CSE 331
Software Design & Implementation

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Fall 2022
Lecture 4½ – Reasoning Wrap-up
Administrivia

• HW2 to be released tonight
  – includes coding part
  – (also has a written problem, independent of the rest)

• Section tomorrow will get you started on coding part

• Bring your laptop (if that is where you plan to work)
  – go through the pre-section setup beforehand
A Harder Example
Example: Dutch National Flag

Given an array of red, white, and blue pebbles, sort the array so the red pebbles are at the front, the white pebbles are in the middle, and the blue pebbles are at the end

Edsgar Dijkstra
Pre- and post-conditions

Precondition: Any mix of red, white, and blue

Mixed colors: red, white, blue

Postcondition:

– red then white then blue
– number of each color is unchanged
Pre- and post-conditions

Precondition: Any mix of red, white, and blue

Mixed colors: red, white, blue

Postcondition:
  – red then white then blue
  – number of each color is unchanged

Want an invariant with
  – postcondition as a special case
  – precondition as a special case (or easy to change to one)
Example: Dutch National Flag

The first idea that comes to mind:

- like postcondition
- like initial condition
Example: Dutch National Flag

The first idea that comes to mind works.

Initial:

Iter 5:

Iter 10:

Iter 15:

Post:
Other potential invariants

Any of these choices work, making the array more-and-more partitioned as you go:

- Red White Blue Mixed
- Red White Mixed Blue
- Red Mixed White Blue
- Mixed Red White Blue
Precise Invariant

Need indices to refer to the split points between colors
  – call these i, j, k

Loop Invariant:
• 0 <= i <= j <= k <= n <= A.length
• A[0], …, A[i-1] are red
• A[i], …, A[j-1] are white
• A[k], …, A[n-1] are blue

No constraints on A[j], …, A[k-1]
Dutch National Flag Code

Invariant:  

Red White Mixed Blue

0 i j k n

Initialization?
Dutch National Flag Code

Invariant:

```
0  i  j  k  n
```

Initialization:
- \( i = j = 0 \) and \( k = n \)
Dutch National Flag Code

Invariant:

\[
\begin{array}{cccc}
Red & White & Mixed & Blue \\
0 & i & j & k & n
\end{array}
\]

Initialization:

- \( i = j = 0 \) and \( k = n \)

Termination condition?
Dutch National Flag Code

Invariant:

Initialization:
• $i = j = 0$ and $k = n$

Termination condition:
• $j = k$
Dutch National Flag Code

```c
int i = 0, j = 0;
int k = n;
{{ Inv: 0 <= i <= j <= k <= n and A[0], ..., A[i-1] are red and ... }}
while (j != k) {

    ??

    need to get j closer to k...
    let's try increasing j by 1

}
```
Dutch National Flag Code

Three cases depending on the value of $A[j]$:

- Red
- White
- Mixed
- Blue

$A[j]$ is either red, white, or blue
Dutch National Flag Code

Three cases depending on the value of $A[j]$:

white

\[
\begin{array}{cccccccc}
0 & i & j & k & n \\
\end{array}
\]

red

\[
\begin{array}{cccccccc}
0 & i & j & k & n \\
\end{array}
\]

blue

\[
\begin{array}{cccccccc}
0 & i & j & k & n \\
\end{array}
\]
Dutch National Flag Code

```c
int i = 0, j = 0;
int k = n;
{{ Inv: 0 <= i <= j <= k <= n and A[0], ..., A[i-1] are red and ... }}
while (j != k) {
    if (A[j] is white) {
        j = j+1;
    } else if (A[j] is blue) {
        swap A[j], A[k-1];
        k = k - 1;
    } else { // A[j] is red
        swap A[i], A[j];
        i = i + 1;
        j = j + 1;
    }
}
```
Binary Search
Example: Binary Search

**Problem:** Given a sorted array $A$ and a number $x$, find index of $x$ (or where it would be inserted) in $A$.

**Idea:** Look at $A[n/2]$ to figure out if $x$ is in $A[0], A[1], ..., A[n/2]$ or in $A[n/2+1], ..., A[n-1]$. Narrow the search for $x$ on each iteration.

(This is an algorithm where you probably still need to go line-by-line even as you get faster at reasoning...)

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Example: Binary Search

**Problem**: Given a sorted array A and a number x, find index of x (or where it would be inserted) in A.

**Idea**: Look at A[n/2] to figure out if x is in A[0], A[1], ..., A[n/2] or in A[n/2+1], ..., A[n-1]. Narrow the search for x on each iteration.

Loop Invariant: A[0], ..., A[i-1] ≤ x < A[j], ..., A[n-1]
- A[i], ..., A[j-1] is the part where we don’t know relation to x
Binary Search Code

Initialization?
Binary Search Code

Initialization:
• i = 0 and j = n
• white region is the whole array
Binary Search Code

Initialization:
• $i = 0$ and $j = n$
• white region is the whole array

Termination condition:
• $i = j$
• white region is empty
• if x is in the array, it is $A[i-1]$
  – if there are multiple copies of x, this returns the last
Binary Search Code

int i = 0;
int j = n;
{{ Inv: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1] and A is sorted }}
while (i != j) {

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    // need to bring i and j closer together...
    // (e.g., increase i or decrease j)

}

{{ A[0], ..., A[i-1] <= x < A[i], ..., A[n-1] }}
Binary Search Code

```c
int i = 0;
int j = n;
{{ Inv: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1] and A is sorted }}
while (i != j) {
    int m = (i + j) / 2;
    if (A[m] <= x) {
    } else {
    }
}
{{ A[0], ..., A[i-1] <= x < A[i], ..., A[n-1] }}
```

Look at the element half way between i and j
Binary Search Code

```c
int i = 0;
int j = n;
{{ Inv: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1] and A is sorted }}
while (i != j) {
    int m = (i + j) / 2;
    if (A[m] <= x) {
        ??
    } else {
    }
}
{{ A[0], ..., A[i-1] <= x < A[i], ..., A[n-1] }}
```

What goes here?
Binary Search Code

```java
int i = 0;
int j = n;
{{ Inv: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1] and A is sorted }}
while (i != j) {
    int m = (i + j) / 2;
    if (A[m] <= x) {
        i = m + 1;
    } else {
        j = m;
    }
}
{{ A[0], ..., A[i-1] <= x < A[i], ..., A[n-1] }}

Since i-1 = m, we have A[i-1] = A[m] <= x
Why do we have A[0] <= ... <= A[i-1]?
Binary Search Code

```c
int i = 0;
int j = n;
{{ Inv: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1] and A is sorted }}
while (i != j) {
    int m = (i + j) / 2;
    if (A[m] <= x) {
        i = m + 1;
    } else {
        j = m;
    }
}
{{ A[0], ..., A[i-1] <= x < A[i], ..., A[n-1] }}
```

invariant satisfied since A[i-1] = A[m] <= x and A is sorted so A[0] <= ... <= A[m]
Binary Search Code

```c
int i = 0;
int j = n;
{{ Inv: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1] and A is sorted }}
while (i != j) {
    int m = (i + j) / 2;
    if (A[m] <= x) {
        i = m + 1;
    } else {
        ??
    }
}
{{ A[0], ..., A[i-1] <= x < A[i], ..., A[n-1] }}
```

What goes here?
Binary Search Code

```c
int i = 0;
int j = n;
{{ Inv: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1] and A is sorted }}
while (i != j) {
    int m = (i + j) / 2;
    if (A[m] <= x) {
        i = m + 1;
    } else {
        j = m;
    }
}
{{ A[0], ..., A[i-1] <= x < A[i], ..., A[n-1] }}
```

invariant satisfied since $x < A[m] = A[j]$ (and A is sorted so $A[m] <= ... <= A[n-1]$)
Binary Search Code

```java
int i = 0;
int j = n;
{{ Inv: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1] and A is sorted }}
while (i != j) {
    int m = (i + j) / 2;
    if (A[m] <= x) {
        i = m + 1;
    } else {
        j = m;
    }
}
{{ A[0], ..., A[i-1] <= x < A[i], ..., A[n-1] }}
```

Does this always terminate?
int i = 0;
int j = n;
{{ Inv: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1] and A is sorted }}
while (i != j) {
    int m = (i + j) / 2;
    if (A[m] <= x) {
        i = m + 1;
    } else {
        j = m;
    }
}
{{ A[0], ..., A[i-1] <= x < A[i], ..., A[n-1] }}
Binary Search Code

```c
int i = 0;
int j = n;
{{ Inv: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1] and A is sorted }}
while (i != j) {
    int m = (i + j) / 2;
    if (A[m] <= x) {
        i = m + 1;
    } else {
        j = m;
    }
}
{{ A[0], ..., A[i-1] <= x < A[i], ..., A[n-1] }}
```

Must satisfy i <= m < j so i increases or j decreases on every iteration.
Binary Search Code

```c
int i = 0;
int j = n;
{{ Inv: A[0], ..., A[i-1] \leq x < A[j], ..., A[n-1] and A is sorted }}
while (i != j) {
    int m = (i + j) / 2;
    if (A[m] \leq x) {
        i = m + 1;
    } else {
        j = m;
    }
}

{{ A[0], ..., A[i-1] \leq x < A[i], ..., A[n-1] }}
```

Is that all we need to do?
Reasoning Summary
Reasoning Summary

• Checking correctness can be a mechanical process
  – using forward or backward reasoning

• This requires that loop invariants are provided
  – those cannot be produced automatically

• Provided you document your loop invariants,
  it should not be too hard for someone else to review your code
Documenting Loop Invariants

• Write down loop invariants for all non-trivial code

• They are often best avoided for “for each” loops:

```java
{{ Inv: printed all the strings seen so far }}
for (String s : L)
    System.out.println(s);
```
Documenting Loop Invariants

- Write down loop invariants for all non-trivial code
- They are often best avoided for “for each” loops:

```java
// Print the strings in L, one per line.
for (String s : L)
    System.out.println(s);
```
Documenting Loop Invariants

• Write down loop invariants for all non-trivial code

• They are often best avoided for “for each” loops.

• Invariants are more helpful when a variable incorporates information from multiple iterations
  – e.g., {{ s = A[0] + … + A[i-1] }}

• Use your best judgement!
Reasoning Summary

• Correctness: tools, inspection, testing
  – need all three to ensure high quality
  – especially cannot leave out inspection

• Inspection (by reasoning) means
  – reasoning through your own code
  – do code reviews

• Practice!
  – essential skill for professional programmers
Reasoning Summary

• You will eventually do this in your head for most code

• Formalism remains useful
  – especially tricky problems
  – interview questions (often tricky)
    • see last example…
Next Topic…
A Problem

“Complete this method such that it returns the location of the largest value in the first $n$ elements of the array $arr$.”

```java
int maxLoc(int[] arr, int n) {
    ...
    }
```
One Solution

```java
int maxLoc(int[] arr, int n) {
    int maxIndex = 0;
    int maxValue = arr[0];
    // Inv: maxValue = max of arr[0] .. arr[i-1] and // maxValue = arr[maxIndex]
    for (int i = 1; i < n; i++) {
        if (arr[i] > maxValue) {
            maxIndex = i;
            maxValue = arr[i];
        }
    }
    return maxIndex;
}
```

What if n = 0?
What if n > arr.length?
What if there are two maximums?

Is this code correct?
A Problem

“Complete this method such that it returns the location of the largest value in the first \( n \) elements of the array \( \text{arr} \).”

```java
int maxLoc(int[] arr, int n) {
    ...
}
```

Could we write a specification so that this is a correct solution?

- precondition that \( n > 0 \)
- throw `ArrayOutOfBoundsException` if \( n > \text{arr}\).length
- return smallest index achieving maximum
Morals

• You can all write the code correctly

• Writing the specification was harder than the code
  – multiple choices for the “right” specification
    • must carefully think through corner cases
  – once the specification is chosen, code is straightforward
  – (both of those will be recurrent themes)

• Some math (e.g. “if n <= 0”) often shows up in specifications
  – English (“if n is less or equal to than 0”) is often worse
How to Check Correctness

• Step 1: need a **specification** for the function
  – can’t argue correctness if we don’t know what it should do
  – surprisingly difficult to write!

• Step 2: determine whether the code meets the specification
  – apply **reasoning**
  – usually easy with the tools we learned
Interview Question
Sorted Matrix Search

Problem Description

Given a matrix M (of size m x n), where every row and every column is sorted, find out whether a given number x is in the matrix.
Sorted Matrix Search

Given a sorted matrix $M$ (of size $m \times n$), where every row and every column is sorted, find out whether a given number $x$ is in the matrix.

(darker color means larger)
Sorted Matrix Search

Given a sorted matrix $M$ (of size $m \times n$), where every row and every column is sorted, find out whether a given number $x$ is in the matrix.

(One) **Idea:** Trace the contour between the numbers $\leq x$ and $> x$ in each row to see if $x$ appears.
Sorted Matrix Search Code

Partial Invariant: $M[i,0], ..., M[i,j-1] < x \leq M[i,j], ..., M[i,n-1]$

- for each $i$, holds for exactly one $j$
- holds when we are in the right spot in row $i$

“...” notation automatically handles special cases:

- if $j = 0$, nothing to the left ("<" constraint is vacuous)
- if $j = n$, nothing to the right ("\leq" constraint is vacuous)
Sorted Matrix Search Code

Initialization:

Partial Invariant: \( \text{M}[i,0], \ldots, \text{M}[i,j-1] < x \leq \text{M}[i,j], \ldots, \text{M}[i,n-1] \)

How do we get the invariant to hold with \( i = 0 \)?
• no easy way to initialize it so the invariant holds
• we need to search...
Sorted Matrix Search Code

Initialization:

New goal: M[0,0], ..., M[0,j-1] < x ≤ M[0,j], ..., M[0,n-1]

- will need a loop to find j
- new loop invariant: x ≤ M[0,j], ..., M[0,n-1]
  - weakening of the new goal
  - decrease j until we get M[0,j-1] to also hold
Sorted Matrix Search Code

Initialization:

```
int i = 0;
int j = ?
{% Inv: x ≤ M[i,j], ..., M[i,n-1] %}
while ( ?? )
  ??
{% M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] %}
```

What is the easiest way to make this hold initially?
Sorted Matrix Search Code

Initialization:

```c
int i = 0;
int j = n;
{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}
while ( ?? )
  ??
{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}
```
Sorted Matrix Search Code

Initialization:

```java
int i = 0;
int j = n;

{{ Inv: x \leq M[i,j], \ldots, M[i,n-1] }}
```

while ( ?? )

```java
{{ M[i,0], \ldots, M[i,j-1] < x \leq M[i,j], \ldots, M[i,n-1] }}
```

When does the postcondition hold? (Careful!)
Sorted Matrix Search Code

Initialization:

```c
int i = 0;
int j = n;
{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}
while (j > 0 && x ≤ M[i,j-1])
  ??
{{ M[i,0], ..., M[i,j] < x ≤ M[i,j], ..., M[i,n-1] }}
```
Sorted Matrix Search Code

Initialization: $i = 0, j = n;$

```c
while (j > 0 && x <= M[i,j-1]) {
    ??
    j = j - 1;
}
```

What goes here?

```c
{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}
```
Sorted Matrix Search Code

Initialization:

```c
int i = 0, j = n;

while (j > 0 && x <= M[i,j-1]) {
    j = j - 1;
}
```

```c
{{ i, 0, ..., i, j-1 < x ≤ M[i,j], ..., M[i,n-1] }}
```
Sorted Matrix Search Code

Initialization:

```cpp
int i = 0, j = n;
{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}
while (j > 0 && x <= M[i,j-1]) {
    j = j - 1;
}
{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}
```

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Sorted Matrix Search Code

Initialization:

\[
\begin{align*}
\text{int } i & = 0; \\
\text{int } j & = n; \\
\{ & \text{ Inv: } x \leq M[i,j], ..., M[i,n-1] \}\end{align*}
\]

\[
\text{while (j > 0 && x } \leq M[i,j-1]) \\
\quad j = j - 1;
\]

\[
\{ & \text{ M[i,0], ..., M[i,j-1] } < x \leq M[i,j], ..., M[i,n-1] \}\]
Sorted Matrix Search Code

\[
\{ \ M[i,0], \ldots, M[i,j-1] < x \leq M[i,j], \ldots, M[i,n-1] \ \}\n\]

That finds the right column in row 0

• can now check \( M[0,j] = x \) (if \( j < n \))
• if not, we can move onto the next row
  – set \( i = i + 1 \)
  – same idea on each row thereafter…
Sorted Matrix Search Code

• Make progress by setting $i = i + 1$
• When $i$ increases, the invariant may be broken
  – we have $x \leq M[i,j] \leq M[i+1,j]$ since columns are sorted
  – and $M[i+1,j] \leq M[i+1,j+1], \ldots, M[i+1,n-1]$ since rows are sorted
  – so we get $x \leq M[i+1,j], \ldots, M[i+1,n-1]$
Sorted Matrix Search Code

• Make progress by setting $i = i + 1$
• When $i$ increases, the invariant may be broken
  – we have $x \leq M[i + 1,j], \ldots, M[i + 1,n - 1]$
  – may need to restore invariant for $M[i,0], \ldots, M[i,j - 1] < x$
  – decrease $j$ until it holds again...
    • when have we seen this before?
    • initialization
Sorted Matrix Search Code

- Make progress by setting $i = i + 1$
- When $i$ increases, the invariant may be broken
  - we have $x \leq M[i+1,j]$, $\ldots$, $M[i+1,n-1]$
  - may need to restore invariant for $M[i,0]$, $\ldots$, $M[i,j-1] < x$
  - could copy and paste the same loop
    - or you can do it with one copy

Don’t try this at home!
Sorted Matrix Search Code

instead of

```c
int i = 0, j = n;
[move j left]
{{ Inv: M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}
while (i != n) {
    i = i + 1;
    [move j left]
}
```

we can write

```c
int i = 0, j = n;
while (i != n) {
    [move j left]
    {{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}
    i = i + 1;
}
```
int i = 0;
int j = n;

while (i != n) {
    {{ Inv: x \leq M[i,j], ..., M[i,n-1] }}
    while (j > 0 && x <= M[i][j-1])
        j = j - 1;

    {{ M[i,0], ..., M[i,j-1] < x \leq M[i,j], ..., M[i,n-1] }}
    if (j < n && x == M[i][j])
        return true;
    i = i + 1;
}
return false;

How do we know from Inv that this is correct?
int i = 0;
int j = n;

while (i != n) {
    {{ Inv: \( x \leq M[i,j], \ldots, M[i,n-1] \) }}
    while (j > 0 && x <= M[i][j-1])
        j = j - 1;
    \{\{ M[i,0], \ldots, M[i,j-1] < x \leq M[i,j], \ldots, M[i,n-1] \}\}
    if (j < n && x == M[i][j])
        return true;
    i = i + 1;
}
return false;

How do we know from Inv that this is correct?

We don't! Something is missing…
int i = 0;
int j = n;
{{ Inv: x not in M[k,l] for k < i and x ≤ M[i,j], ..., M[i,n-1] }}
while (i != n) {
    {{ Inv: x not in M[k,l] for k < i and x ≤ M[i,j], ..., M[i,n-1] }}
    while (j > 0 && x <= M[i][j-1])
        j = j - 1;
    {{ x not in M[k,l] for k < i and M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}
    if (j < n && x == M[i][j])
        return true;
    i = i + 1;
}
return false;