CSE 331
Software Design & Implementation

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Fall 2020

Lecture 4½ – Reasoning Wrap-up

Interview Question

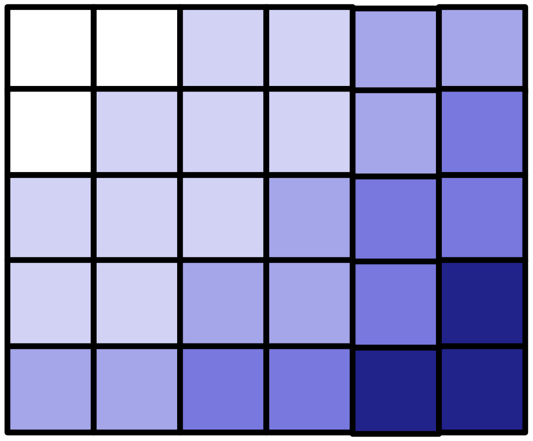
Sorted Matrix Search

Problem Description

Given a matrix M (of size m x n), where every row and every column is sorted, find out whether a given number x is in the matrix.

Sorted Matrix Search

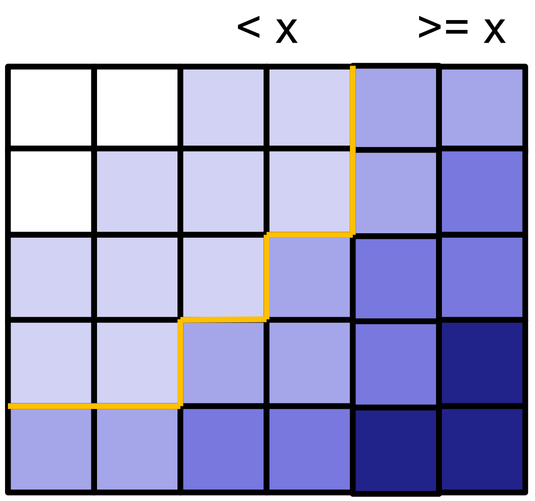
Given a sorted matrix M (of size m x n), where every row and every column is sorted, find out whether a given number x is in the matrix.



(darker color means larger)

Sorted Matrix Search

Given a sorted matrix M (of size m x n), where every row and every column is sorted, find out whether a given number x is in the matrix.



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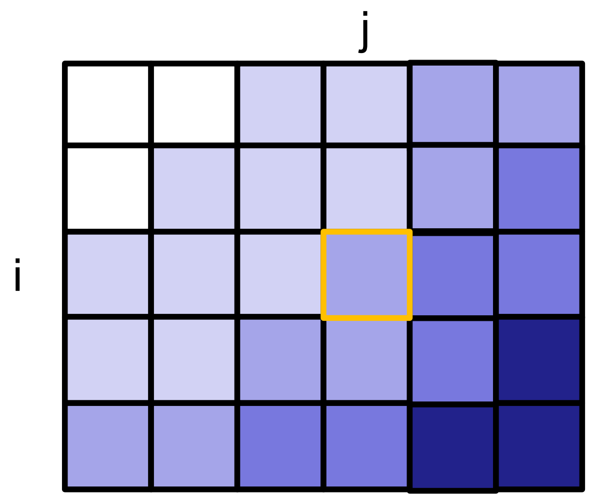
(One) **Idea**: Trace the contour between the numbers ≤ x and > x in each row to see if x appears.

Sorted Matrix Search Code

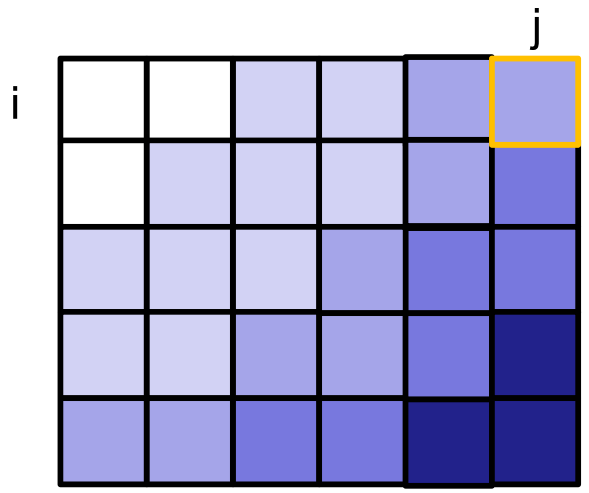


Partial Invariant: M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1]

- for each i, holds for exactly one j
- holds when we are in the right spot in row i

Sorted Matrix Search Code

Initialization:



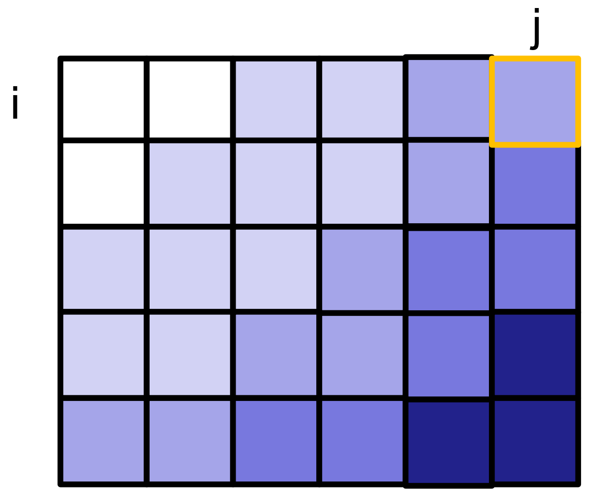
Partial Invariant: M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1]

How do we get the invariant to hold with i = 0?

- no easy way to initialize it so the invariant holds
- we need to search...

Sorted Matrix Search Code

Initialization:

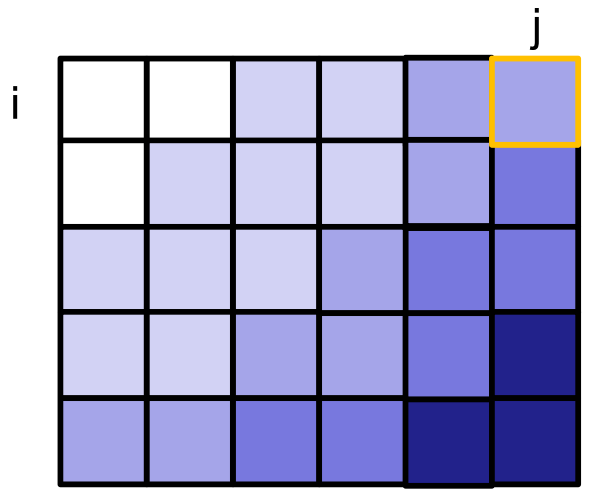


New goal: M[0,0], ..., M[0,j-1] < x ≤ M[0,j], ..., M[0,n-1]

- will need a loop to find j
- Loop invariant: x ≤ M[0,j], ..., M[0,n-1]
  - weakening of the new goal
  - decrease j until we get M[0,j-1] to also hold

Sorted Matrix Search Code

Initialization:

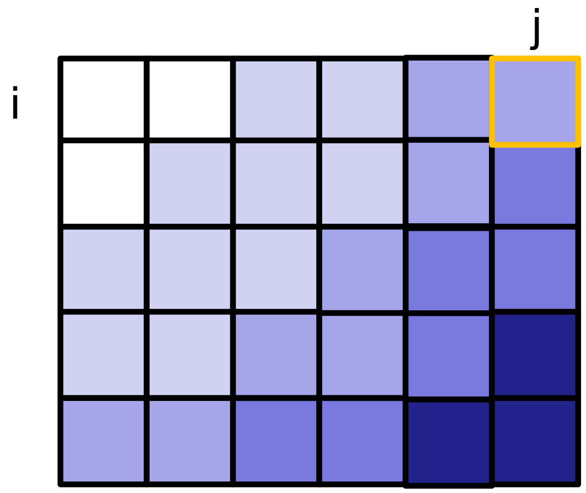


```
int i = 0;
int j = ?
```
{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}
```
while ( ?? )
   ??
```
{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

What is the easiest way to make this hold initially?

Sorted Matrix Search Code

Initialization:



```
int i = 0;
int j = n;
```

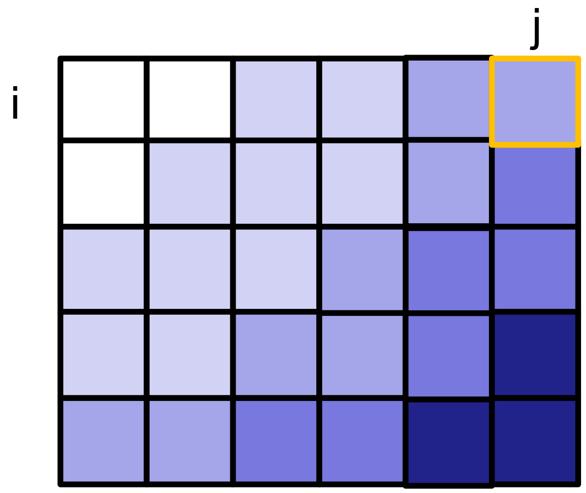
{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}

```
while ( ?? )
  ??
```

{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

Sorted Matrix Search Code

Initialization:



```
int i = 0;
int j = n;
```

{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}

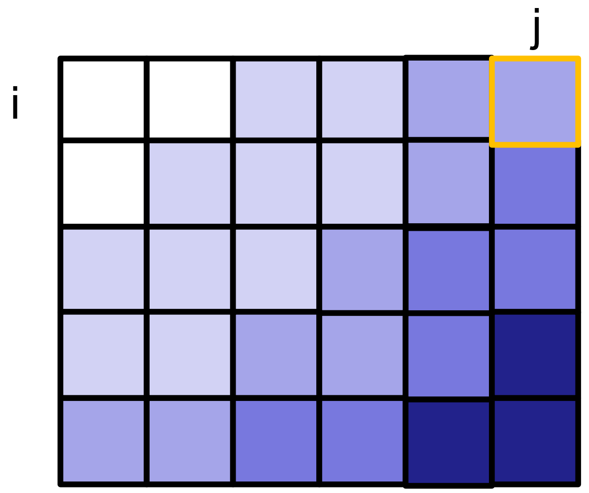
```
while ( ?? )
  ??
```

{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

When does the postcondition hold? (Careful!)

Sorted Matrix Search Code

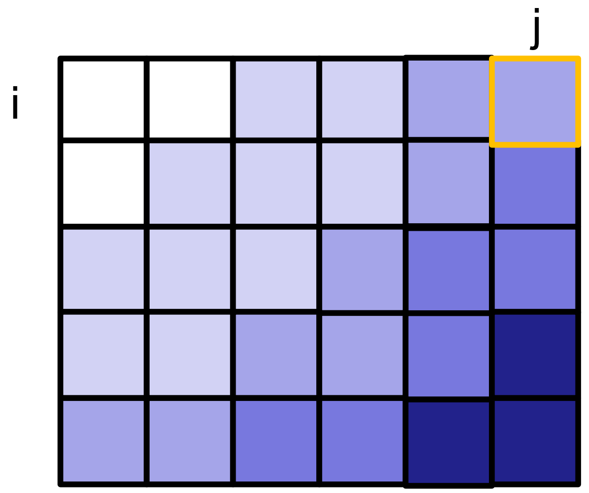
Initialization:



```
int i = 0;
int j = n;
```
{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}
```
while (j > 0 && x <= M[i,j-1])
  ??
```
{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

Sorted Matrix Search Code

Initialization:

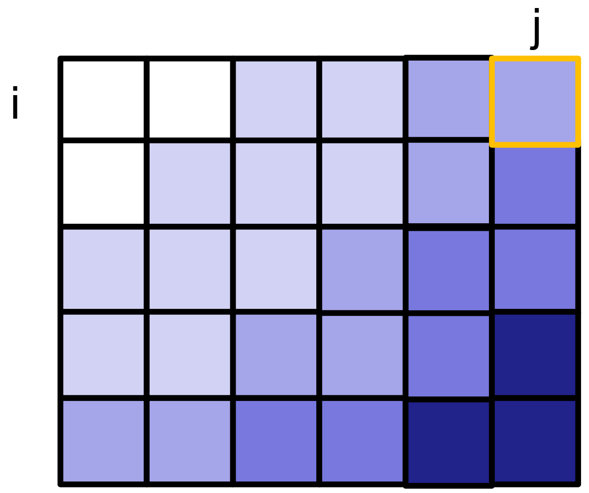


```
int i = 0, j = n;
```
{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}
```
while (j > 0 && x <= M[i,j-1]) {
   ??
   j = j - 1;
}
```
What goes here?

{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

Sorted Matrix Search Code

Initialization:



```
int i = 0, j = n;
```
{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}
```
while (j > 0 && x <= M[i,j-1]) {
  ??
  j = j - 1;
}
```
{{ x ≤ M[i,j], ..., M[i,n-1] and x ≤ M[i,j-1] }}

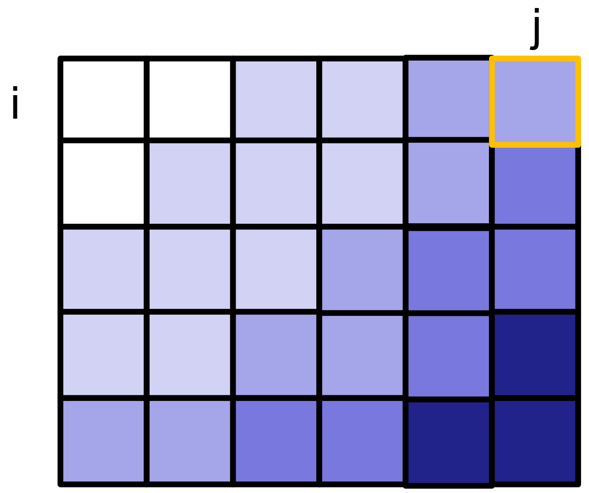
{{ x ≤ M[i,j-1], ..., M[i,n-1] }}

{{ x ≤ M[i,j], ..., M[i,n-1] }}

{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

Sorted Matrix Search Code

Initialization:



```
int i = 0, j = n;
```

{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}

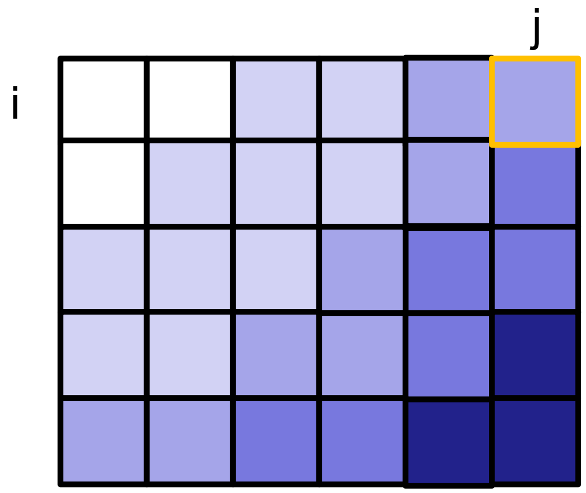
```
while (j > 0 && x <= M[i,j-1]) {

  j = j - 1;
}
```

What goes here? Nothing!

{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

Sorted Matrix Search Code

Initialization:



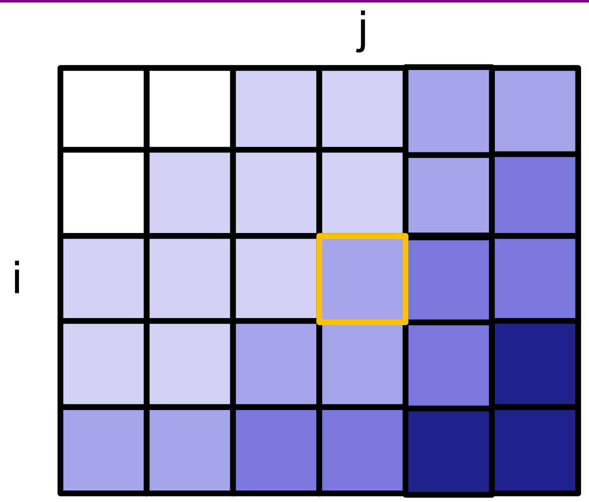
```
int i = 0;
int j = n;
```

{{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}

```
while (j > 0 && x <= M[i,j-1])
  j = j - 1;
```

{{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}

Sorted Matrix Search Code

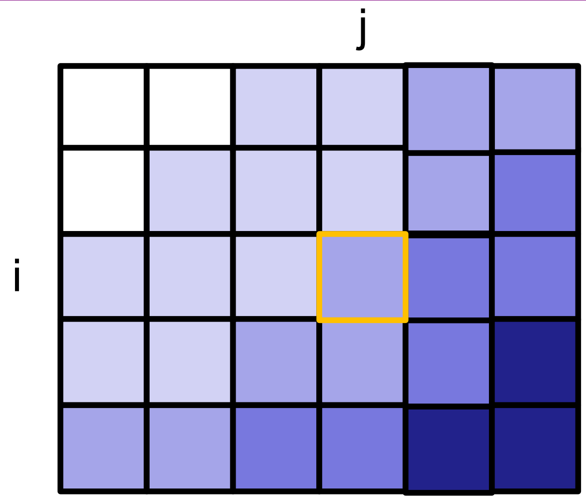


That finds the right column in row 0

- can now check M[0,j] = x (if j < n)
- if not, we can move onto the next row
  - x cannot be anywhere in the row if it's not at M[i,j]
  - set `i = i + 1`

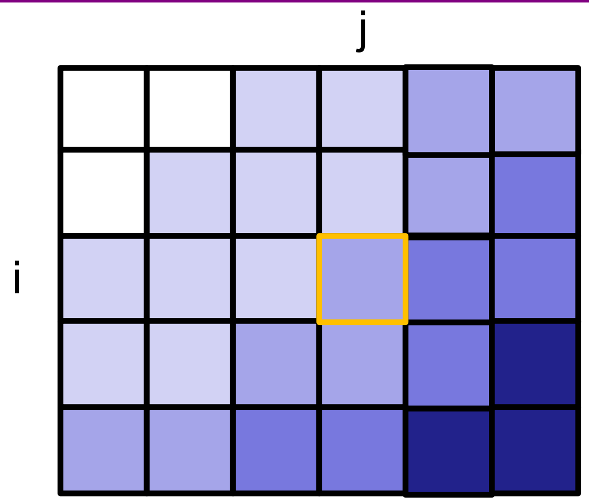
Process continues in each row thereafter...

Sorted Matrix Search Code



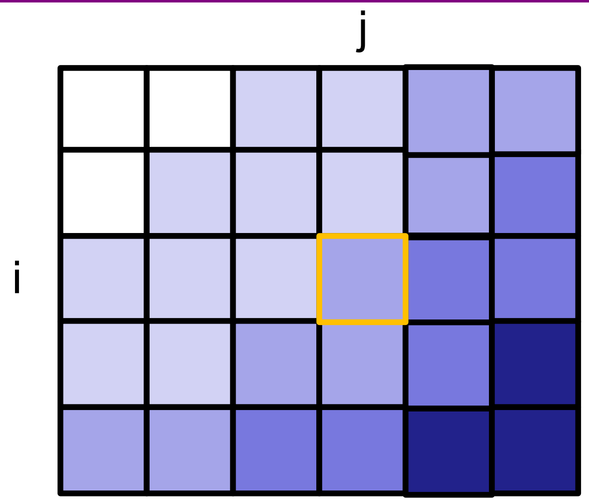
- Make progress by setting `i = i + 1`
- When i increases, the invariant may be broken
  - we have x ≤ M[i,j] ≤ M[i+1,j] since columns are sorted
  - and M[i+1,j] ≤ M[i +1,j+1], .., M[i +1,n-1] since rows are sorted
  - so we get x ≤ M[i +1,j], .., M[i +1,n-1]

Sorted Matrix Search Code



- Make progress by setting `i = i + 1`
- When i increases, the invariant may be broken
  - we have x <= M[i +1,j], .., M[i +1,n-1]
  - may need to restore invariant for M[i,0], ..., M[i,j-1] < x
  - decrease j until it holds again...
    - when have we seen this before?
    - initialization

Sorted Matrix Search Code



- Make progress by setting `i = i + 1`
- When i increases, the invariant may be broken
  - we have x <= M[i +1,j], .., M[i +1,n-1]
  - may need to restore invariant for M[i,0], ..., M[i,j-1] < x
  - could copy and paste the same loop
    - or you can do it with one copy

Don't try this at home!

Sorted Matrix Search Code

instead of

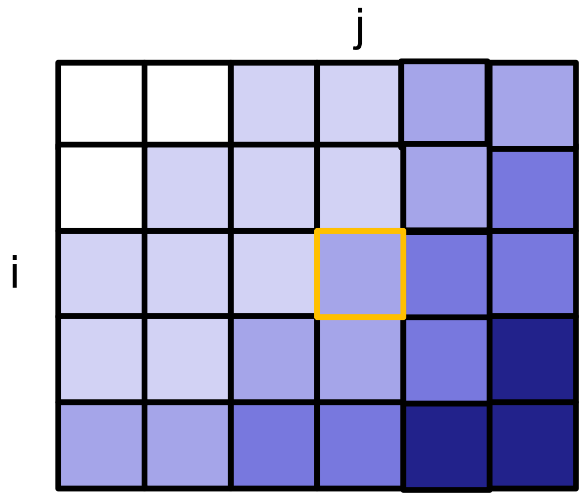
```
int i = 0, j = n;
[move j left]
{{ Inv: M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}
while (i != n) {
   i = i + 1;
   [move j left]
}
```

we can write

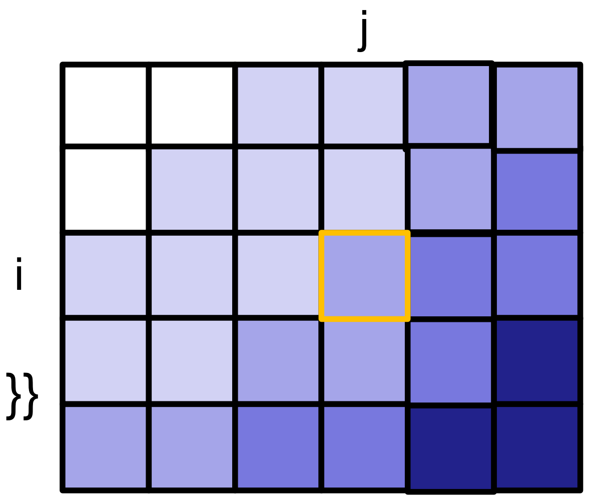
```
int i = 0, j = n;
while (i != n) {
   [move j left]
   {{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}
   i = i + 1;
}
```

Sorted Matrix Search Code

```
int i = 0;
int j = n;

while (i != n) {
  {{ Inv: x ≤ M[i,j], ..., M[i,n-1] }}
  while (j > 0 && x <= M[i,j-1])
    j = j - 1;

  {{ M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}
  if (j < n && x == M[i,j])
    return true;
  i = i + 1;
}
return false;
```



Sorted Matrix Search Code



```
int i = 0;
int j = n;
{{ Inv: x not in M[k,l] for k < i and x ≤ M[i,j], ..., M[i,n-1] }}
while (i != n) {
   {{ Inv: x not in M[k,l] for k < i and x ≤ M[i,j], ..., M[i,n-1] }}
   while (j > 0 && x <= M[i,j-1])
      j = j - 1;

   {{ x not in M[k,l] for k < i and M[i,0], ..., M[i,j-1] < x ≤ M[i,j], ..., M[i,n-1] }}
   if (j < n && x == M[i,j])
      return true;
   i = i + 1;
}
return false;
```

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Lecture 4½ – Reasoning Wrap-up

Reasoning Summary

Reasoning Summary

- Checking correctness can be a mechanical process
  - using forward or backward reasoning

- This requires that loop invariants are provided
  - those cannot be produced automatically

- As long as you document your loop invariants,
  it should not be too hard for someone else to review your code

Documenting Loop Invariants

- Write down loop invariants for all non-trivial code

- They are often best avoided for “for each” loops:

```
{{ Inv: printed all the strings removed so far }}
for (String s : L)
  System.out.println(s);
```

Documenting Loop Invariants

- Write down loop invariants for all non-trivial code
- They are often best avoided for “for each” loops:

```
// Print the strings in L, one per line.
for (String s : L)
  System.out.println(s);
```

Documenting Loop Invariants

- Write down loop invariants for all non-trivial code
- They are often best avoided for “for each” loops:

```
{{ Inv: B has 2*x + 1 for each element x removed so far }}
for (int x : A)
  B.add(2*x + 1);
```

Documenting Loop Invariants

- Write down loop invariants for all non-trivial code
- They are often best avoided for "for each" loops:

```
// Set B = 2*A + 1 (element-wise)
for (int x : A)
  B.add(2*x + 1);
```

Documenting Loop Invariants

- Write down loop invariants for all non-trivial code
- They are often best avoided for “for each” loops.
- Invariants are more helpful when a variable incorporates information from multiple iterations
  - e.g., {{ s = A[0] + … + A[i-1] }}
- *Use your best judgement!*

Reasoning Summary

- You can check correctness by reasoning alone

- Correctness: tools, inspection, testing
  - reasoning through your own code
  - do code reviews

- Practice!
  - essential skill for professional programmers

Reasoning Summary

- You will eventually do this in your head for most code
- Formalism remains useful
  - especially tricky problems
  - interview questions (often tricky)
    - see last example…

Next Topic…

A Problem

“Complete this method such that it returns the location of the largest value in the first **`n`** elements of the array **`arr`**.”

```
int maxLoc(int[] arr, int n) {
    ...
}
```

One Solution

```
int maxLoc(int[] arr, int n) {
  int maxIndex = 0;
  int maxValue = arr[0];
  // Inv: maxValue = max of arr[0] .. arr[i-1] and
  //      maxValue = arr[maxIndex]
  for (int i = 1; i < n; i++) {
    if (arr[i] > maxValue) {
      maxIndex = i;
      maxValue = arr[i];
    }
  }
  return maxIndex;
}
```

Is this code correct?

What if n = 0?

What if n > arr.length?

What if there are two maximums?

A Problem

"Complete this method such that it returns the location of the largest value in the first **`n`** elements of the array **`arr`**."

```
int maxLoc(int[] arr, int n) {
   ...
}
```

Could we write a specification so that this is a **correct** solution?

- throw `IllegalArgumentException` if n <= 0
- throw `ArrayOutOfBoundsException` if n > arr.length
- return smallest index achieving maximum

Morals

- You can all write the code correctly

- Writing the specification was harder than the code
  - multiple choices for the “right” specification
    - have to carefully think through corner cases
  - once the specification is chosen, code is straightforward
  - (both of those will be recurrent themes)

- Some math (e.g. “if n <= 0”) often shows up in specifications
  - English (“if n is less or equal to than 0”) is often worse

How to Check Correctness

- Step 1: need a **specification** for the function
  - can’t argue correctness if we don’t know what it should do
  - surprisingly difficult to write!

- Step 2: determine whether the code meets the specification
  - apply **reasoning**
  - usually easy with the tools we learned