CSE 331
Software Design & Implementation

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Spring 2014

Lecture 2 – Reasoning About Code With Logic

Reasoning about code

Determine what facts are true as a program executes

- Under what assumptions

Examples:

- If **`x`** starts positive, then **`y`** is **`0`** when the loop finishes
- Contents of the array **`arr`** refers to are sorted
- Except at one code point, **`x + y == z`**
- For all instances of **`Node n`**,
  `n.next == null` ∨ **`n.next.prev == n`**
- …

Why do this?

- Essential complement to *testing*, which we will also study
  - Testing: Actual results for some actual inputs
  - Logical reasoning: Reason about whole classes of inputs/ states at once ("If **`x > 0`**, …")
    - *Prove* a program correct (or find bugs trying)
    - Understand *why* code is correct

- Stating assumptions is the essence of specification
  - "Callers must not pass **`null`** as an argument"
  - "Callee will always return an unaliased object"
  - …

Our approach

- Hoare Logic: a 1970s approach to logical reasoning about code
  - For now, consider just variables, assignments, if-statements, while-loops
    - So no objects or methods

- This lecture: The idea, without loops, in 3 passes
  1. High-level intuition of forward and backward reasoning
  2. Precise definition of logical assertions, preconditions, etc.
  3. Definition of weaker/stronger and weakest-precondition

- Next lecture: Loops

Why?

- Programmers rarely “use Hoare logic” like in this lecture
  - For simple snippets of code, it’s overkill
  - Gets very complicated with objects and aliasing
  - But is occasionally useful for loops with subtle *invariants*
    - Examples: Homework 0, Homework 2

- Also it’s an ideal setting for the right logical foundations
  - How can logic “talk about” program states?
  - How does code execution “change what is true”?
  - What do “weaker” and “stronger” mean?

This is all essential for *specifying library-interfaces*, which *does* happen All the Time in The Real World (coming lectures)

Example

Forward reasoning:

– Suppose we initially know (or assume) `w > 0`

```
// w > 0
x = 17;
// w > 0  ∧  x == 17
y = 42;
// w > 0  ∧  x == 17 ∧  y == 42
z = w + x + y;
// w > 0 ∧ x == 17 ∧ y == 42 ∧ z > 59
…
```

– Then we know various things after, including `z > 59`

Example

Backward reasoning:

- Suppose we want **`z`** to be negative at the end

```
// w + 17 + 42 < 0
x = 17;
// w + x + 42 < 0
y = 42;
// w + x + y < 0
z = w + x + y;
// z < 0
```

- Then we know initially we need to know/assume **`w < -59`**
  - Necessary and sufficient

Forward vs. Backward, Part 1

- Forward reasoning:
  - Determine what follows from initial assumptions
  - Most useful for *maintaining an invariant*

- Backward reasoning
  - Determine sufficient conditions for a certain result
    - If result desired, the assumptions suffice for correctness
    - If result undesired, the assumptions suffice to trigger bug

Forward vs. Backward, Part 2

- Forward reasoning:
  - Simulates the code (for many "inputs" "at once")
  - Often more intuitive
  - But introduces [many] facts irrelevant to a goal

- Backward reasoning
  - Often more useful: Understand what each part of the code contributes toward the goal
  - "Thinking backwards" takes practice but gives you a powerful new way to reason about programs

Conditionals

```
// initial assumptions
if(…) {
  … // also know test evaluated to true
} else {
  … // also know test evaluated to false
}
// either branch could have executed
```

Two key ideas:

1. The precondition for each branch includes information about the result of the test-expression
2. The overall postcondition is the disjunction (“or”) of the postcondition of the branches

Example (Forward)

Assume initially **`x >= 0`**

```
// x >= 0
z = 0;
// x >= 0 ∧ z == 0
if(x != 0) {
  // x >= 0 ∧ z == 0 ∧ x != 0 (so x > 0)
  z = x;
  // … ∧ z > 0
} else {
  // x >= 0 ∧ z == 0 ∧ !(x!=0) (so x == 0)
  z = x + 1;
  // … ∧ z == 1
}
// ( … ∧ z > 0) v (… ∧ z == 1)  (so z > 0)
```

Our approach

- Hoare Logic, a 1970s approach to logical reasoning about code
  - [Named after its inventor, Tony Hoare]
  - Considering just variables, assignments, if-statements, while-loops
    - So no objects or methods

- This lecture: The idea, without loops, in 3 passes
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- Next lecture: Loops

Some notation and terminology

- The "assumption" before some code is the precondition
- The "what holds after (given assumption)" is the postcondition

- Instead of writing pre/postconditions after //, write them in {…}
  - This is not Java
  - How Hoare logic has been written "on paper" for 40ish years

```
{ w < -59 }
x = 17;
{ w + x < -42 }
```

  - In pre/postconditions, = is equality, not assignment
    - Math's "=", which for numbers is Java's **==**

```
{ w > 0  ∧  x = 17 }
y = 42;
{ w > 0  ∧  x = 17 ∧  y = 42 }
```

What an assertion means

- An *assertion* (pre/postcondition) is a logical formula that can refer to program state (e.g., contents of variables)
- A *program state* is something that “given” a variable can “tell you” its contents
  - Or any expression that has no *side-effects*
- An assertion *holds* for a program state, if evaluating using the program state produces *true*
  - Evaluating a program variable produces its contents in the state
  - Can think of an assertion as representing the *set* of (exactly the) states for which it holds

A Hoare Triple

- A Hoare triple is two assertions and one piece of code:

$$\{P\}\ S\ \{Q\}$$

  - *P* the precondition
  - *S* the code (statement)
  - *Q* the postcondition

- A Hoare triple $\{P\}\ S\ \{Q\}$ is (by definition) valid if:
  - For all states for which *P* holds, executing *S* always produces a state for which *Q* holds
  - Less formally: If *P* is true before *S*, then *Q* must be true after
  - Else the Hoare triple is invalid

Examples

Valid or invalid?

– (Assume all variables are integers without overflow)

- `{x != 0} y = x*x; {y > 0}`
- `{z != 1} y = z*z; {y != z}`
- `{x >= 0} y = 2*x; {y > x}`
- `{true} (if(x > 7) {y=4;} else {y=3;}) {y < 5}`
- `{true} (x = y; z = x;) {y=z}`
- `{x=7 ∧ y=5}`
  `(tmp=x; x=tmp; y=x;)`
  `{y=7 ∧ x=5}`

Examples

Valid or invalid?

- (Assume all variables are integers without overflow)

- `{x != 0} y = x*x; {y > 0}` valid
- `{z != 1} y = z*z; {y != z}` invalid
- `{x >= 0} y = 2*x; {y > x}` invalid
- `{true} (if(x > 7) {y=4;} else {y=3;}) {y < 5}` valid
- `{true} (x = y; z = x;) {y=z}` valid
- `{x=7 ∧ y=5}` invalid
  `(tmp=x; x=tmp; y=x;)`
  `{y=7 ∧ x=5}`

Aside: assert in Java

- An assertion in Java is a statement with a Java expression, e.g.,

```
assert x > 0 && y < x;
```

- Similar to our assertions
  - Evaluate using a program state to get **`true`** or **`false`**
  - Uses Java syntax

- In Java, this is a run-time thing: Run the code and raise an exception if assertion is violated
  - Unless assertion-checking is disabled
  - Later course topic

- This week: we are reasoning about the code, not running it on some input

The general rules

- So far: Decided if a Hoare triple was valid by using our understanding of programming constructs

- Now: For each kind of construct there is a general rule
  - A rule for assignment statements
  - A rule for two statements in sequence
  - A rule for conditionals
  - [next lecture:] A rule for loops
  - …

Assignment statements

`{P} x = e; {Q}`

- Let **`Q′`** be like **`Q`** except replace every **`x`** with **`e`**
- Triple is valid if:

  For all program states, if **`P`** holds, then **`Q′`** holds
  - That is, **`P`** implies **`Q′`**, written **`P => Q′`**

- Example: **`{z > 34} y=z+1; {y > 1}`**
  - **`Q′ is {z+1 > 1}`**

Sequences

`{P} S1;S2 {Q}`

- Triple is valid if and only if there is an assertion **`R`** such that
  - **`{P}S1{R}`** is valid, and
  - **`{R}S2{Q}`** is valid
- Example: **`{z >= 1} y=z+1; w=y*y; {w > y}`** (integers)
  - Let **`R`** be **`{y > 1}`**
  - Show **`{z >= 1} y=z+1; {y > 1}`**
    - Use rule for assignments: **`z >= 1`** implies **`z+1 > 1`**
  - Show **`{y > 1} w=y*y; {w > y}`**
    - Use rule for assignments: **`y > 1`** implies **`y*y > y`**

Conditionals

`{P} if(b) S1 else S2 {Q}`

- Triple is valid if and only if there are assertions `Q1,Q2` such that
  - `{P ∧ b}S1{Q1}` is valid, and
  - `{P ∧ !b}S2{Q2}` is valid, and
  - `Q1 ∨ Q2` implies `Q`

- Example: `{true} (if(x > 7) y=x; else y=20;) {y > 5}`
  - Let `Q1` be `{y > 7}` (other choices work too)
  - Let `Q2` be `{y = 20}` (other choices work too)
  - Use assignment rule to show `{true ∧ x > 7}y=x;{y>7}`
  - Use assignment rule to show `{true ∧ x <= 7}y=20;{y=20}`
  - Indicate `y>7 ∨ y=20` implies `y>5`

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Weaker vs. Stronger

If P1 implies P2 (written P1 => P2), then:

- P1 is stronger than P2
- P2 is weaker than P1

- Whenever P1 holds, P2 also holds
- So it is more (or at least as) "difficult" to satisfy P1
  - The program states where P1 holds are a subset of the program states where P2 holds
- So P1 puts more constraints on program states
- So it's a stronger set of obligations/requirements

Examples

- **`x = 17`** is stronger than **`x > 0`**
- **`x is prime`** is neither stronger nor weaker than **`x is odd`**
- **`x is prime and x > 2`** is stronger than
  `x is odd and x > 2`
- …

Why this matters to us

- Suppose:
  - **`{P}S{Q}`**, and
  - **`P`** is weaker than some **`P1`**, and
  - **`Q`** is stronger than some **`Q1`**

- Then: **`{P1}S{Q}`** and **`{P}S{Q1}`** and **`{P1}S{Q1}`**

- Example:
  - **`P`** is **`x >= 0`**
  - **`P1`** is **`x > 0`**
  - **`S`** is **`y = x+1`**
  - **`Q`** is **`y > 0`**
  - **`Q1`** is **`y >= 0`**

So…

- For backward reasoning, if we want `{P}S{Q}`, we could instead:
  - Show `{P1}S{Q}`, and
  - Show `P => P1`
- Better, we could just show `{P2}S{Q}` where `P2` is the weakest precondition of `Q` for `S`
  - Weakest means the most lenient assumptions such that `Q` will hold
  - Any precondition `P` such that `{P}S{Q}` is valid will be stronger than `P2`, i.e., `P => P2`
- Amazing (?): Without loops/methods, for any `S` and `Q`, there exists a unique weakest precondition, written wp(`S`,`Q`)
  - Like our general rules with backward reasoning

Weakest preconditions

- wp(`x = e;`, `Q`) is `Q` with each `x` replaced by `e`
  - Example: wp(`x = y*y;`, `x > 4`) = `y*y > 4`, i.e., `|y| > 2`
- wp(`S1;S2`, `Q`) is wp(`S1`,wp(`S2`,`Q`))
  - I.e., let `R` be wp(`S2`,`Q`) and overall wp is wp(`S1`,`R`)
  - Example: wp(`(y=x+1; z=y+1;)`, `z > 2`) =
    `(x + 1)+1 > 2`, i.e., `x > 0`
- wp(`if b S1 else S2`, `Q`) is this logic formula:

  `(b` $\wedge$ wp(`S1,Q`)`) v (!b` $\wedge$ wp(`S2,Q`)`)`

  - (In any state, b will evaluate to either true or false…)
  - (You can sometimes then simplify the result)

Simple examples

- If `S` is `x = y*y` and `Q` is `x > 4`,
  then wp(`S`,`Q`) is `y*y > 4`, i.e., `|y| > 2`

- If `S` is `y = x + 1; z = y – 3;` and `Q` is `z = 10`,
  then wp(`S`,`Q`) …
  = wp(`y = x + 1; z = y – 3;`, `z = 10`)
  = wp(`y = x + 1;`, wp(`z = y – 3;`, `z = 10`))
  = wp(`y = x + 1;`, wp(`z = y – 3;`, `z = 10`))
  = wp(`y = x + 1;`, `y-3 = 10`)
  = wp(`y = x + 1;`, `y = 13`)
  = `x+1 = 13`
  = `x = 12`

Bigger example

```
S is if (x < 5) {
        x = x*x;
      } else {
        x = x+1;
      }
Q is x >= 9
```

wp(S, `x >= 9`)

= (`x < 5` ∧ wp(`x = x*x;`, `x >= 9`))

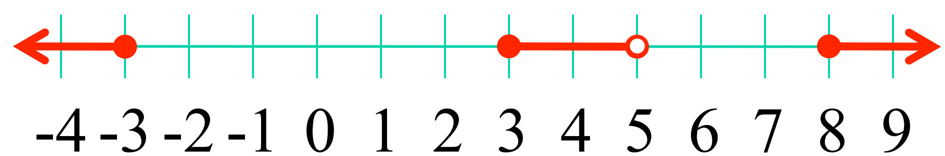
∨ (`x >= 5` ∧ wp(`x = x+1;`, `x >= 9`))

= (`x < 5 ∧ x*x >= 9`)

∨ (`x >= 5 ∧ x+1 >= 9`)

= (`x <= -3`) ∨ (`x >= 3 ∧ x < 5`)

∨ (`x >= 8`)



If-statements review

Forward reasoning

$\{P\}$
`if B`
  $\{P \land B\}$
  `S1`
  $\{Q1\}$
`else`
  $\{P \land !B\}$
  `S2`
  $\{Q2\}$
$\{Q1 \lor Q2\}$

Backward reasoning

$\{ (B \land wp(S1, Q))$
  $\lor\ (!B \land wp(S2, Q)) \}$
`if B`
  $\{wp(S1, Q)\}$
  `S1`
  $\{Q\}$
`else`
  $\{wp(S2, Q)\}$
  `S2`
  $\{Q\}$
$\{Q\}$

“Correct”

- If wp(**`S`**,**`Q`**) is **`true`**, then executing **`S`** will always produce a state where **`Q`** holds
  - **`true`** holds for every program state

One more issue

- With forward reasoning, there is a problem with assignment:
  - Changing a variable can affect other assumptions

- Example:

```
{true}
w=x+y;
{w = x + y;}
x=4;
{w = x + y ∧ x = 4}
y=3;
{w = x + y ∧ x = 4 ∧ y = 3}
```

But clearly we do not know **`w=7`**!

The fix

- When you assign to a variable, you need to replace all other uses of the variable in the post-condition with a different variable
  - So you refer to the “old contents”

- Corrected example:

```
{true}
w=x+y;
{w = x + y;}
x=4;
{w = x1 + y ∧ x = 4}
y=3;
{w = x1 + y1 ∧ x = 4 ∧ y = 3}
```

Useful example

- Swap contents
  - Give a name to initial contents so we can refer to them in the post-condition
  - Just in the formulas: these “names” are not in the program
  - Use these extra variables to avoid “forgetting” “connections”

```
{x = x_pre ∧ y = y_pre}
tmp = x;
{x = x_pre ∧ y = y_pre ∧ tmp=x}
x = y;
{x = y ∧ y = y_pre ∧ tmp=x_pre}
y = tmp;
{x = y_pre ∧ y = tmp ∧ tmp=x_pre}
```