Background

In this problem, we will verify the correctness of operations that work with finite sets of real numbers stored in a sorted array. We will also include the number -∞ as the first element in the array and +∞ as the last element in the array. This removes some special cases that would otherwise need to be handled in the code.

In summary, the array S represents a set in this manner iff it satisfies the constraint that

$$-\infty = S[0] < S[1] < S[2] < \ldots < S[n] < S[n+1] = +\infty$$

where n is the number of elements in the set (which is 2 less than the length of S due to the extra elements at the front and back). This array represents the set {S[1], S[2], .., S[n]}.

In other words, the constraint above is the *representation invariant*, and for an array S satisfying this invariant, its *abstract value* is {S[1], S[2], .., S[n]}.

We will represent real numbers using Java's float type. In addition to being able to represent a wide range of numbers (e.g., -10^{100}, $+10^{100}$, and $1/10^{100}$), a float can also represent -∞ and +∞. Although we will not need it in the code below, there is also a special float value called NaN ("not a number"), which is produced for invalid operations. For example, the calculation 1.0 / 0.0 produces +∞, while the calculation 0.0 / 0.0 produces NaN.

In these problems, we will not require you to *show* every line of reasoning, only the most important ones. However, you will likely want to fill in the important assertions by working through the code line-by-line as before.

Hints: The following facts may be useful:
- The assertion "a < min(b, c)" is equivalent to (another way of writing) "a < b and a < c".
- The assertion "max(a, b) < c" is equivalent to "a < c and b < c".
- Together, "max(a, b) < min(c, d)" is equivalent to "a < c and a < d and b < c and b < d".
- These equivalences also hold with "<" replaced by "≤".

Verifying Correctness of Union

Fill in the indicated assertions by reasoning in the direction indicated by the arrows. We will address the places where "?"s appear on the *page 4*.

Notation: we will use "n" and "m" to refer to the number of points in sets S and T, respectively.

{{ Pre: -∞ = S[0] < S[1] < ... < S[n] < S[n+1] = ∞ and -∞ = T[0] < T[1] < ... < T[m] < T[m+1] = ∞ and
        U is an array containing at least n+m+2 elements }}
↓ U[0] = Float.NEGATIVE_INFINITY;
↓ int i = 1, j = 1, k = 1;

{{ Pre and _____ }}

? answer 1 on a separate page

{{ Inv: Pre and -∞ = U[0] < U[1] < ... < U[k-1] < min(S[i], T[j]) and
    {U[1], U[2], ..., U[k-1]} = {S[1], S[2], ..., S[i-1]} ∪ {T[1], T[2], ..., T[j-1]} }}
while (S[i] < Float.POSITIVE_INFINITY || T[j] < Float.POSITIVE_INFINITY) {
↓   if (S[i] < T[j]) {
    {{ Inv and _____ }}

    U[k] = S[i]; // ? answer 2 on the page 4

    {{ _____
       _____ }}
↑   i = i + 1;
↑   k = k + 1;
  } else if (S[i] > T[j]) {
    {{ Inv and _____ }}

    U[k] = T[j]; // ? answer 3 on the page 4

    {{ _____
       _____ }}
↑   j = j + 1;
↑   k = k + 1;
  } else {
    {{ Inv and _____ }}

    U[k] = S[i]; // ? answer 4 on the page 4

    {{ _____
       _____ }}

  // continued on the next page…

↑   i = i + 1;
↑   j = j + 1;
↑   k = k + 1;
  }
 }

↓ U[k] = Float.POSITIVE_INFINITY;

{{ Inv and _____ }}

? answer 5 on the next page

{{ Post: -∞ = U[0] < U[1] < ... < U[k] = ∞ and
            {U[1], U[2], ..., U[k-1]} = {S[1], S[2], ..., S[n]} ∪ {T[1], T[2], ..., T[m]} }}

Explanations

Explain why the invariant holds initially (answer 1) and why the postcondition holds (answer 5):

Comparing Assertions

The three "?"s in the middle of the loop (answers 2-4) appear next to an assignment and between two assertions that you filled in on the previous page. In each case, compare the two assertions and explain **which facts in the bottom assertion are missing in the top assertion**. In other words, which facts are not stated in the top assertion but are necessary to show that the top assertion implies the bottom? (Do not yet worry about whether these facts hold.)

More Explanations

For the three cases above, explain why the additional facts must hold from what we know in the top assertion and the assignment statement in between them.

Addendum

Congratulations on checking the correctness of a complex algorithm! While arrays are familiar data structures, the constraints added to them with this representation are tricky.

As you continue programming, you will find that this example is **typical** of what it is like to check the correctness of complex algorithms. They typically have invariants with many facts to keep track of and loop bodies with several cases to consider. We check their correctness exactly as you did above: by reasoning forward and backward into each case, comparing the facts known before and needed after to identify the additional facts that need to hold, and then figuring out why the known facts in that specific case ensure that they do always hold.

If you were able to work through this example, I think you can check the correctness of many other complex algorithms. With that in mind, let's do some more practice…

Verifying Correctness of Intersection

Fill in the indicated assertions by reasoning in the direction indicated by the arrows. We will address the places where "?"s appear on the *next page*.

{{ Pre: -∞ = S[0] < S[1] < … < S[n] < S[n+1] = ∞ and -∞ = T[0] < T[1] < … < T[m] < T[m+1] = ∞ and

      U is an array containing at least min(n,m)+2 elements }}

↓ U[0] = Float.NEGATIVE_INFINITY;

↓ int i = 1, j = 1, k = 1;

  ? answer 1 on the next page

{{ Inv: Pre and -∞ = U[0] < U[1] < … < U[k-1] < min(S[i], T[j]) and max(S[i-1], T[j-1]) < min(S[i], T[j]) and

    {U[1], U[2], …, U[k-1]} = {S[1], S[2], …, S[i-1]} ∩ {T[1], T[2], …, T[j-1]} }}

  while (S[i] < Float.POSITIVE_INFINITY || T[j] < Float.POSITIVE_INFINITY) {

↓   if (S[i] < T[j]) {

     {{ Inv and _____ }}

       ? answer 2 on the next page

        {{ _____

                        _____ }}

↑   i = i + 1;

    } else if (S[i] > T[j]) {

     {{ Inv and _____ }}

       ? answer 3 on the next page

        {{ _____

                        _____ }}

↑   j = j + 1;

    } else {

     {{ Inv and _____ }}

     U[k] = S[i]; // ? answer 4 on the next page

        {{ _____

                        _____ }}

↑   i = i + 1;

↑   j = j + 1;

↑   k = k + 1;

    }

  }

↓ U[k] = Float.POSITIVE_INFINITY;

{{ Inv and _____ }}

? answer 5 on the next page

{{ Post: -∞ = U[0] < U[1] < … < U[k] = ∞ and

      {U[1], U[2], …, U[k-1]} = {S[1], S[2], …, S[n]} ∩ {T[1], T[2], …, T[m]} }}

Explanations

Explain why the invariant holds initially (answer 1) and why the postcondition holds (answer 5):

Comparing Assertions

The three "?"s in the middle of the loop (answers 2-4) appear next to an assignment and between two assertions that you filled in on the previous page. In each case, compare the two assertions and explain **which facts in the bottom assertion are missing in the top assertion**. In other words, which facts are not stated in the top assertion but are necessary to show that the top assertion implies the bottom? (Do not yet worry about whether these facts hold.)

More Explanations

For the three cases above, explain why the additional facts must hold from what we know in the top assertion and the assignment statement in between them.

Code Review

For which of the five cases we considered earlier (before the loop, after the loop, and the three parts of the if/else if/else statement), do you think the author should have included comments?