CSE 331
Software Design & Implementation

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Summer 2017

Lecture 4 – Writing Loops

(Based on slides by Mike Ernst, Dan Grossman, David Notkin, Hal Perkins, Zach Tatlock)

Reminders

- Quiz 1 due tonight
- HW2 is posted
  - due by 11pm on Wed
  - one tip for problems 1–3...

Checking correctness of loops

Not just about finding in assertions after each line…

Also need to check that loop invariant:

1. holds initially
2. is preserved by the loop body
3. implies postcondition upon termination

Problems 1–3 on HW2 ask you to fill in the assertions and also Check that 1–2 hold

- (I didn't ask you to do 3 since it's obvious each time, but you convince yourself it holds too)

Example: sum of array

The following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
{{ ____________ }}
i = 0;
{{ ____________ }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  {{ ____________ }}
  s = s + b[i];
  {{ ____________ }}
  i = i + 1;
  {{ ____________ }}
}
{{ ____________ }}
{{ s = b[0] + ... + b[n-1] }}
```

Example: sum of array

The following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
{{ s = 0 }}
i = 0;
{{ s = 0 and i = 0 }}
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while (i != n) {
  {{ ____________ }}
  s = s + b[i];
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  i = i + 1;
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{{ s = b[0] + ... + b[n-1] }}
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while (i != n) {
  {{ s = b[0] + ... + b[i-1] and i != n }}
  s = s + b[i];
  {{ s = b[0] + ... + b[i-1] + b[i] and i != n }}
  i = i + 1;
  {{ s = b[0] + ... + b[i-2] + b[i-1] and i-1 != n }}
}
{{ ______________ }}
{{ s = b[0] + ... + b[n-1] }}
```

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  {{ s = b[0] + ... + b[i-2] + b[i-1] and i-1 != n }}
}
{{ ______________ }}
{{ s = b[0] + ... + b[n-1] }}
```

```
{{ s + b[i] = b[0] + ... + b[i] }}
s = s + b[i];
{{ s = b[0] + ... + b[i] }}
i = i + 1
{{ s = b[0] + ... + b[i-1] }}
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  i = i + 1;
  {{ s = b[0] + ... + b[i-2] + b[i-1] and i-1 != n }}
}
{{ s = b[0] + ... + b[i-1] and not (i != n) }}
{{ s = b[0] + ... + b[n-1] }}
```

```
{{ s + b[i] = b[0] + ... + b[i] }}
s = s + b[i];
{{ s = b[0] + ... + b[i] }}
i = i + 1
{{ s = b[0] + ... + b[i-1] }}
```

Example: sum of array

The following code to compute `b[0] + … + b[n-1]`:

Are we done?

```
{{ }}
s = 0;
{{ s = 0 }}
i = 0;
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  {{ s = b[0] + ... + b[i-1] and i != n }}
  s = s + b[i];
  {{ s = b[0] + ... + b[i-1] + b[i] and i != n }}
  i = i + 1;
  {{ s = b[0] + ... + b[i-2] + b[i-1] and i-1 != n }}
}
{{ s = b[0] + ... + b[i-1] and not (i != n) }}
{{ s = b[0] + ... + b[n-1] }}
```

```
{{ s + b[i] = b[0] + ... + b[i] }}
s = s + b[i];
{{ s = b[0] + ... + b[i] }}
i = i + 1
{{ s = b[0] + ... + b[i-1] }}
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Example: sum of array

The following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
{{ s = 0 }}
i = 0;
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  {{ s = b[0] + ... + b[i-1] and i != n }}
  s = s + b[i];
  {{ s = b[0] + ... + b[i-1] + b[i] and i != n }}
  i = i + 1;
  {{ s = b[0] + ... + b[i-2] + b[i-1] and i-1 != n }}
}
{{ s = b[0] + ... + b[i-1] and not (i != n) }}
{{ s = b[0] + ... + b[n-1] }}
```

Does invariant hold initially?

Are we done?
No, we need to check 1-3

```
{{ s + b[i] = b[0] + ... + b[i] }}
s = s + b[i];
{{ s = b[0] + ... + b[i] }}
i = i + 1
{{ s = b[0] + ... + b[i-1] }}
```

Example: sum of array

i = 3: s = b[0] + b[1] + b[2]
i = 2: s = b[0] + b[1]
i = 1: s = b[0]
i = 0: s = 0

The following code to compute `b[0] + … + b[n-1]`:

Are we done?
No, we need to check 1-3

```
{{ }}
s = 0;
{{ s = 0 }}
i = 0;
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  {{ s = b[0] + ... + b[i-1] and i != n }}
  s = s + b[i];
  {{ s = b[0] + ... + b[i-1] + b[i] and i != n }}
  i = i + 1;
  {{ s = b[0] + ... + b[i-2] + b[i-1] and i-1 != n }}
}
{{ s = b[0] + ... + b[i-1] and not (i != n) }}
{{ s = b[0] + ... + b[n-1] }}
```

Holds initially? Yes: i = 0 implies s = b[0] + … + b[-1] = 0

```
{{ s + b[i] = b[0] + ... + b[i] }}
s = s + b[i];
{{ s = b[0] + ... + b[i] }}
i = i + 1
{{ s = b[0] + ... + b[i-1] }}
```

Example: sum of array

The following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
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while (i != n) {
  {{ s = b[0] + ... + b[i-1] and i != n }}
  s = s + b[i];
  {{ s = b[0] + ... + b[i-1] + b[i] and i != n }}
  i = i + 1;
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}
{{ s = b[0] + ... + b[i-1] and not (i != n) }}
{{ s = b[0] + ... + b[n-1] }}
```

Are we done?
No, we need to check 1-3

```
{{ s + b[i] = b[0] + ... + b[i] }}
s = s + b[i];
{{ s = b[0] + ... + b[i] }}
i = i + 1
{{ s = b[0] + ... + b[i-1] }}
```

Does postcondition hold on termination?

Example: sum of array

The following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
{{ s = 0 }}
i = 0;
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  {{ s = b[0] + ... + b[i-1] and i != n }}
  s = s + b[i];
  {{ s = b[0] + ... + b[i-1] + b[i] and i != n }}
  i = i + 1;
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}
{{ s = b[0] + ... + b[i-1] and not (i != n) }}
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```

Are we done?
No, we need to check 1-3

```
{{ s + b[i] = b[0] + ... + b[i] }}
s = s + b[i];
{{ s = b[0] + ... + b[i] }}
i = i + 1
{{ s = b[0] + ... + b[i-1] }}
```

Postcondition holds? Yes, since i = n.

Example: sum of array

The following code to compute `b[0] + … + b[n-1]`:

```
{{ }}
s = 0;
{{ s = 0 }}
i = 0;
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  {{ s = b[0] + ... + b[i-1] and i != n }}
  s = s + b[i];
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Are we done?
No, we need to check 1-3

Does loop body preserve invariant?

```
{{ s + b[i] = b[0] + ... + b[i] }}
s = s + b[i];
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Example: sum of array

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  {{ s = b[0] + ... + b[i-1] and i != n }}
  s = s + b[i];
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  i = i + 1;
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Are we done?
No, we need to check 1-3

Does loop body preserve invariant?

```
{{ s + b[i] = b[0] + ... + b[i] }}
s = s + b[i];
{{ s = b[0] + ... + b[i] }}
i = i + 1
{{ s = b[0] + ... + b[i-1] }}
```

Yes. Weaken by dropping "i-1 != n"

Example: sum of array

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s = 0;
{{ s = 0 }}
i = 0;
{{ s = 0 and i = 0 }}
{{ Inv: s = b[0] + ... + b[i-1] }}
while (i != n) {
  {{ s = b[0] + ... + b[i-1] and i != n }}
  s = s + b[i];
  {{ s = b[0] + ... + b[i-1] + b[i] and i != n }}
  i = i + 1;
  {{ s = b[0] + ... + b[i-2] + b[i-1] and i-1 != n }}
}
{{ s = b[0] + ... + b[i-1] and not (i != n) }}
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No, we need to check 1-3

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```
{{ s + b[i] = b[0] + ... + b[i] }}
s = s + b[i];
{{ s = b[0] + ... + b[i] }}
i = i + 1
{{ s = b[0] + ... + b[i-1] }}
```

Yes. If Inv holds, then so does this (just add b[i] to both sides of Inv)

Loop Invariants

- There is no general way to deduce the invariant from the code

- Why would we ever need to do this?
- It suggests coding like this:

Idea ➡ Code ➡ Invariant ➡ Proof

Loop Invariants

- There is no general way to deduce the invariant from the code

- Why would we ever need to do this?
- Don't do this:

Idea ➡ Code ➡ Invariant ➡ Proof

Loop Invariants

- There is no general way to deduce the invariant from the code.

- Don't do this:

Idea ➡ Code ➡ Invariant ➡ Proof

- Instead, do this:

Idea ➡ Invariant ➡ Code ➡ Proof

Loop Invariants Before Code

- Loop invariant comes out of the algorithm idea
  - describes partial progress toward the goal
    - how you will get from start to end
  - contains the essence of the algorithm idea

- A good invariant will make the code easier to write
  - a great invariant makes the code “write itself”
  - (we will see the same thing with invariants for ADTs etc.)

Loop Invariants in this Course

- We advocate writing invariants before the code
  - if the code is there, the invariant should be there too

- You will not be asked to find the invariant for the code
- Types of problems in HW2:
  - given invariant and code, prove it correct
  - given invariant, write code
  - write invariant and (then) code [for simple algorithms]

- When writing code, document your loop invariants (if nontrivial)
  - don't make readers re-discover them
  - improves changeability and understandability

Loop Invariant Design Patterns

- Often loop invariant is a weakening of the postcondition
  - partial progress with completion a special case

- Example: finding the maximum value in an array
  - postcondition: m = max(A[0], …, A[n-1])
  - loop invariant: m = max(A[0], ..., A[i-1]) for some i
    - postcondition is special case i = n

- Only *slightly* weakened postcondition: I and not `cond` implies Q
- Stronger is usually better
  - if it is strong enough, there is only one way to write body
  - (but if it's too strong, there may be no way to write the body!)

Filling in code, given invariant

Can often deduce correct code directly from loop invariant

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Can often deduce correct code directly from loop invariant:

- what is the easiest way to satisfy the loop invariant?
  - this gives you the initialization code

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- when does loop invariant satisfy the postcondition?
  - this gives you the termination condition

Filling in code, given invariant

Can often deduce correct code directly from loop invariant:

- what is the easiest way to satisfy the loop invariant?
  - this gives you the initialization code
- when does loop invariant satisfy the postcondition?
  - this gives you the termination condition
- how do you make progress toward termination?
  - if condition is i != n (and i <= n), try i = i + 1
  - if condition is i != j (and i <= j), try i = i + 1 or j = j – 1
  - write out the new invariant with this change (e.g. i+1 for i)
  - figure out code needed to make the new invariant hold
    - usually just a small change (since Inv change is small)

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
??
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (?) {

   ??

}
```

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

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{{ b.length >= n and n > 0 }}

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??
```

Easiest way to make this hold?

{{ Inv: m = max(b[0], ..., b[i-1]) }}

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Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
??
```

Easiest way to make this hold?
Take i = 1 and m = max(b[0])

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (?) {

   ??

}
```

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

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{{ b.length >= n and n > 0 }}

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int i = 1;
int m = b[0];
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{{ m = max(b[0], ..., b[n-1]) }}

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{{ m = max(b[0], ..., b[n-1]) }}

When does Inv imply postcondition?

Example: max of array

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{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (?) {

   ??

}
```

{{ m = max(b[0], ..., b[n-1]) }}

When does Inv imply postcondition?
Happens when i = n

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {

   ??

}
```

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

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int i = 1;
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while (i != n) {

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```

How do we progress toward termination?

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
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{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {

   ??

}
```

How do we progress toward termination?
We start at i = 1 and end at i = n, so…

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {

   ??
   i = i + 1;
}
```

How do we progress toward termination?
We start at i = 1 and end at i = n, so
Try this.

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {

  ??
  i = i + 1;
}
```

When i becomes i+1, Inv becomes:
m = max(b[0], …, b[i])

{{ m = max(b[0], ..., b[n-1]) }}

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {

   ??
   i = i + 1;
}
```

{{ m = max(b[0], ..., b[n-1]) }}

How do we get
from m = max(b[0], …, b[i-1])
to m = max(b[0], …, b[i])?

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {

   ??
   i = i + 1;
}
```

{{ m = max(b[0], ..., b[n-1]) }}

How do we get
from m = max(b[0], …, b[i-1])
to m = max(b[0], …, b[i])?

Set m = max(m, b[i])

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
int m = b[0];
```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {
  if (b[i] > m)
    m = b[i];
  i = i + 1;
}
```

{{ m = max(b[0], ..., b[n-1]) }}

How do we get
from m = max(b[0], …, b[i-1])
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Set m = max(m, b[i])

Example: max of array

Write code to compute max(b[0], …, b[n-1]):

{{ b.length >= n and n > 0 }}

```
int i = 1;
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```

{{ Inv: m = max(b[0], ..., b[i-1]) }}

```
while (i != n) {
  if (b[i] > m)
    m = b[i];
  i = i + 1;
}
```

{{ m = max(b[0], ..., b[n-1]) }}

Filling in code, given invariant

As you saw, we can often deduce correct code directly from Inv

- cases where this happens are the best invariants

The invariant is *often* the essence of the algorithm **idea**

- then rest is just details that follow from the invariant

Finding the loop invariant

Not every loop invariant is simple weakening of postcondition, but…

- that is the easiest case
- it happens a lot

In this class (e.g., exams):

- if I ask you to find the invariant, it will *very likely* be of this type
- I may ask you to inspect code with more complex invariants
- to learn about more ways of finding invariants: CSE 421

Examples: finding loop invariants

1. sum of array
   - postcondition: s = b[0] + b[1] + … + b[n-1]

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   - postcondition: s = b[0] + b[1] + … + b[n-1]
   - loop invariant: s = b[0] + b[1] + … + b[i-1]
     - gives postcondition when i = n
     - gives s = 0 when i = 0

Examples: finding loop invariants

1. sum of array
   - postcondition: s = b[0] + b[1] + … + b[n-1]
   - loop invariant: s = b[0] + b[1] + … + b[i-1]
     - gives postcondition when i = n
     - gives s = 0 when i = 0

2. max of array
   - postcondition: m = max(b[0], b[1], …, b[n-1])

Examples: finding loop invariants

1. sum of array
   - postcondition: s = b[0] + b[1] + … + b[n-1]
   - loop invariant: s = b[0] + b[1] + … + b[i-1]
     - gives postcondition when i = n
     - gives s = 0 when i = 0

2. max of array
   - postcondition: m = max(b[0], b[1], …, b[n-1])
   - loop invariant: m = max(b[0], b[1], ..., b[i-1])
     - gives postcondition when i = n
     - gives m = b[0] when i = 1

More Examples

Example: quotient and remainder

Problem: Set q to be the quotient of x/y and r to be the remainder

Precondition: x >= 0 and y > 0

Postcondition: q*y + r = x and 0 <= r < y

- i.e., y doesn't go into x any more times

Example: quotient and remainder

Problem: Set q to be the quotient of x/y and r to be the remainder

Precondition: x >= 0 and y > 0

Postcondition: q*y + r = x and 0 <= r < y

- i.e., y doesn't go into x any more times

Loop invariant: q*y + r = x and r >= 0

- postcondition is special case when we also have r < y
- this suggests a loop condition…

Example: quotient and remainder

We want “r < y” when the conditions fails

- so the condition is r >= y
- can see immediately that the postcondition holds on loop exit

{{ Inv: q*y + r = x and 0 <= r }}

```
while (r >= y) {


}
```

{{ q*y + r = x and 0 <= r < y }}

Example: quotient and remainder

Need to make the invariant hold initially…

- search for the simplest way that works
- can only have r (= q*y – x) >= 0 for all y if we take q = 0

```
int q = 0;
int r = x;
{{ Inv: q*y + r = x and 0 <= r }}
while (r >= y) {



}
{{ q*y + r = x and 0 <= r < y }}
```

Example: quotient and remainder

We have r large initially.

Need to shrink r on each iteration in order to terminate…

- if r >= y, then y goes into x at least one more time

```
int q = 0;
int r = x;
{{ Inv: q*y + r = x and 0 <= r }}
while (r >= y) {
  q = q + 1;
  r = r - y;
}
{{ q*y + r = x and 0 <= r < y }}
```

Example: quotient and remainder

We have r large initially.

Need to shrink r on each iteration in order to terminate…

- if r >= y, then y goes into x at least one more time

```
int q = 0;
int r = x;
```

{{ Inv: q*y + r = x and 0 <= r }}

```
while (r >= y) {
  q = q + 1;
  r = r - y;
}
```

{{ q*y + r = x and 0 <= r < y }}

(+y and -y cancel)

{{ (q+1)*y + r-y = x and y <= r }}
{{ q*y + r-y = x and 0 <= r-y }}
{{ q*y + r = x and 0 <= r }}

Example: Dutch National Flag

Given an array of red, white, and blue pebbles, sort the array so the red pebbles are at the front, the white pebbles are in the middle, and the blue pebbles are at the end

Edsgar Dijkstra

Pre- and post-conditions

Precondition: Any mix of red, white, and blue

Postcondition:

- Red, then white, then blue
- Number of each color same as in original array

Pre- and post-conditions

Precondition: Any mix of red, white, and blue

Postcondition:

- Red, then white, then blue
- Number of each color same as in original array

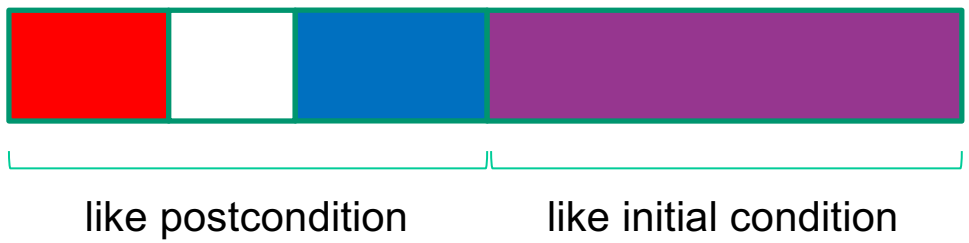
Loop invariant should (essentially) have

- postcondition as a special case
- initial condition as a special case

Loop invariant describes continuum of partial progress

Example: Dutch National Flag

The first idea that comes to mind:



Example: Dutch National Flag

The first idea that comes to mind works.

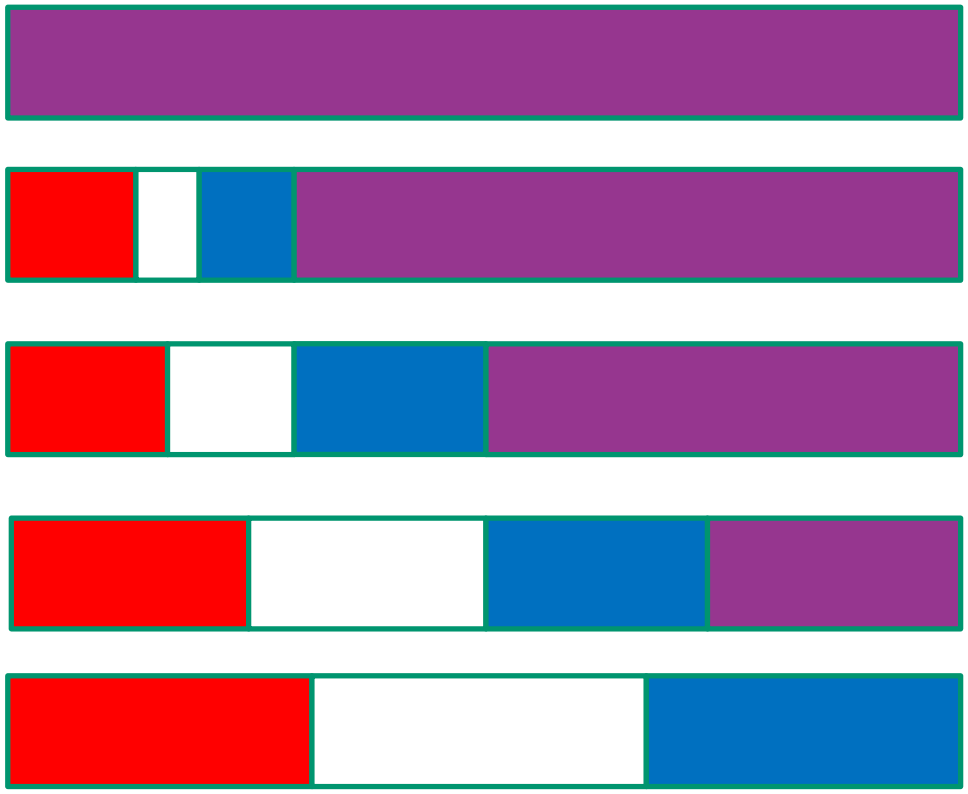
Initial:

Iter 5:

Iter 10:

Iter 15:

Post:



Other potential invariants

Any of these choices work, making the array more-and-more partitioned as you go:

(Middle two have one less line of code… only matters on slides)

Precise Invariant

Need indices to refer to the split points between colors

– call these i, j, k

Loop Invariant:

- 0 <= i <= j <= k <= A.length
- A[0], A[1], …, A[i-1] is red
- A[i], A[i+1], …, A[j-1] is white
- A[k], A[k+1], …, A[n-1] is blue

No constraints on A[j], A[j+1], ..., A[k-1]

Dutch National Flag Code

Invariant:

Initialization?

Dutch National Flag Code

Invariant:

Initialization:

- i = j = 0 and k = n

Dutch National Flag Code

Invariant:

Initialization:

- i = j = 0 and k = n

Termination condition?

Dutch National Flag Code

Invariant:

Initialization:

- i = j = 0 and k = n

Termination condition:

- j = k

Dutch National Flag Code

```
int i, j = 0;
int k = n;
```

{{ Inv: 0 <= i <= j <= k <= n and A[0], …, A[i-1] is red and ... }}

```
while (j != k) {

    // need to get j closer to k
    // let's increase j...

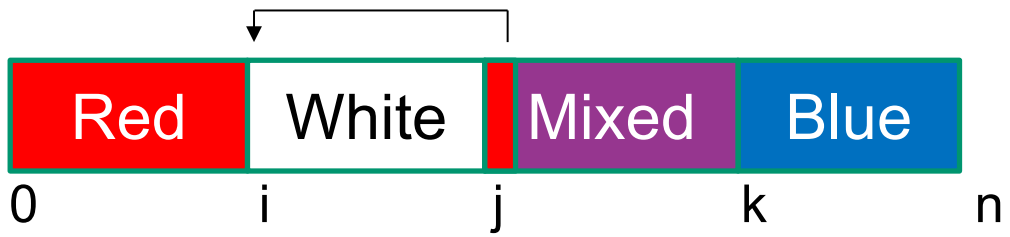
}
```

Dutch National Flag Code

Three cases depending on the value of A[j]:

white

red



blue

Dutch National Flag Code

```
int I = 0, j = 0;
int k = n;
```

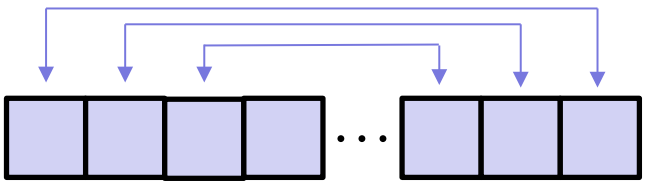
{{ Inv: 0 <= i <= j <= k <= n and A[0], …, A[i-1] is red and ... }}

```
while (j != k) {
  if (A[j] is white) {
     j = j+1;
  } else if (A[j] is blue) {
     swap A[j], A[k-1];
     k = k - 1;
  } else { // A[j] is red
     swap A[i], A[j];
     i = i + 1;
     j = j + 1;
  }
}
```

Example: Reverse an Array

Problem: Given array A of length n, put elements in reverse order.

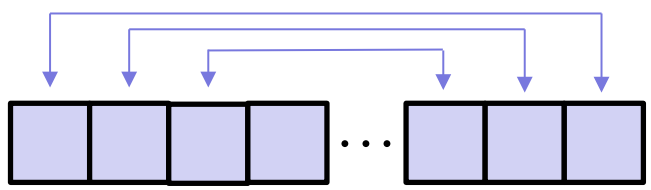
Idea: Swap A[0] and A[n-1] then A[1] and A[n-2] etc.



Example: Reverse an Array

Problem: Given array A of length n, put elements in reverse order.

Idea: Swap A[0] and A[n-1] then A[1] and A[n-2] etc.



Invariant: A[0], A[1], …, A[i-1] swapped with A[j], A[j+1], ..., A[n-1]

- (here, “swapped” means swapped with reverse of)
- indices i and j keep track of what has been swapped

Reverse an Array Code

Invariant: A[0], A[1], …, A[i-1] swapped with A[j], A[j+1], ..., A[n-1]

Initialization?

Reverse an Array Code

Invariant: A[0], A[1], …, A[i-1] swapped with A[j], A[j+1], ..., A[n-1]

Initialization:

- i = 0 and j = n

Reverse an Array Code

Invariant: A[0], A[1], …, A[i-1] swapped with A[j], A[j+1], ..., A[n-1]

Initialization:

- i = 0 and j = n

Termination condition:

- i != j and i != j - 1
- if i = j - 1, then A[i] = A[j-1] stays in place when A is reversed

Reverse an Array Code

```
int i = 0;
int j = n;
```

{{ Inv: A[0], A[1], …, A[i-1] swapped with A[j], A[j+1], ..., A[n-1] }}

```
while (i != j and i != j - 1) {

    // perform another swap...

}
```

{{ A[0], ..., A[n/2] swapped with A[n-n/2], ..., A[n-1] }}

Reverse an Array Code

```
int i = 0;
int j = n;
```

{{ Inv: A[0], A[1], …, A[i-1] swapped with A[j], A[j+1], ..., A[n-1] }}

```
while (i != j and i != j - 1) {
  swap A[i], A[j-1];
  i = i + 1;
  j = j - 1;
}
```

{{ A[0], ..., A[n/2] swapped with A[n-n/2], ..., A[n-1] }}

Reverse an Array Code

```
int i = 0;
int j = n;
```

{{ Inv: A[0], A[1], …, A[i-1] swapped with A[j], A[j+1], ..., A[n-1] }}

```
while (i != j and i != j - 1) {
  swap A[i], A[j-1];
  i = i + 1;
  j = j - 1;
}
```

{{ A[0], …, A[i] swapped with A[j-1], ..., A[n-1] }}

{{ A[0], ..., A[n/2] swapped with A[n-n/2], ..., A[n-1] }}

Example: Binary Search

Problem: Given a sorted array A and a number x, find index of x (or where it would be inserted) in A.

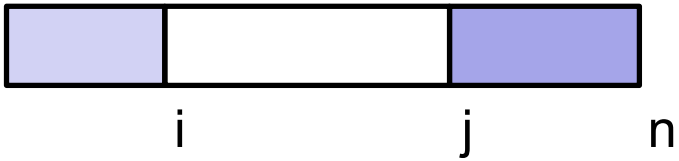
Idea: Look at A[n/2] to figure out if x is in A[0], A[1], ..., A[n/2] or in A[n/2+1], ..., A[n-1]. Narrow the search for x on each iteration.

(This is an algorithm where you probably still need to go line-by-line even as you get faster at reasoning...)

Example: Binary Search

Problem: Given a sorted array A and a number x, find index of x (or where it would be inserted) in A.

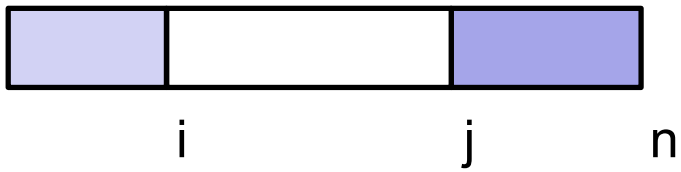
Idea: Look at A[n/2] to figure out if x is in A[0], A[1], ..., A[n/2] or in A[n/2+1], ..., A[n-1]. Narrow the search for x on each iteration.



Loop Invariant: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1]

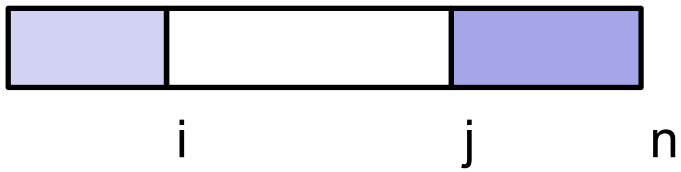
- so x must lie in A[i], ..., A[j-1]
- A[i], ..., A[j-1] is the part where we don't know relation to x

Binary Search Code



Initialization?

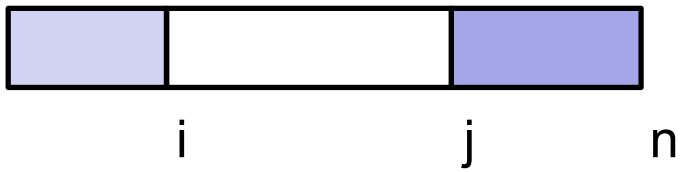
Binary Search Code



Initialization:

- i = 0 and j = n
- white region is the whole array

Binary Search Code



Initialization:

- i = 0 and j = n
- white region is the whole array

Termination condition:

- i = j
- white region is empty
- if x is in the array, it is A[i-1]
  - if there are multiple copies of x, this returns the *last*

Binary Search Code

```
int i = 0;
int j = n;
```

{{ Inv: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1] }}

```
while (i != j) {

    // need to bring i and j closer together...
    // (e.g., increase i or decrease j)

}
```

{{ A[0], ..., A[i-1] <= x < A[i], ..., A[n-1] }}

Binary Search Code

```
int i = 0;
int j = n;
```

{{ Inv: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1] }}

```
while (i != j) {
  int m = (i + j) / 2;
  if (A[m] <= x) {
    i = m + 1;
  } else {
    j = m;
  }
}
```

{{ A[0], ..., A[i-1] <= x < A[i], ..., A[n-1] }}

Binary Search Code

```
int i = 0;
int j = n;
```

{{ Inv: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1] }}

```
while (i != j) {
  int m = (i + j) / 2;
  if (A[m] <= x) {
    i = m + 1;
  } else {
    j = m;
  }
}
```

{{ A[0], ..., A[i-1] <= x < A[i], ..., A[n-1] }}

Binary Search Code

```
int i = 0;
int j = n;
```

{{ Inv: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1] }}

```
while (i != j) {
  int m = (i + j) / 2;
  if (A[m] <= x) {
    i = m + 1;
  } else {
    j = m;
  }
}
```

invariant satisfied since A[i-1] = A[m] <= x
(and A is sorted so A[0] <= … <= A[m])

{{ A[0], ..., A[i-1] <= x < A[i], ..., A[n-1] }}

Binary Search Code

```
int i = 0;
int j = n;
```

{{ Inv: A[0], ..., A[i-1] <= x < A[j], ..., A[n-1] }}

```
while (i != j) {
  int m = (i + j) / 2;
  if (A[m] <= x) {
    i = m + 1;
  } else {
    j = m;
  }
}
```

invariant satisfied since x < A[m] = A[j]
(and A is sorted so A[m] <= ... <= A[n-1])

{{ A[0], ..., A[i-1] <= x < A[i], ..., A[n-1] }}

Aside on Termination

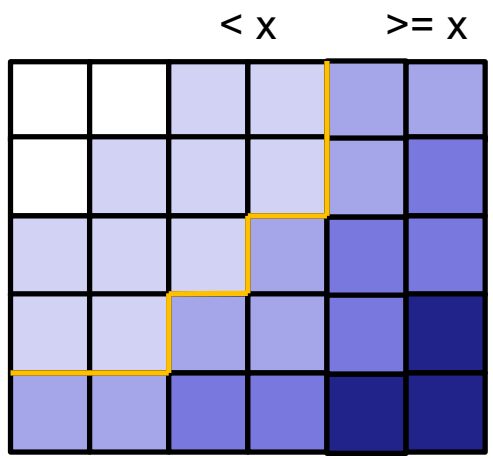
- Most often correctness is harder work than termination
  - the latter follows from running time bound

- But also examples where termination is more interesting
  - (cases with variable progress toward termination condition)
  - quotient and remainder (Inv: q*y + r == x and r >= 0)
  - binary search
- It's easy to make a mistake and have no progress
  - then the code may loop forever

- See 16su HW2 for a problem where correctness is trivial and the *only* difficult part is checking that it terminates

Example: Sorted Matrix Search

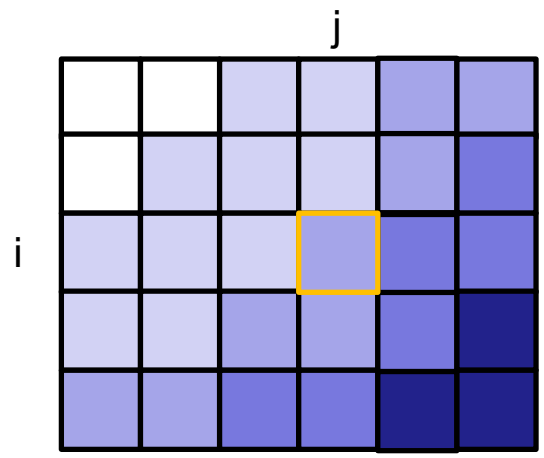
Problem: Given a sorted a matrix M (of size m x n), where every row and every column is sorted, find out whether a given number x is in the matrix.



(darker color means larger)

(One) **Idea**: Trace the contour between the numbers <= x and > x on each row to see if x appears.

Sorted Matrix Search Code

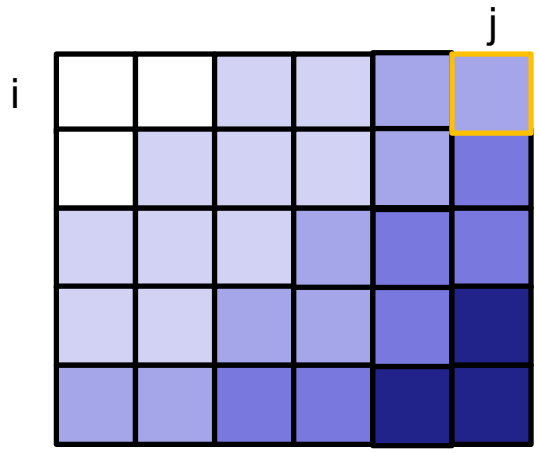


Loop Invariant: M[i,0], ..., M[i,j-1] < x <= M[i,j], ..., M[i,n-1]

- will increase i from 0 to m
- for each i, need to find the right j

Sorted Matrix Search Code

Initialization:

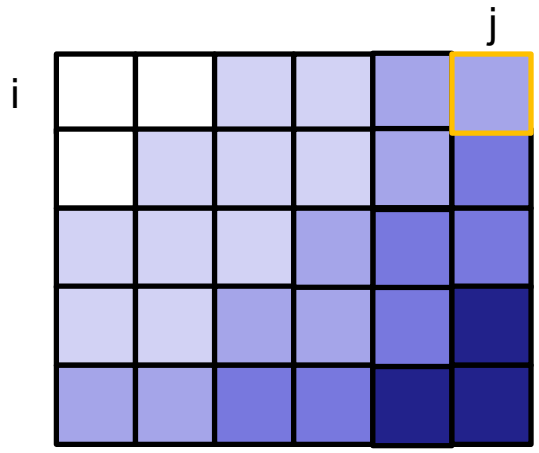


No obvious way to initialize so the invariant holds

To start in row 0 (i = 0), we need to search...

Sorted Matrix Search Code

Initialization:



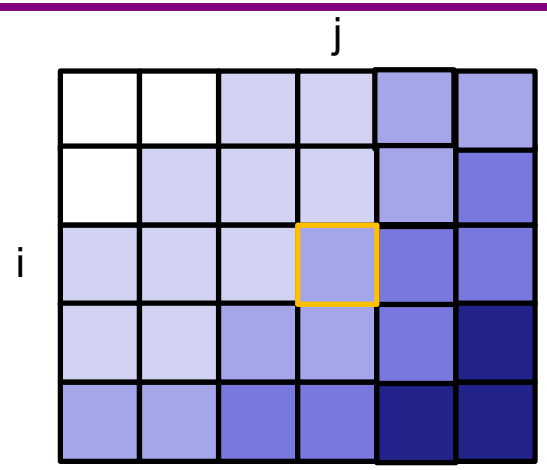

```
int i = 0;
int j = n;
```

{{ Inv: x <= M[i,j], ..., M[i,n-1] }}

```
while (j > 0 and x <= M[i,j-1])
  j = j - 1;
```

{{ j = 0 or M[i,j-1] < x <= M[i,j], ..., M[i,n-1] }}

Sorted Matrix Search Code



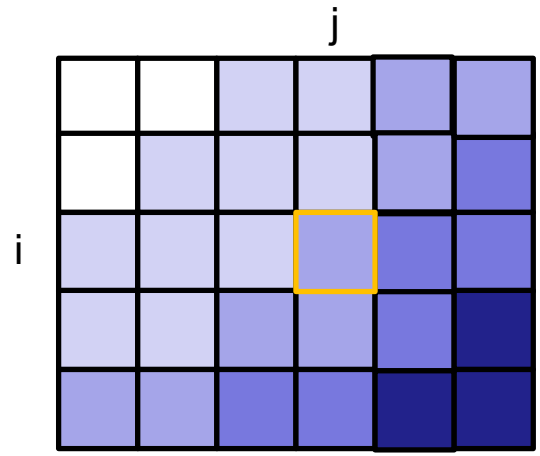
Loop body:

- when i increases, the invariant may be broken
  - we have M[i,j] <= M[i+1,j], so everything to right is still bigger
  - may need to decrease j to restore invariant for M[i,0], ..., M[i,j-1]
  - this is the same issue came up in initialization

Sorted Matrix Search Code

```
int i = 0;
int j = n;
while (i < n) {
  {{ Inv: x <= M[i,j], ..., M[i,n-1] }}
  while (j > 0 and x <= M[i,j-1])
    j = j - 1;

  {{ M[i,0], ..., M[i,j-1] < x <= M[i,j], ..., M[i,n-1] }}
  if (j <= n-1 and x == M[i,j])
    return true;
  i = i + 1;
}
return false;
```



Example: Special Composites

Problem: Find the N-th largest number of the form $2^a3^b5^c$, for some exponents a, b, c >= 0.

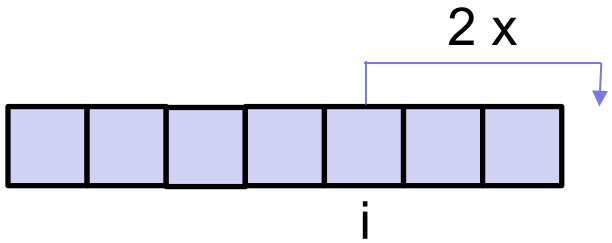
Idea: Generate these numbers in order ($1 = 2^03^05^0$, $2 = 2^13^05^0$, ...) until we get to the N-th.

Subproblem: given the first m numbers of this form, find m+1st.

Idea: Multiply every number by 2, 3, 5. Take the smallest result that is larger than the m-th number.

- $O(n^2)$ if implemented naively
- O(n log n) if implemented using binary search for 2, 3, and 5
- O(n) if optimized

Example: Special Composites



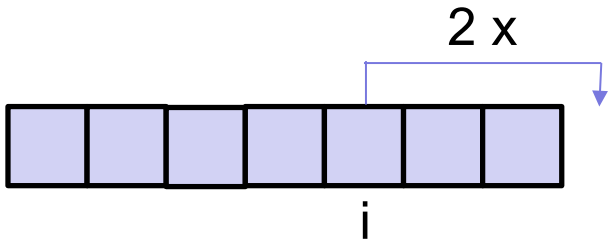
Optimization:

- Keep track of smallest index i such that 2 * A[i] > A[m-1]
- Do the same for 3 and 5. Call these indexes j and k
- Each iteration, we just need the smallest of these 3 numbers

Invariant:

- P2: 2*A[0], ..., 2*A[i-1] <= A[m-1] < 2*A[i], ..., 2*A[m-1]
- P3 (using j) and P5 (using k)

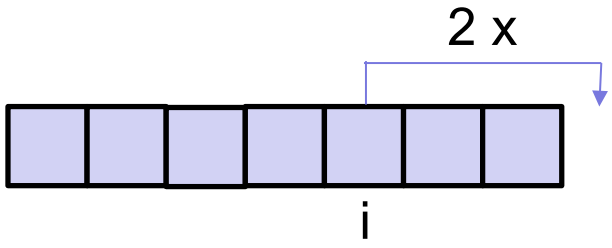
Special Composites Code



Initalization:

- Let A = [1] and m = 1
  - (note that array A also changes in this algorithm)
- Then i = j = k = 0 since 1 < 2, 3, 5

Special Composites Code



Termination:

- stop when m = N
- the N-th largest special composite is in A[m-1]

Special Composites Code

```
int[] A = new int[N]; A[0] = 1;
int i = 0, j = 0, k = 0, m = 1;
{{ Inv: A[m-1] < 2*A[i], 3*A[j], 5*A[k] (... abridged ...) }}
while (m < N) {
  A[m] = min(2*A[i], 3*A[j], 5*A[k]);
  if (2*A[i] == A[m])
    i = i + 1;
  if (3*A[j] == A[m])
    j = j + 1;
  if (5*A[k] == A[m])
    k = k + 1;
  m = m + 1;
}
return A[m-1];
```

Why not ”else if” ?

Special Composites Code

```
int[] A = new int[N]; A[0] = 1;
int i = 0, j = 0, k = 0, m = 1;
{{ Inv: A[m-1] < 2*A[i], 3*A[j], 5*A[k] (... abridged ...) }}
while (m < N) {
  A[m] = min(2*A[i], 3*A[j], 5*A[k]);
  if (2*A[i] == A[m])
    i = i + 1;
  if (3*A[j] == A[m])
    j = j + 1;
  if (5*A[k] == A[m])
    k = k + 1;
  m = m + 1;
}
return A[m-1];
```

Invariant says this is next

Preserves invariant:

- if 2*A[i] != A[m], then 2*A[i] > A[m]
- if 2*A[i] = A[m], then increasing i means we move to 2*A[i+1], which is > A[m]