CSE 331
Software Design & Implementation

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Spring 2020

Lecture 4½ – An Interview Question

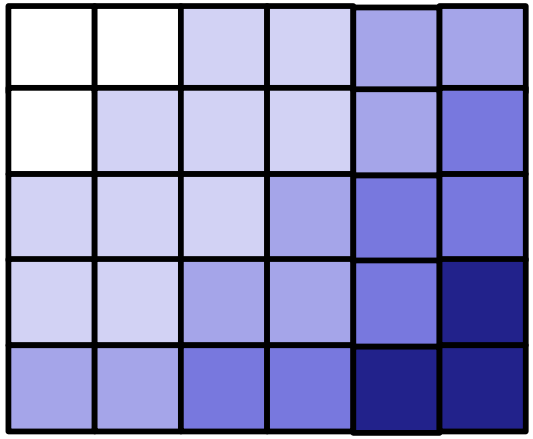
Sorted Matrix Search

Problem Description

Given a matrix M (of size m x n), where every row and every column is sorted, find out whether a given number x is in the matrix.

Sorted Matrix Search

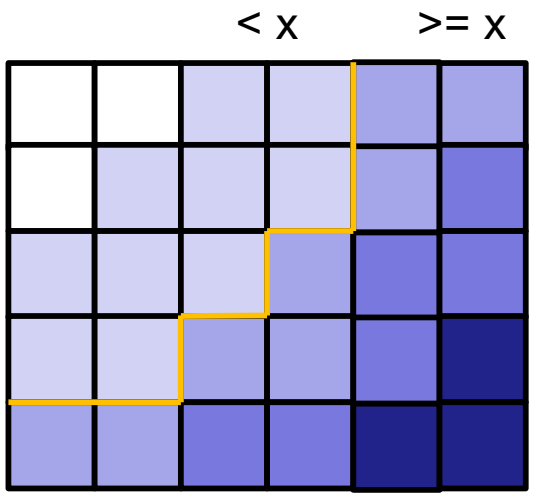
Given a sorted matrix M (of size m x n), where every row and every column is sorted, find out whether a given number x is in the matrix.



(darker color means larger)

Sorted Matrix Search

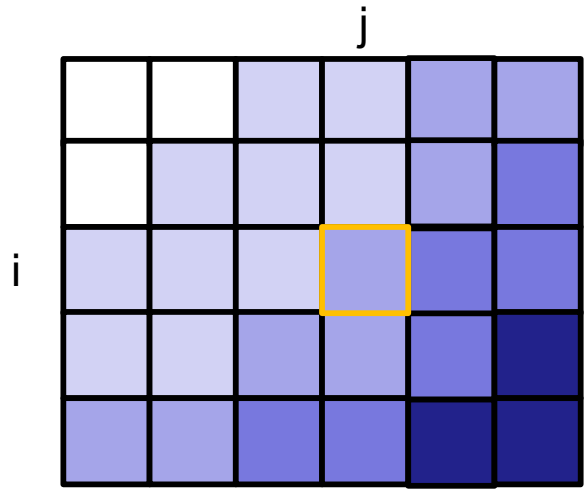
Given a sorted matrix M (of size m x n), where every row and every column is sorted, find out whether a given number x is in the matrix.



(darker color means larger)

(One) **Idea**: Trace the contour between the numbers ≤ x and > x in each row to see if x appears.

Sorted Matrix Search Code

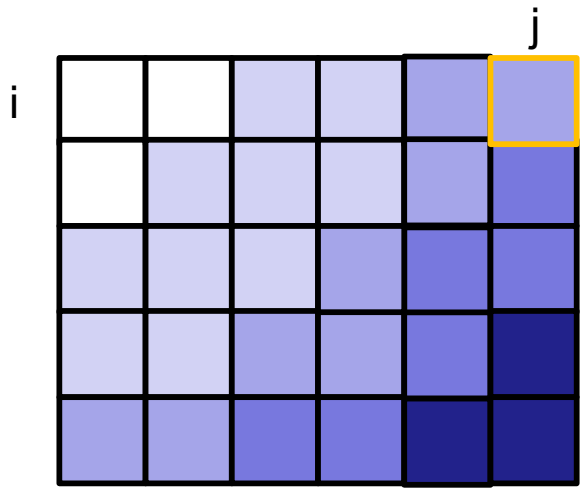


Partial Invariant: M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1]

- for each i, holds for exactly one j
- holds when we are in the right spot in row i

Sorted Matrix Search Code

Initialization:



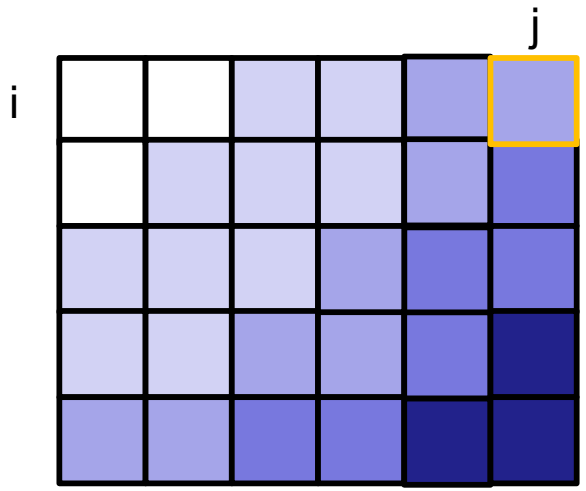
Partial Invariant: M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1]

How do we get the invariant to hold with i = 0?

- no easy way to initialize it so the invariant holds
- we need to search...

Sorted Matrix Search Code

Initialization:

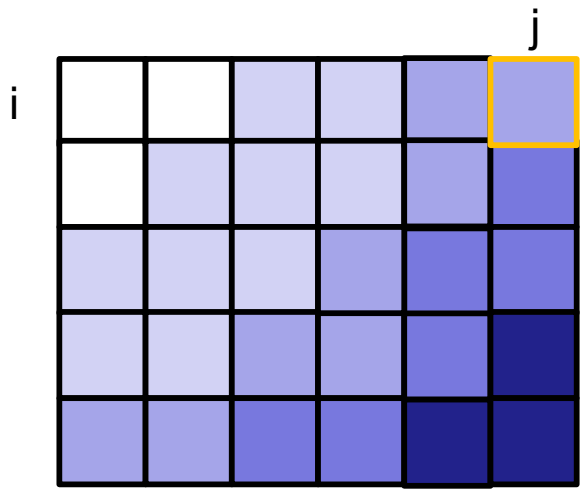


New goal: M[0,0], ..., M[0,j-1] < x ≤ M[0,j], ..., M[0,n-1]

- will need a loop to find j
- Loop invariant: x ≤ M[0,j], ..., M[0,n-1]
  - weakening of the new goal
  - decrease j until we get M[0,j-1] to also hold

Sorted Matrix Search Code

Initialization:




```
int i = 0;
int j = ?
```

{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}

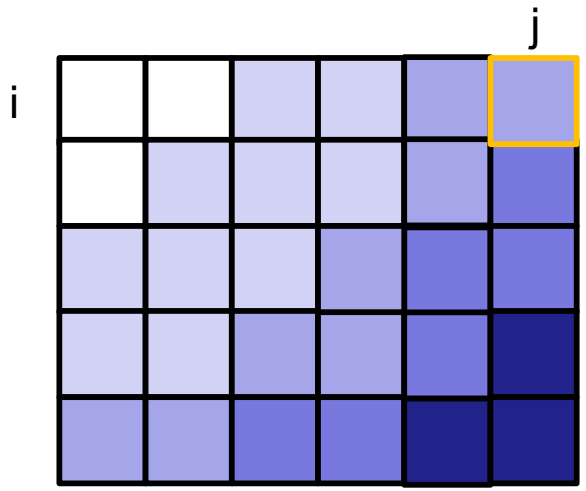
What is the easiest way to make this hold initially?

```
while ( ?? )
   ??
```

{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

Sorted Matrix Search Code

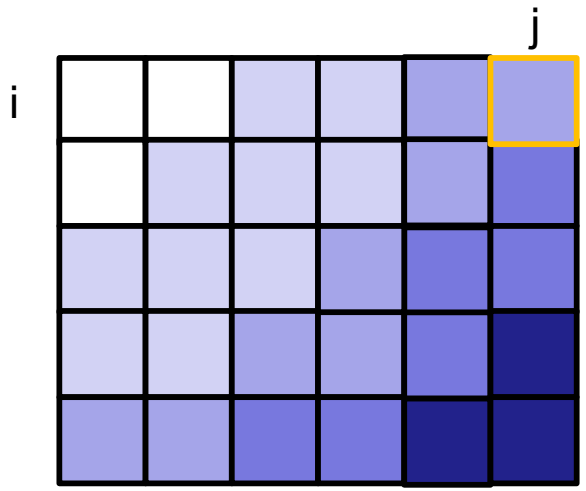
Initialization:




```
int i = 0;
int j = n;
```
{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}
```
while ( ?? )
   ??
```
{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

Sorted Matrix Search Code

Initialization:




```
int i = 0;
int j = n;
```

{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}

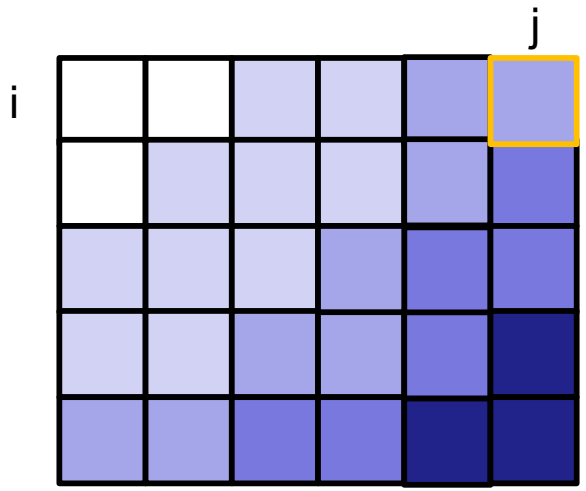
```
while ( ?? )
  ??
```

When does the postcondition hold? (Careful!)

{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

Sorted Matrix Search Code

Initialization:




```
int i = 0;
int j = n;
```

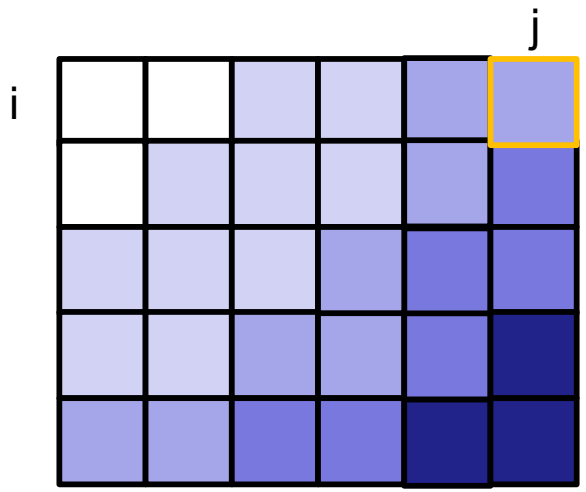
{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}

```
while (j > 0 && x <= M[i,j-1])
   ??
```

{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

Sorted Matrix Search Code

Initialization:




```
int i = 0, j = n;
```

{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}

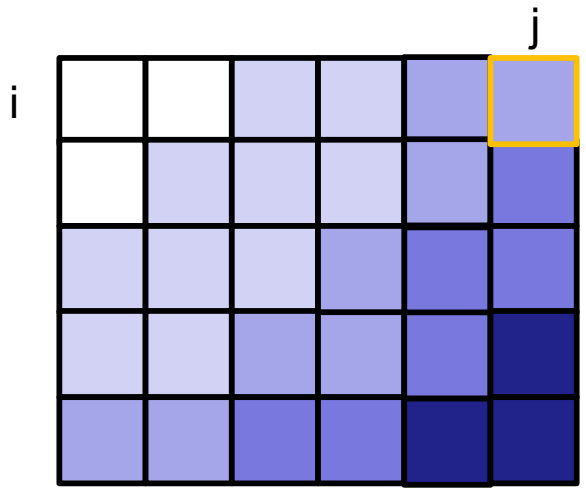
```
while (j > 0 && x <= M[i,j-1]) {
  ??
  j = j - 1;
}
```

What goes here?

{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

Sorted Matrix Search Code

Initialization:




```
int i = 0, j = n;
```

{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}

```
while (j > 0 && x <= M[i,j-1]) {
```

{{ x ≤ M[i,j], ..., M[i,n-1] and x ≤ M[i,j-1] }}

```
  ??
```

{{ x ≤ M[i,j-1], ..., M[i,n-1] }}

```
  j = j - 1;
```

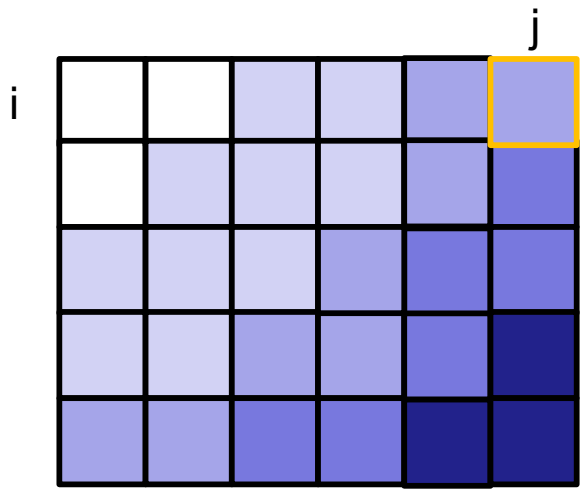
{{ x ≤ M[i,j], ..., M[i,n-1] }}

```
}
```

{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

Sorted Matrix Search Code

Initialization:




```
int i = 0, j = n;
```

{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}

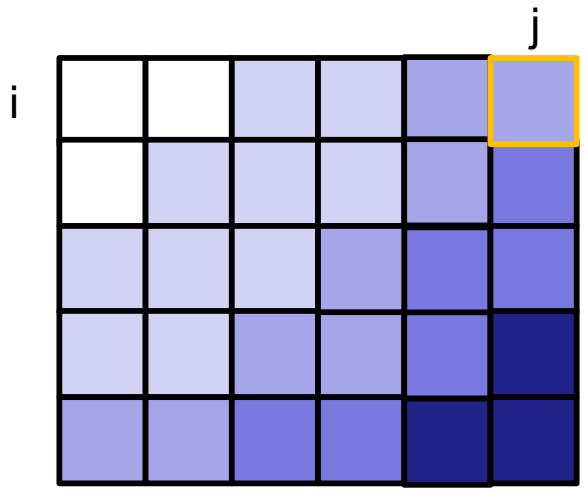
```
while (j > 0 && x <= M[i,j-1]) {

  j = j - 1;
}
```

What goes here?
Nothing!

{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

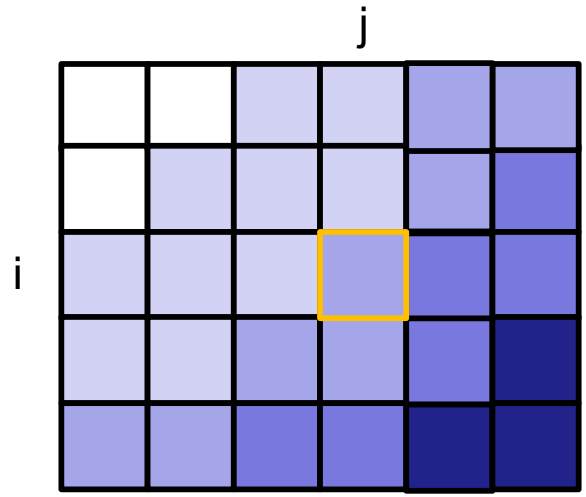
Sorted Matrix Search Code

Initialization:




```
int i = 0;
int j = n;
```
{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}
```
while (j > 0 && x <= M[i,j-1])
  j = j - 1;
```
{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

Sorted Matrix Search Code

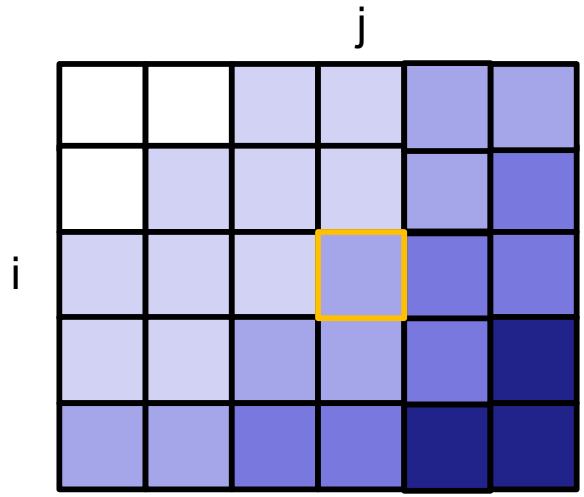


That finds the right column in row 0

- can now check M[0,j] = x (if j < n)
- if not, we can move onto the next row
  - x cannot be anywhere in the row if it's not at M[i,j]
  - set `i = i + 1`

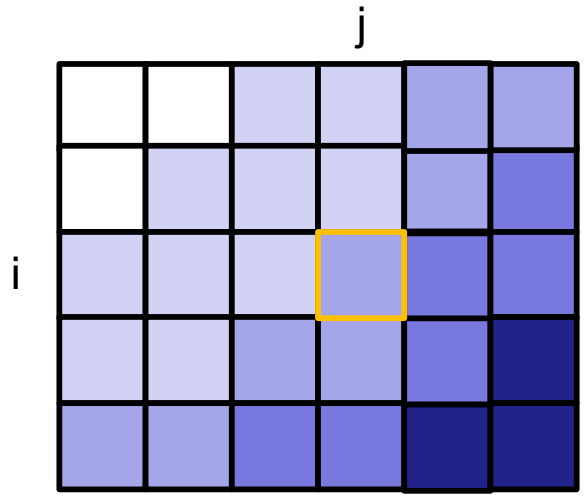
Process continues in each row thereafter...

Sorted Matrix Search Code



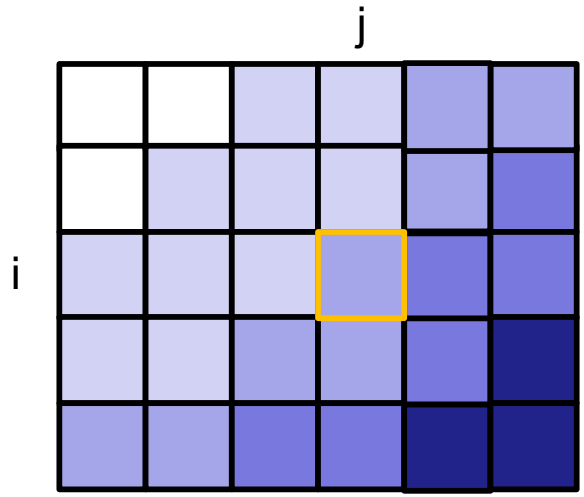
- Make progress by setting `i = i + 1`
- When i increases, the invariant may be broken
  - we have x ≤ M[i,j] ≤ M[i+1,j] since columns are sorted
  - and M[i+1,j] ≤ M[i +1,j+1], .., M[i +1,n-1] since rows are sorted
  - so we get x ≤ M[i +1,j], .., M[i +1,n-1]

Sorted Matrix Search Code



- Make progress by setting `i = i + 1`
- When i increases, the invariant may be broken
  - we have x <= M[i +1,j], .., M[i +1,n-1]
  - may need to restore invariant for M[i,0], ..., M[i,j-1] < x
  - decrease j until it holds again...
    - when have we seen this before?
    - initialization

Sorted Matrix Search Code



- Make progress by setting `i = i + 1`
- When i increases, the invariant may be broken
  - we have x <= M[i +1,j], .., M[i +1,n-1]
  - may need to restore invariant for M[i,0], ..., M[i,j-1] < x
  - could copy and paste the same loop
    - or you can do it with one copy

Don't try this at home!

Sorted Matrix Search Code

instead of

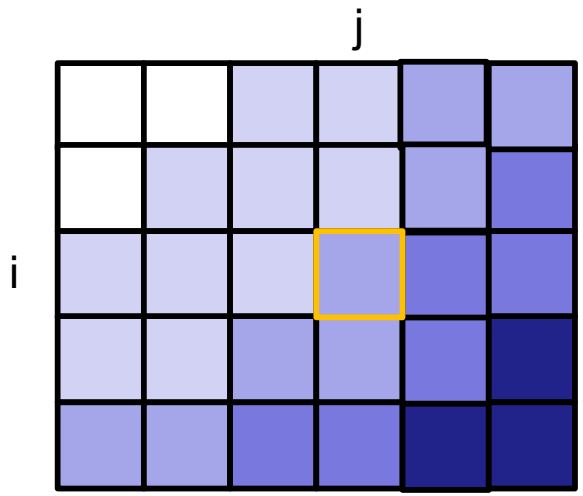
```
int i = 0, j = n;
[move j left]
{{ Inv: M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}
while (i != n) {
  i = i + 1;
  [move j left]
}
```

we can write

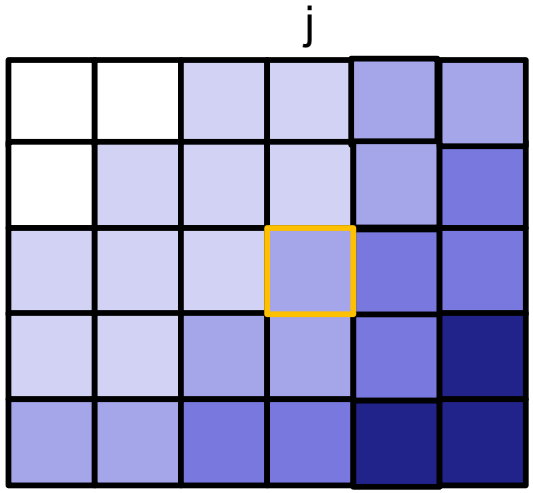
```
int i = 0, j = n;
while (i != n) {
  [move j left]
  {{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}
  i = i + 1;
}
```

Sorted Matrix Search Code

```
int i = 0;
int j = n;

while (i != n) {
   {{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}
   while (j > 0 && x <= M[i,j-1])
     j = j - 1;

   {{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}
   if (j < n && x == M[i,j])
     return true;
   i = i + 1;
}
return false;
```



Sorted Matrix Search Code




```
int i = 0;
int j = n;
{{ Inv: x not in M[k,l] for k < i and x ≤ M[i,j], ..., M[i,n-1] }}
while (i != n) {
   {{ Inv: x not in M[k,l] for k < i and x ≤ M[i,j], ..., M[i,n-1] }}
   while (j > 0 && x <= M[i,j-1])
      j = j - 1;

   {{ x not in M[k,l] for k < i and M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}
   if (j < n && x == M[i,j])
      return true;
   i = i + 1;
}
return false;
```