CSE 331
Software Design & Implementation

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Fall 2017

Lecture 2 – Reasoning About Code With Logic

(Based on slides by Mike Ernst, Dan Grossman, David Notkin, Hal Perkins, Zach Tatlock)

Announcements

- Sign up for the discussion board (link also on the web site):
  - https://piazza.com/washington/fall2017/cse331

- Will post reasoning notes from previous quarters on the web

- HW1 posted
  - practice applying these ideas
  - builds up to verifying correctness of short, non-loop code
  - due on Tuesday by 11pm

- Reading quiz 1 posted
  - due next Friday by 11pm
  - no late days for these but I will drop lowest score

A Problem

"Write a method to return the index of the max of the first **`n`** elements of the array **`arr`**."

```
int indexOfMaximum(int[] arr, int n) {
    ...
}
```

Take a minute to think about how you'd write this…

A Solution?

Is this solution **correct**?

```
int indexOfMaximum(int[] arr, int n) {
    int maxValue = arr[0];
    int maxIndex = 0;
    for (int i = 1; i < n; i++) {
      if (arr[i] > maxValue) {
        maxValue = arr[i];
        maxIndex = i;
      }
    }
    return maxIndex;
}
```

A Solution?

Is this solution **correct**?

```
int indexOfMaximum(int[] arr, int n) {
    int maxValue = arr[0];
    int maxIndex = 0;
    for (int i = 1; i < n; i++) {
      if (arr[i] > maxValue) {
        maxValue = arr[i];
        maxIndex = i;
      }
    }
    return maxIndex;
}
```

Corner cases:
- What if there are ties?
- What if `n` is 0?

Error cases:
- What if `arr.length < n`?
- What if `arr` is null?

Morals

- You can all write the code

- **Step 1**: what does it mean to be correct?
  - that is called the “specification” for the function
  - can’t argue correctness if we don’t know what is correct

- Specifications are hard to write
  - there can be many corner cases and error cases
  - do we even want to specify behavior for all of these?
    - depends on the situation (more next time...)

- Takes work to show that the code is correct
  - we will learn how to make this easy
  - this is reasoning (and inspection)

Reasoning about code

Idea: determine what *facts* are true at each line of the program

- We would like to know:
  - at the end, `maxIndex` is index of the maximum element
  - at the end, negatives before zeros before positives in `arr`
- Get there by understanding what is true at each line until end
  - then check that those facts that are true at the end include all the things required by the *specification*

Why do this?

- Essential for building **high quality** programs
  - allows us to inspect code to check correctness
  - need all three: tools, *inspection*, & testing
  - inspection is even the most effective of the three

- Essential for building **high complexity** programs
  - allows us to build modular programs
    - each module has assumptions about how it will be used
  - misunderstandings btw module writers will cause bugs
  - assumptions must be clearly stated (and inspected)

Approaches

- We will discuss two approaches
  - forward reasoning: start at the top and work down
  - backward reasoning: start at the end and work up

- Plan:
  1. intuitive version (as seen in section)
  2. formal definitions & rules

Example of Forward Reasoning

Suppose we initially know (or assume) **`w >= 1`**

```
x = 2 * w;

y = x + 2;

z = y / 2;
```

What can we say at the end about **`z`**?

Example of Forward Reasoning

Suppose we initially know (or assume) **`w >= 1`**

```
x = 2 * w;
  // w >= 1 and x = 2 * w
y = x + 2;

z = y / 2;
```

What can we say at the end about **`z`**?

Example of Forward Reasoning

Suppose we initially know (or assume) **`w >= 1`**

```
x = 2 * w;
  // w >= 1 and x = 2 * w  => x >= 2 * 1 = 2
y = x + 2;

z = y / 2;
```

What can we say at the end about **`z`**?

Example of Forward Reasoning

Suppose we initially know (or assume) **`w >= 1`**

```
x = 2 * w;
  // w >= 1 and x = 2 * w  => x >= 2 * 1 = 2
y = x + 2;
  // w >= 1 and x = 2 * w and y = x + 2
z = y / 2;
```

What can we say at the end about **`z`**?

Example of Forward Reasoning

Suppose we initially know (or assume) **`w >= 1`**

```
x = 2 * w;
  // w >= 1 and x = 2 * w  => x >= 2 * 1 = 2
y = x + 2;
  // w >= 1 and x = 2 * w and y = x + 2
  //  => y >= 2 + 2 = 4
z = y / 2;
```

What can we say at the end about **`z`**?

Example of Forward Reasoning

Suppose we initially know (or assume) **`w >= 1`**

```
x = 2 * w;
  // w >= 1 and x = 2 * w  => x >= 2 * 1 = 2
y = x + 2;
  // w >= 1 and x = 2 * w and y = x + 2
  //  => y >= 2 + 2 = 4
z = y / 2;
  // w >= 1 and x = 2*w and y = x+2 and z = y/2
```

What can we say at the end about **`z`**?

Example of Forward Reasoning

Suppose we initially know (or assume) **`w >= 1`**

```
x = 2 * w;
  // w >= 1 and x = 2 * w  => x >= 2 * 1 = 2
y = x + 2;
  // w >= 1 and x = 2 * w and y = x + 2
  //  => y >= 2 + 2 = 4
z = y / 2;
  // w >= 1 and x = 2*w and y = x+2 and z = y/2
  //  => z >= 4/2 = 2
```

What can we say at the end about **`z`**?

Example of Forward Reasoning

Suppose we initially know (or assume) **`w >= 1`**

```
x = 2 * w;
  // w >= 1 and x = 2 * w  => x >= 2 * 1 = 2
y = x + 2;
  // w >= 1 and x = 2 * w and y = x + 2
  //  => y >= 2 + 2 = 4
z = y / 2;
  // w >= 1 and x = 2*w and y = x+2 and z = y/2
  //  => z >= 4/2 = 2
```

What can we say at the end about **`z`**? `z >= 2`

Forward Reasoning

- Forward reasoning:
  - informally, simulates the code (for all inputs at once)
  - formally, determine what follows from initial assumptions

- This is the way most programmers *inspect* their code

- Advantages and disadvantages:
  - intuitive
  - introduces (many) irrelevant facts
    - (more on ways to deal with this later...)

Example of Backward Reasoning

Suppose we want to show that **`z >= 1`** (at the end)

What needs to be true about **`w`**?

```
x = 2 * w;

y = x + 2;

z = y / 2;
  // z >= 1
```

Example of Backward Reasoning

Suppose we want to show that **`z >= 1`** (at the end)

What needs to be true about **`w`**?

```
x = 2 * w;

y = x + 2;
  // y / 2 >= 1  or equivalently  y >= 2
z = y / 2;
  // z >= 1
```

Example of Backward Reasoning

Suppose we want to show that `z >= 1` (at the end)

What needs to be true about `w`?

```
x = 2 * w;
  // x + 2 >= 2  or equivalently  x >= 0
y = x + 2;
  // y / 2 >= 1  or equivalently  y >= 2
z = y / 2;
  // z >= 1
```

Example of Backward Reasoning

Suppose we want to show that **`z >= 1`** (at the end)

What needs to be true about **`w`**?

```
  // 2 * w >= 0  or equivalently  w >= 0
x = 2 * w;
  // x + 2 >= 2  or equivalently  x >= 0
y = x + 2;
  // y / 2 >= 1  or equivalently  y >= 2
z = y / 2;
  // z >= 1
```

Backward Reasoning

- Backward reasoning:
  - determines sufficient conditions for end result
    - e.g., assumptions needed for correctness

- Advantages and disadvantages:
  - less intuitive
  - determines exactly what is necessary to achieve the goal
  - gives you another (powerful) way to reason about code

Our approach

- We will take a **methodical** approach to reasoning about code
  - spell everything out in detail to avoid any misunderstanding
  - (you can move more quickly as you get practice)

- Hoare Logic
  - named after its inventor, Sir Anthony Hoare (inventor of quicksort)
  - considers just assignments, if-statements, and while-loops
    - everything else can be built out of these
  - we will consider just integer-valued variables
    - for Java, we will need floats, strings, objects, etc.

- This lecture: assignments & if-statements; Next lecture: loops

Terminology

- The *program state* is the values of all the (relevant) variables
- An *assertion* is a logical formula referring to the program state (e.g., contents of variables) at a given point
- An assertion *holds* for a program state if the formula is true when those values are substituted for the variables
- An assertion before the code is a *precondition*
  – these represent assumptions about when that code is used
- An assertion after the code is a *postcondition*
  – these represent what we want the code to accomplish

Notation

- Instead of writing assertions as comments, Hoare logic uses {..}
  - since Java code also has {..}, I will use {{…}}
  - e.g., `{{ w >= 1 }} x = 2 * w; {{ x >= 2 }}`
- Assertions are math not Java
  - you can use the usual math notation
    - (e.g., = instead of == for equals)
  - purpose is communication with other humans (not computers)
  - we will need **`and`**, **`or`**, **`not`** as well
    - can also write use ∧ (and) ∨ (or) etc.
- The Java language also has assertions (**`assert`** statements)
  - throws an exception if the condition does not evaluate true
  - we will discuss these more later in the course

Hoare Logic

- A Hoare triple is two assertions and one piece of code:

  {{ *P* }} *S* {{ *Q* }}

  – *P* the precondition
  – *S* the code
  – *Q* the postcondition

- A Hoare triple **{{ *P* }} *S* {{ *Q* }}** is called valid if:
  – in any state where P holds, executing S produces a state where Q holds
  – i.e., if *P* is true before *S*, then *Q* must be true after it
  – otherwise the triple is called invalid

- We will use this to argue correctness with S an entire method.

Example 1

Is the following Hoare triple valid or invalid?

- assume all variables are integers and there is no overflow

```
{{ x != 0 }} y = x*x; {{ y > 0 }}
```

Example 1

Is the following Hoare triple valid or invalid?

- assume all variables are integers and there is no overflow

```
{{ x != 0 }} y = x*x; {{ y > 0 }}
```

Valid

- **`y`** could only be zero if **`x`** were zero (which it isn't)

Example 2

Is the following Hoare triple valid or invalid?

- assume all variables are integers and there is no overflow

```
{{ z != 1 }} y = z*z; {{ y != z }}
```

Example 2

Is the following Hoare triple valid or invalid?

– assume all variables are integers and there is no overflow

```
{{ z != 1 }} y = z*z; {{ y != z }}
```

Invalid

- counterexample: `z = 0`

Example 3

Is the following Hoare triple valid or invalid?

- assume all variables are integers and there is no overflow

```
{{ x >= 0 }} y = 2*x; {{ y > x }}
```

Example 3

Is the following Hoare triple valid or invalid?

- assume all variables are integers and there is no overflow

```
{{ x >= 0 }} y = 2*x; {{ y > x }}
```

Invalid

- counterexample: `x = 0`

Example 4

Is the following Hoare triple valid or invalid?

```
{{ }}
if (x > 7) {
  y = 4;
} else {
  y = 3;
}
{{ y < 5 }}
```

Example 4

Is the following Hoare triple valid or invalid?

```
{{ }}
if (x > 7) {
  y = 4;
} else {
  y = 3;
}
{{ y < 5 }}
```

Valid

- `y` is either 3 or 4; in either case, it is less than 5

Example 5

Is the following Hoare triple valid or invalid?

```
{{ }}
x = y;
z = x;
{{ y = z }}
```

Example 5

Is the following Hoare triple valid or invalid?

```
{{ }}
x = y;
z = x;
{{ y = z }}
```

Valid

Example 6

Is the following Hoare triple valid or invalid?

```
{{ x = 7 and y = 5 }}
// swap x and y
tmp = x;
x = tmp;
y = x;
{{ x = 5 and y = 7 }}
```

Example 6

Is the following Hoare triple valid or invalid?

```
{{ x = 7 and y = 5 }}
// swap x and y
tmp = x;
x = tmp;
y = x;
{{ x = 5 and y = 7 }}
```

Invalid

- first two lines leave x unchanged, so we get **`x = y = 7`**

The general rules

- Some of these require some thought
  - it would be preferable to do this without (much) thought
  - fortunately, there is a “turn the crank” way of doing these

- For each kind of construct, there is a general rule
  - assignment statements
  - two statements in sequence
  - conditionals
  - loops (next lecture)

Assignment Rule

`{{ P }} x = e; {{ Q }}`

- Let **`Q[x=e]`** be like **`Q`** except replace every **`x`** with **`e`**
  - after "**`x = e;`**", **`Q`** and **`Q[x=e]`** are equivalent
  - but **`Q[x=e]`** does not involve **`x`** so it holds after "**`x = e;`**" if and only if it holds before
  - so we can consider **`P`** and **`Q[x=e]`** w/out the assignment
  - (This is backward reasoning.)

- This triple is valid iff: whenever **`P`** holds, **`Q[x=e]`** also holds
  - in logic, we'd say it is valid if **`P`** implies **`Q[x=e]`**

Assignment Rule Example

`{{ z > 34 }} y = z + 1; {{ y > 1 }}`

- **`Q[y=z+1]`** is **`z + 1 > 1`**
  - this is equivalent to **`z > 0`**
  - whenever **`z > 34`**, we also have **`z > 0`**
  - this is valid

Sequence Rule

`{{ P }} S1;S2 {{ Q }}`

- Triple is valid iff: there is an assertion **`R`** such that
  - `{{ P }} S1 {{ R }}` is valid and
  - `{{ R }} S2 {{ Q }}` is valid
- For now, we will need to guess **`R`**
  - we will see shortly that we can find an **`R`** without guessing

Sequence Rule Example

`{{ z >= 1 }} y = z+1; w = y*y; {{ w > y }}`

- Choose **`R`** to be **`y > 1`**
- Show **`{{ z >= 1 }} y=z+1; {{ y > 1 }}`**
  - use assignment rule: **`z >= 1`** implies **`z+1 > 1`**?
  - equivalently, **`z >= 1`** implies **`z > 0`**? Valid.
- Show **`{{ y > 1 }} w=y*y; {{ w > y }}`**
  - use assignment rule: **`y > 1`** implies **`y*y > y`**
  - requires some thought, but valid

- Both of these are triples valid, so the triple at the top is valid

Conditional Rule

`{{ P }} if (b) {S1} else {S2} {{ Q }}`

- When S1 executes, we know **`P and b`**
- When S2 executes, we know **`P and not b`**

- Triple is valid iff: there are assertions **`Q1`** and **`Q2`** such that
  - **`{{ P and b }} S1 {{ Q1 }}`** is valid and
  - **`{{ P and not b }} S2 {{ Q2 }}`** is valid and
  - **`Q1 or Q2`** implies **`Q`**
    - (i.e., **`Q1`** implies **`Q`** and **`Q2`** implies **`Q`**)

Conditional Rule

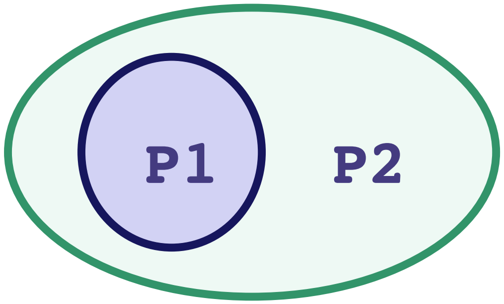
```
{{ }} if (x > 7) {y=x;} else {y=20;} {{ y > 5 }}
```

- Let `Q1` be `y > 7` (other choices work too)
  - use assignment rule to show `{{ x > 7 }} y=x; {{ y > 7 }}`
- Let `Q2` be `y = 20` (other choices work too)
  - use assignment rule to show `{{ x <= 7 }} y=20; {{ y = 20 }}`
- Check that `y > 7 or y = 20` implies `y > 5`

Weaker vs Stronger

If "whenever P1 holds, P2 also holds", then:

- P1 is called stronger than P2
- P2 is called weaker than P1



- It is more (or at least as) "difficult" to satisfy P1
  - the program states where P1 holds are a subset of the states where P2 holds
- P1 puts more constraints on program states
- P1 is a stronger set of requirements

- We do not always have P1 stronger than P2 or vice versa!
  - most assertions are incomparable

Examples

- **`x = 17`** is stronger than **`x > 0`**
- **`x is prime`** is neither stronger nor weaker than **`x is odd`**
  - these two statements are incomparable
- **`x is prime and x > 2`** is stronger than
  `x is odd and x > 2`
- Many other examples...

Applications to Method Design

- When writing a method, you decide the preconditions
  - e.g., a parameter may be assumed positive
  - e.g., an array may be assumed to be non-empty

- There are advantages and disadvantages to weaker vs stronger
  - stronger preconditions make the code easier to **change**
    - there are more allowed implementations
  - weaker preconditions allow more uses
    - there are more allowed calls
  - stronger preconditions may make the code easier to write
  - weaker preconditions are necessary for libraries

- We will discuss this more later on…

Applications to Hoare Logic

- Suppose:
  - `{{ P }} S {{ Q }}` is valid and
  - some `P1` is *stronger* than `P` and
  - some `Q1` is *weaker* than `Q`

- Then these are all valid too:
  - `{{ P1 }} S {{ Q }}`
    - a state where `P1` holds is one where `P` also holds
  - `{{ P }} S {{ Q1 }}`
    - a state where `Q` holds is one where `Q1` also holds
  - `{{ P1 }} S {{ Q1 }}`

Example Applications to Hoare Logic

`{{ x >= 0 }} y = x + 1; {{ y > 0 }}`

- We know this is valid by the assignment rule

- Let **`P1`** be **`x > 0`**
  - stronger since **`x >= 0`** implies **`x > 0`**
- Let **`Q1`** be **`y >= 0`**
  - weaker since **`y >= 0`** implies **`y > 0`**

- Thus, the following is also valid:

`{{ x > 0 }} y = x + 1; {{ y >= 0 }}`

Weakest preconditions

- Suppose we know **`Q`** and **`S`**
- There are potentially many **`P`** such that {{**`P`**}} **`S`** {{**`Q`**}} is valid

- Would be ideal if there were a *unique* weakest precondition **P**
  - most general assumptions under which **`S`** makes **`Q`** hold
  - get a valid triple for **`P1`** if and only if **`P1`** implies **`P`**

- Amazingly, without loops, for any **`S`** and **`Q`**, this exists!
  - we denote this by wp(**`S`**,**`Q`**)
  - can be found by general rules

- Allows you to reason backward *without any guessing*
  - just as you do with forward reasoning

Rules for weakest preconditions

- wp(**`x = e`**, **`Q`**) is **`Q[x=e]`**
  - Example: wp(**`x = 2*y`**, **`x > 4`**) = **`2*y > 4`**, i.e., **`y > 2`**
- wp(**`S1;S2`**, **`Q`**) is wp(**`S1`**, wp(**`S2`**,**`Q`**))
  - i.e., let **`R`** be wp(**`S2`**,**`Q`**) and overall wp is wp(**`S1`**,**`R`**)
  - Example: wp(**`y = x+1`**, wp(**`z = y+1`**, **`z > 2`**)) =
    wp(**`y = x+1`**, **`y+1 > 2`**) =
    `(x+1)+1 > 2` or equivalently **`x > 0`**
- wp(**`if b S1 else S2`**, **`Q`**) is this logic formula:

  `(b and` wp(**`S1`**,**`Q`**)**`) or (!b and`** wp(**`S2`**,**`Q`**)**`)`**

  - you need wp(**`S1`**,**`Q`**) if **`S1`** is executed and wp(**`S2`**,**`Q`**) if **`S2`** is
  - you can often simplify the result considerably

More Examples

- If `S` is `x = y*y` and `Q` is `x > 4`,
  then wp(`S`,`Q`) is `y*y > 4`, i.e., `|y| > 2`

- If `S` is `y = x + 1; z = y – 3;` and `Q` is `z = 10`,
  then wp(`S`,`Q`) …
  = wp(`y = x + 1; z = y – 3`, `z = 10`)
  = wp(`y = x + 1`, wp(`z = y – 3`, `z = 10`))
  = wp(`y = x + 1`, `y-3 = 10`)
  = wp(`y = x + 1`, `y = 13`)
  = `x+1 = 13`
  = `x = 12`

Bigger Example

`S` is `if (y < 5) { x = y*y; } else { x = y+1; }`

wp(`S`, `x >= 9`)

= (`y < 5 and` wp(`x = y*y`, `x >= 9`))

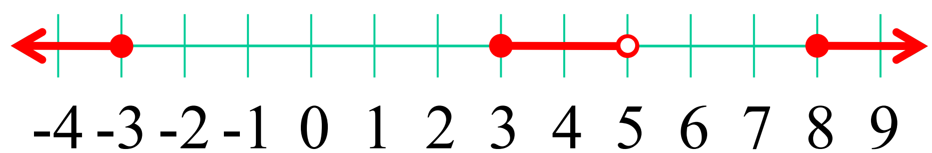
`or` (`y >= 5 and` wp(`x = y+1`, `x >= 9`))

= (`y < 5 and y*y >= 9`)

`or` (`y >= 5 and y+1 >= 9`)

= (`y <= -3`) `or` (`y >= 3 and y < 5`)

`or` (`y >= 8`)



If-statements review

Forward reasoning

```
{{ P }}
if B
  {{ P and B }}
  S1
  {{ Q1 }}
else
  {{ P and not B }}
  S2
  {{ Q2 }}
{{ Q1 or Q2 }}
```

Backward reasoning

```
{{ (B and wp(S1, Q)) or
  (not B and wp(S2, Q)) }}
if B
  {{ wp(S1, Q) }}
  S1
  {{ Q }}
else
  {{ wp(S2, Q) }}
  S2
  {{ Q }}
{{ Q }}
```

One caveat

- With forward reasoning, there is a problem with assignment:
  - changing a variable can affect other assumptions

```
{{ }}
w = x + y;
{{ w = x + y }}
x = 4;
{{ w = x + y and x = 4 }}
y = 3;
{{ w = x + y and x = 4 and y = 3 }}
```

- But clearly we do not know **`w = 7`**!
- The assertion **`w = x + y`** means the *original* values of **`x`** and **`y`**

One Fix

- Use different names for the values at different points
  - common to use subscripts to distinguish these
  - on every assignment, rename references to the old values

$\{\{\ \}\}$
`w = x + y;`
$\{\{\ w = x + y\ \}\}$
`x = 4;`
$\{\{\ w = x_1 + y \text{ and } x = 4\ \}\}$
`y = 3;`
$\{\{\ w = x_1 + y_1 \text{ and } x = 4 \text{ and } y = 3\ \}\}$

Useful example: swap

- Consider code for a swapping **x** and **y**

```
{{ }}
tmp = x;
{{ tmp = x }}
x = y;
{{ tmp = x1 and x = y }}
y = tmp;
{{ tmp = x1 and x = y1 and y = tmp }}
```

- Post condition implies $x = y_1$ and $y = x_1$
- I.e., their final values are equal to the original values swapped