = b[0] + ... + b[i-1] }}
```
while (i != n) {
```
{{ s = b[0] + ... + b[i-1] and i != n }}
```
  s = s + b[i];
  i = i + 1;
```
{{ s = b[0] + ... + b[i-1] }}
```
}
```
{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies **I**
- {{ **I** and i != n }} `S` {{ **I** }} ?

{{ s + b[i] = b[0] + … + b[i] }}
{{ s = b[0] + … + b[i] }}

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = 0;
```

{{ Inv: s = b[0] + ... + b[i-1] }}

```
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[i-1] and not (i != n) }}

{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = 0) implies **`I`**
- {{ **`I`** and i != n }} `S` {{ **`I`** }}
- {{ **`I`** and not (i != n) }} implies
  s = b[0] + … + b[n-1] ?

Example: sum of array

Consider the following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
i = 0;
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  s = s + b[i];
  i = i + 1;
}
{{ s = b[0] + ... + b[n-1] }}
```

- (s = 0 and i = 0) implies **`I`**
- {{ **`I`** and i != n }} `S` {{ **`I`** }}
- {{ **`I`** and i = n }} implies `Q`

These three checks verify that the outermost triple is valid (i.e., that the code is correct).

Termination

- Technically, this analysis does not check that the code **terminates**
  - it shows that the postcondition holds if the loop exits
  - but we never showed that the loop actually exits

- However, that follows from an analysis of the running time
  - e.g., if the code runs in $O(n^2)$ time, then it terminates
  - an infinite loop would be O(infinity)
  - any finite bound on the running time proves it terminates

- It is normal to also analyze the running time of code we write, so we get termination already from that analysis.

Example HW problem

The following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
{{ _____________ }}
i = 0;
{{ _____________ }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  {{ _____________ }}
  s = s + b[i];
  {{ _____________ }}
  i = i + 1;
  {{ _____________ }}
}
{{ _____________ }}
{{ s = b[0] + ... + b[n-1] }}
```

Example HW problem

The following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
{{_____ s = 0 ______________ }}
i = 0;
{{_____ s = 0 and i = 0 ______ }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  {{ ______ s + b[i] = b[0] + ... + b[i] ______ }} or equiv {{ s = b[0] + ... + b[i-1] }}
  s = s + b[i];
  {{ ______ s = b[0] + ... + b[i]  __________ }}
  i = i + 1;
  {{ ______ s = b[0] + ... + b[i-1] _________ }}
}
{{ ______ s = b[0] + ... + b[i-1] and not (i != n) ______ }}
{{ s = b[0] + ... + b[n-1] }}
```

Are we done?

Warning: not just filling in blanks

The following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
{{ s = 0 }}
i = 0;
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
   {{ s = b[0] + ... + b[i-1] }}
   s = s + b[i];
   {{ s = b[0] + ... + b[i] }}
   i = i + 1;
   {{ s = b[0] + ... + b[i-1] }}
}
{{ s = b[0] + ... + b[i-1] and not (i != n) }}
{{ s = b[0] + ... + b[n-1] }}
```

Does invariant hold initially?

Are we done?
No, need to also check...

Warning: not just filling in blanks

The following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
{{ s = 0 }}
i = 0;
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  {{ s = b[0] + ... + b[i-1] }}
  s = s + b[i];
  {{ s = b[0] + ... + b[i] }}
  i = i + 1;
  {{ s = b[0] + ... + b[i-1] }}
}
{{ s = b[0] + ... + b[i-1] and not (i != n) }}
{{ s = b[0] + ... + b[n-1] }}
```

Does loop body preserve invariant?

Are we done?
No, need to also check...

Warning: not just filling in blanks

The following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
```
{{ s = 0 }}
```
i = 0;
```
{{ s = 0 and i = 0 }}

{{ Inv: s = b[0] + ... + b[i-1] }}
```
while (i != n) {
```
{{ s = b[0] + ... + b[i-1] and i != n }}
```
  s = s + b[i];
```
{{ s = b[0] + ... + b[i-1] + b[i] and i != n }}
```
  i = i + 1;
```
{{ s = b[0] + ... + b[i-2] + b[i-1] and i-1 != n }}
```
}
```
{{ s = b[0] + ... + b[i-1] and not (i != n) }}

{{ s = b[0] + ... + b[n-1] }}

Does postcondition hold on termination?

Are we done?
No, need to also check...

Warning: not just filling in blanks

The following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
{{ s = 0 }}
i = 0;
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  {{ s = b[0] + ... + b[i-1] and i != n }}
  s = s + b[i];
  {{ s = b[0] + ... + b[i-1] + b[i] and i != n }}
  i = i + 1;
  {{ s = b[0] + ... + b[i-2] + b[i-1] and i-1 != n }}
}
{{ s = b[0] + ... + b[i-1] and not (i != n) }}
{{ s = b[0] + ... + b[n-1] }}
```

Are we done?
No, need to also check...

HW has “?”s at these three places to indicate a triple that requires explanation

Example: sum of array (attempt 2)

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
i = -1;
```
{{ Inv: s = b[0] + ... + b[i] }}] Changed
```
while (i != n-1) {
  i = i + 1;
  s = s + b[i];
}
```
{{ s = b[0] + ... + b[n-1] }}

Example: sum of array (attempt 2)

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = -1;
```

Changed from i = 0

{{ Inv: s = b[0] + ... + b[i] }}

```
while (i != n-1) {
  i = i + 1;
  s = s + b[i];
}
```

Changed from n

Reordered

{{ s = b[0] + ... + b[n-1] }}

Example: sum of array (attempt 2)

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = -1;
```

{{ Inv: s = b[0] + ... + b[i] }}

```
while (i != n-1) {
  i = i + 1;
  s = s + b[i];
}
```

{{ s = b[0] + ... + b[n-1] }}

Work as before:

- (s = 0 and i = -1) implies **`I`**
  - **`I`** holds initially
- (**`I`** and i = n-1) implies **`Q`**
  - **`I`** implies **`Q`** at exit

Example: sum of array (attempt 2)

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
i = -1;
```
{{ Inv: s = b[0] + ... + b[i] }}
```
while (i != n-1) {
  i = i + 1;
  s = s + b[i];
}
```
{{ s = b[0] + ... + b[n-1] }}

{{ s + b[i+1] = b[0] + ... + b[i+1] }}

{{ s + b[i] = b[0] + … + b[i] }}

{{ s = b[0] + ... + b[i] }}

Example: sum of array (attempt 2)

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = -1;
```

{{ Inv: s = b[0] + ... + b[i] }}

```
while (i != n-1) {
  i = i + 1;
  s = s + b[i];
}
```

{{ s = b[0] + ... + b[n-1] }}

- (s = 0 and i = -1) implies **I**
  - as before
- {{ **I** and i != n-1 }} `S` {{ **I** }}
  - reason backward
- (**I** and i = n-1) implies **Q**
  - as before

Example: sum of array (attempt 3)

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = -1;
```

{{ Inv: s = b[0] + ... + b[i] }}

```
while (i != n-1) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

Suppose we miss-order the assignments to `i` and `s`…

Where does the correctness check fail?

Example: sum of array (attempt 3)

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}

```
s = 0;
i = -1;
```

{{ Inv: s = b[0] + ... + b[i] }}

```
while (i != n-1) {
  s = s + b[i];
  i = i + 1;
}
```

{{ s = b[0] + ... + b[n-1] }}

Suppose we miss-order the assignments to `i` and `s`…

We can spot this bug because the invariant does not hold:

{{ s + b[i] = b[0] + ... + b[i+1] }}
{{ s = b[0] + ... + b[i+1] }}
{{ s = b[0] + ... + b[i] }}

First assertion is not Inv.

Example: sum of array (attempt 3)

Consider the following code to compute `b[0] + … + b[n-1]`:

{{ }}
```
s = 0;
i = -1;
```
{{ Inv: s = b[0] + ... + b[i] }}
```
while (i != n-1) {
  s = s + b[i];
  i = i + 1;
}
```
{{ s = b[0] + ... + b[n-1] }}

Suppose we miss-order the assignments to `i` and `s`…

We can spot this bug because the invariant does not hold:

{{ s = b[0] + ... + b[i-1] + b[i+1] }}

For example, if i = 2, then

s = b[0] + b[1] + b[2] vs
s = b[0] + b[1] + b[3]