HW7, Dijkstra's

CSE 331 – Section 7

02/21/2013

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Modified by David Mailhot,

with much material from Dan Grossman

Homework 7

Modify your graph to use Generics

- Change your hw5 code where it is now
- Will have to update hw5, hw6 tests

Implement Dijkstra’s algorithm

- Alternate search algorithm that uses edge weights
- Apply to Marvel graph, with edge weights reciprocal to number of books in common

Note on folders

MarvelPaths2.java looks in src/hw7/data

HW7TestDriver.java looks in src/hw7/test

Shortest paths

Done: BFS to find the minimum path length from **v** to **u**

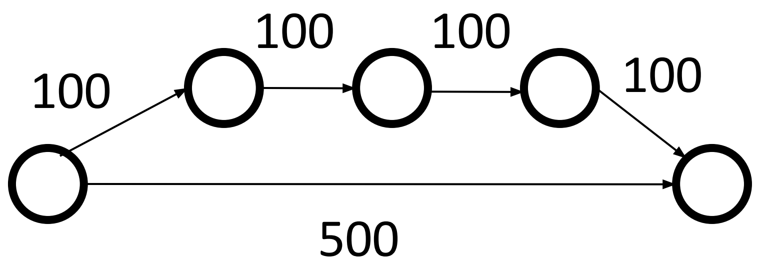
Now: Weighted graphs

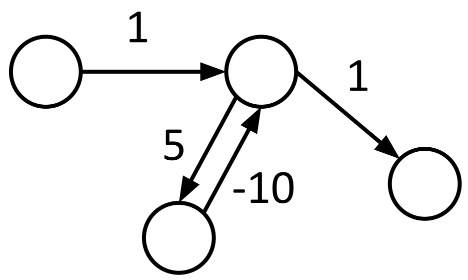
Given a weighted graph and node **v**,

find the minimum-cost path from **v** to every node

Unlike before, BFS will not work

Not as easy





Why BFS won't work:

Smallest-cost path may not have the fewest edges

We will assume there are no negative weights

- Problem is ill-defined if there are negative-cost cycles
- Today's algorithm is wrong if edges can be negative

Dijkstra’s Algorithm

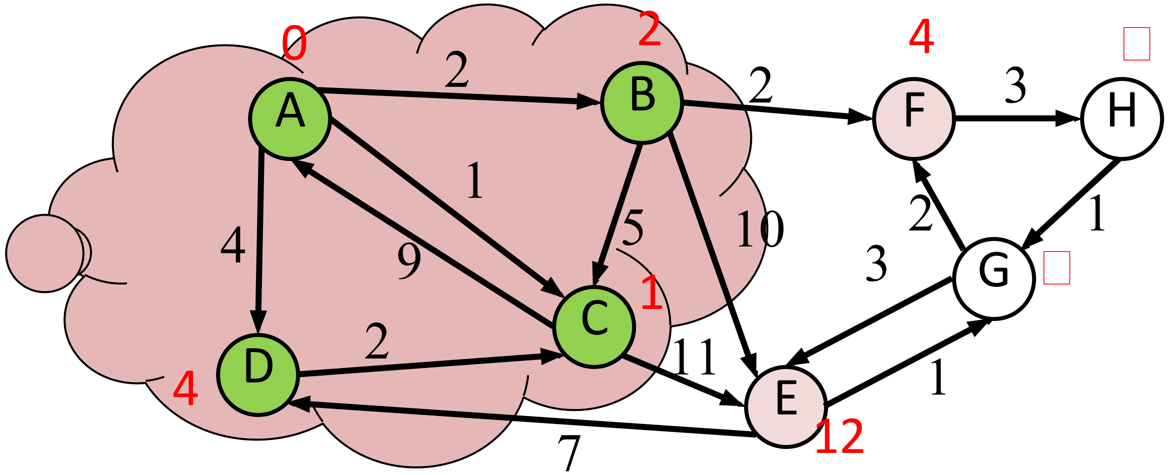
Named after its inventor Edsger Dijkstra (1930-2002)

- Truly one of the “founders” of computer science;
  this is just one of his many contributions

The idea: reminiscent of BFS, but adapted to handle weights

- Grow the set of nodes whose shortest distance has been computed
- Nodes not in the set will have a “best distance so far”
- A priority queue will turn out to be useful for efficiency

Dijkstra’s Algorithm: Idea



Initially, start node has cost 0 and all other nodes have cost ∞

At each step:

- Pick closest unknown vertex **v**
- Add it to the “cloud” of known vertices
- Update distances for nodes with edges from **v**

That’s it!

Aside: weights for Marvel Data

The Marvel data doesn't really have a measure of 'weight' we can use:

- So for HW7 you'll be hacking your own!

Aside: weights for Marvel Data

The idea: the more well-connected two characters are, the lower the weight and the more likely that a path is taken through them.

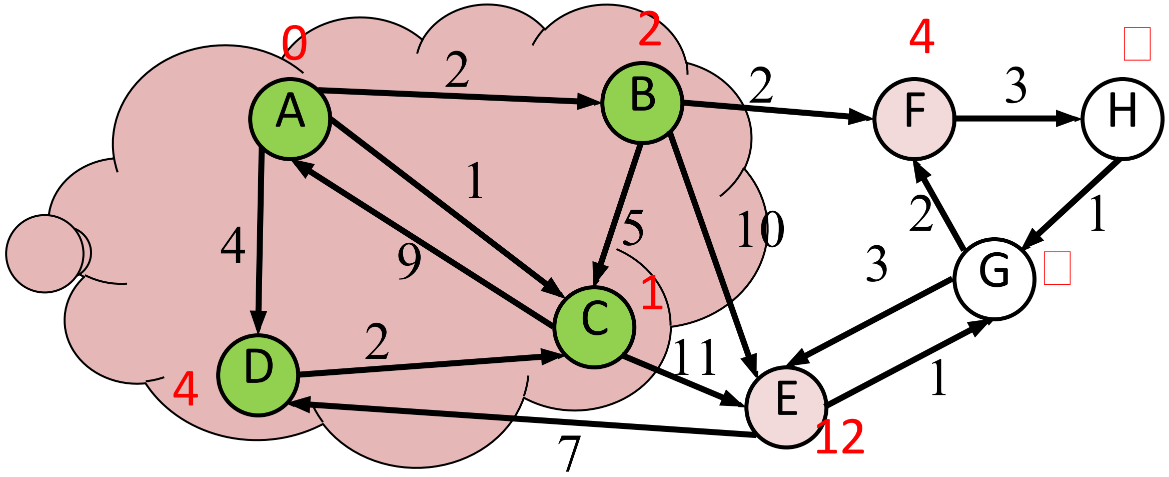
- The weight of the edge between two characters is equal to the inverse of how many comic books those two characters are in together (the 'multiplicative inverse').
- *For example, if Amazing Amoeba and Zany Zebra appeared in 5 comic books together, the weight of the edge between them would be 1/5.*
- No duplicate edges: two characters will have at most one edge between them that is labeled with a cost.

Aside: weights for Marvel Data

You'll be placing your new Marvel application in hw7/MarvelPaths2.java.

Key: You will calculate edge costs when you read in the data and construct your graph using those calculated weights, all in MarvelPaths2.java

Dijkstra’s Algorithm: Idea



Initially, start node has cost 0 and all other nodes have cost ∞

At each step:

- Pick closest unknown vertex **v**
- Add it to the “cloud” of known vertices
- Update distances for nodes with edges from **v**

That’s it!

The Algorithm

1. For each node **`v`**, set **`v.cost = ∞`** and **`v.known = false`**
2. Set **`source.cost = 0`**
3. While there are unknown nodes in the graph
   a. Select the unknown node **`v`** with lowest cost
   b. Mark **`v`** as known
   c. For each edge **`(v,u)`** with weight **`w`**,

```
 c1 = v.cost + w // cost of best path through v  to u
 c2 = u.cost  // cost of best path to u previously known
if(c1 < c2){ // if the path through v is better
   u.cost = c1
   u.path = v // for computing actual paths
}
```

Important features

When a vertex is marked known, the cost of the shortest path to that node is known

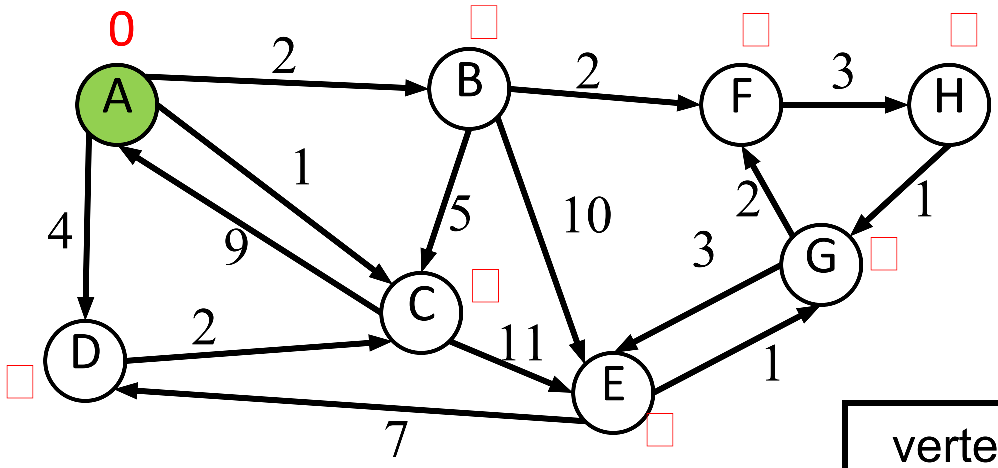
- The path is also known by following back-pointers

While a vertex is still not known, another shorter path to it *might* still be found

e: The “Order Added to Known Set” is not important

- A detail about how the algorithm works (client doesn’t care)
- Not used by the algorithm (implementation doesn’t care)
- It is sorted by path-cost, resolving ties in some way

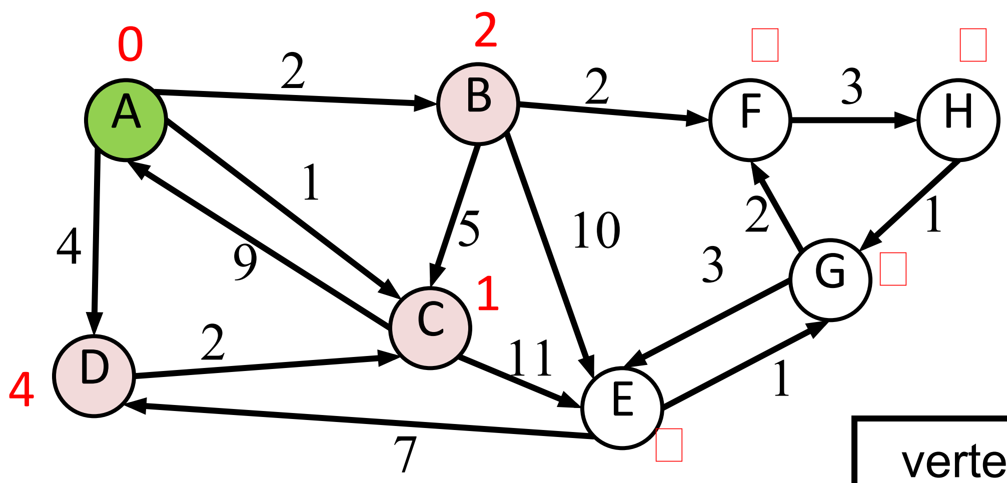
Example #1



Order Added to Known Set:

vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	
H		??	

Example #1

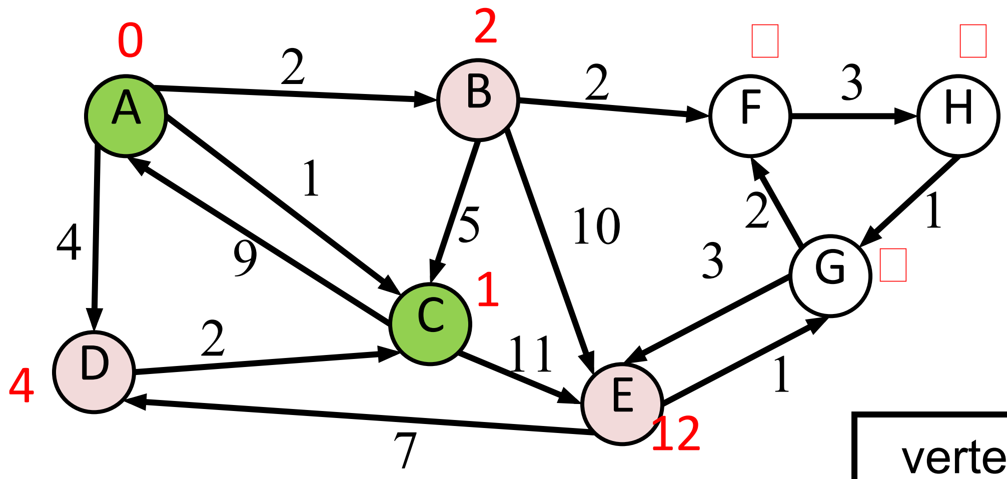


vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C		≤ 1	A
D		≤ 4	A
E		??	
F		??	
G		??	
H		??	

Order Added to Known Set:

A

Example #1

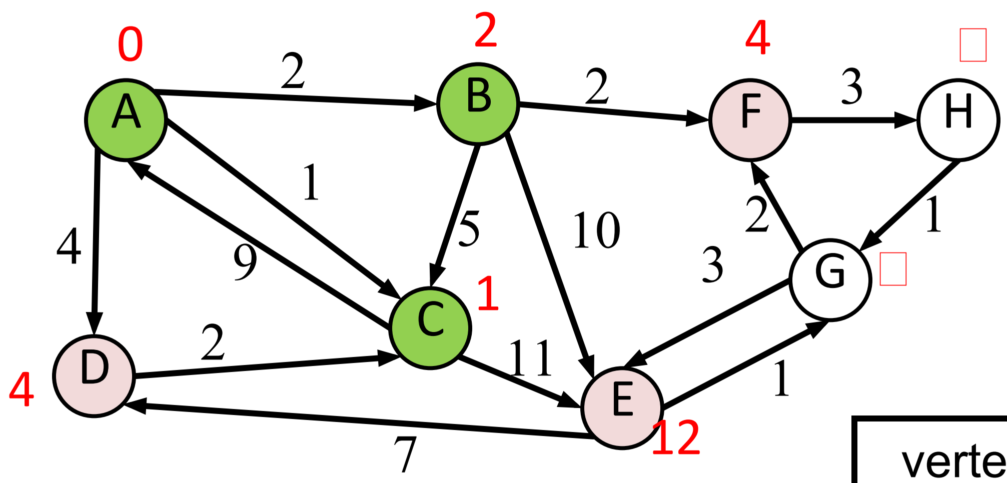


vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		??	
G		??	
H		??	

<u>Order Added to Known Set:</u>

A, C

Example #1

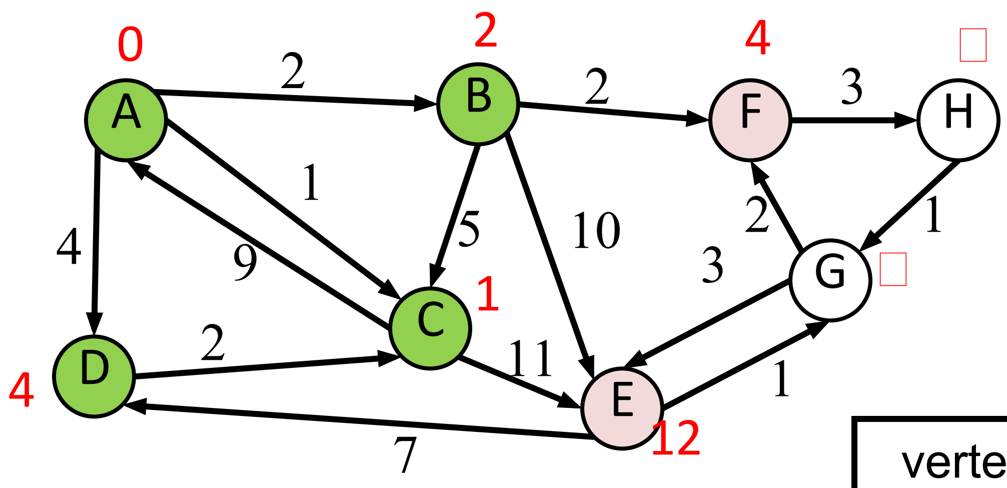


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

Order Added to Known Set:

A, C, B

Example #1

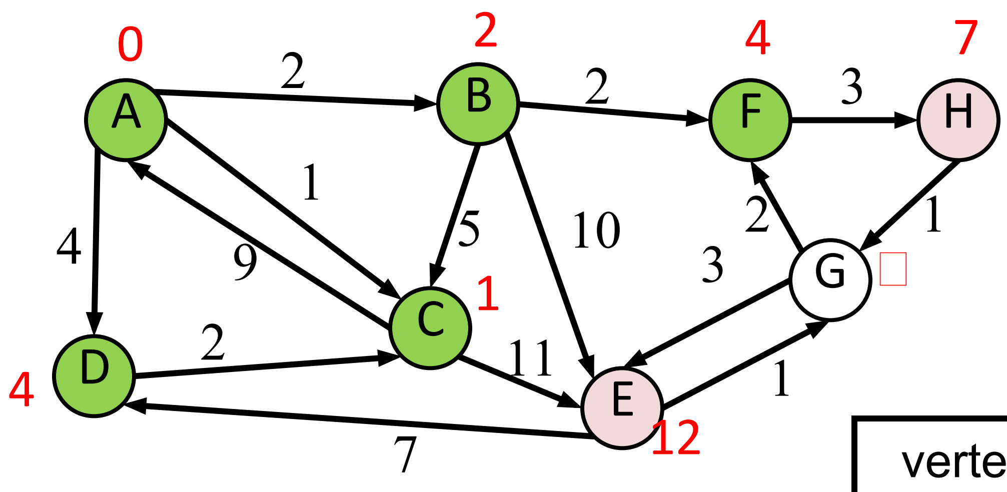


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

Order Added to Known Set:

A, C, B, D

Example #1

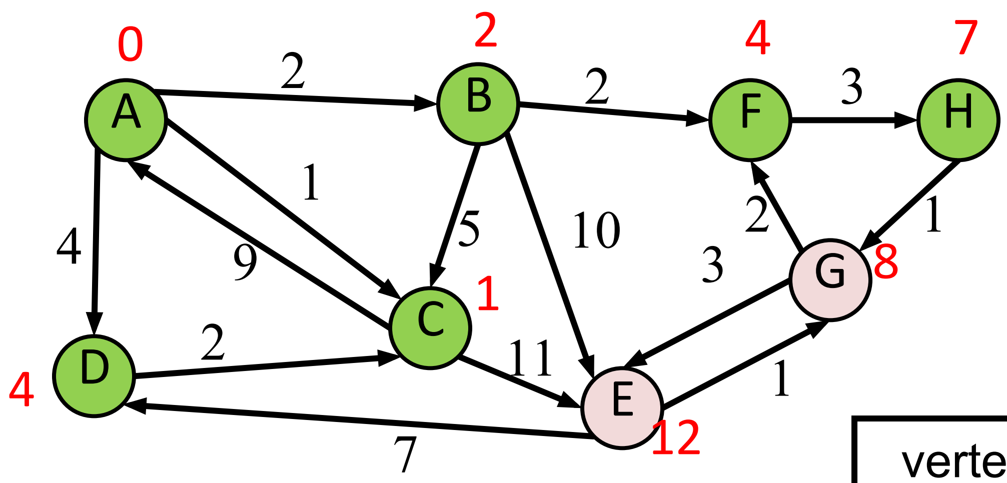


Order Added to Known Set:

A, C, B, D, F

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		??	
H		≤ 7	F

Example #1

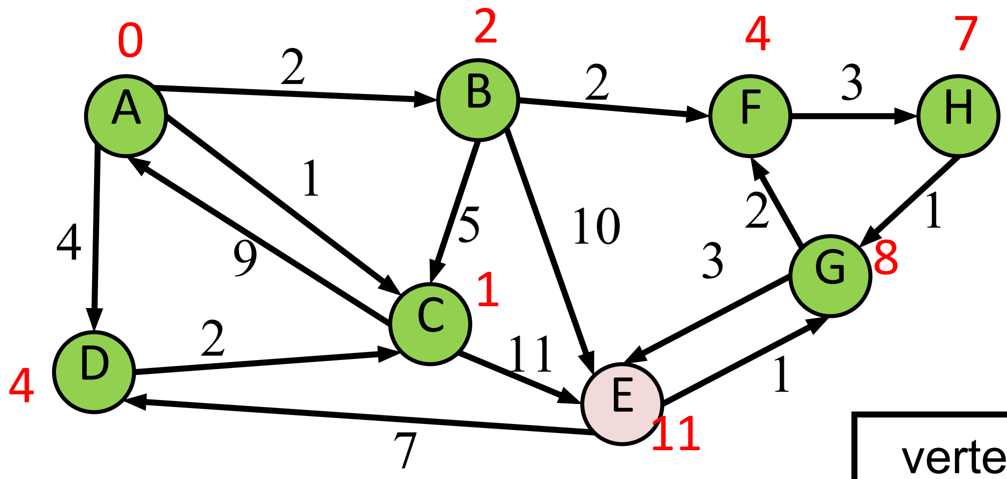


<u>Order Added to Known Set:</u>

A, C, B, D, F, H

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		≤ 8	H
H	Y	7	F

Example #1

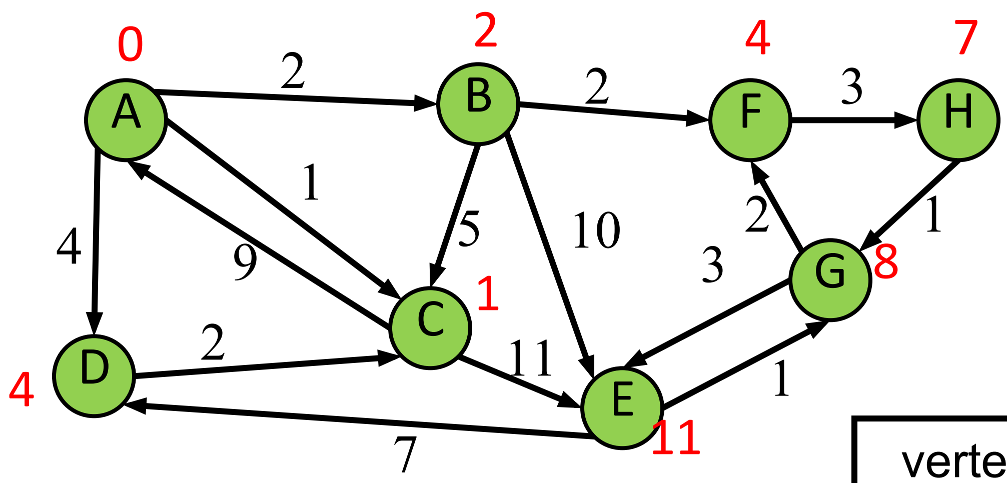


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Order Added to Known Set:

A, C, B, D, F, H, G

Example #1



vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Order Added to Known Set:

A, C, B, D, F, H, G, E

Features

When a vertex is marked known,
the cost of the shortest path to that node is known

- The path is also known by following back-pointers

While a vertex is still not known,
another shorter path to it might still be found

Note: The “Order Added to Known Set” is not important

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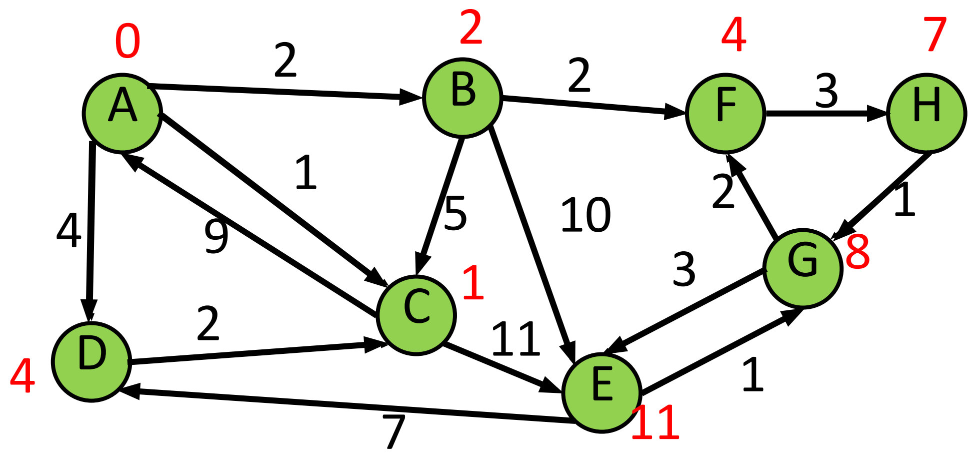
- Not used by the algorithm (implementation doesn’t care)
- It is sorted by path-cost, resolving ties in some way

Interpreting the Results

Now that we're done, how do we get the path from, say, A to E?

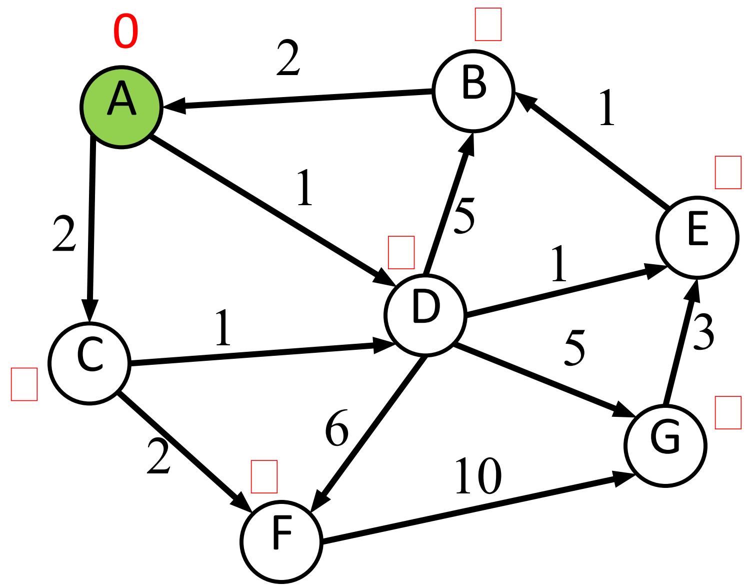


Order Added to Known Set:

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

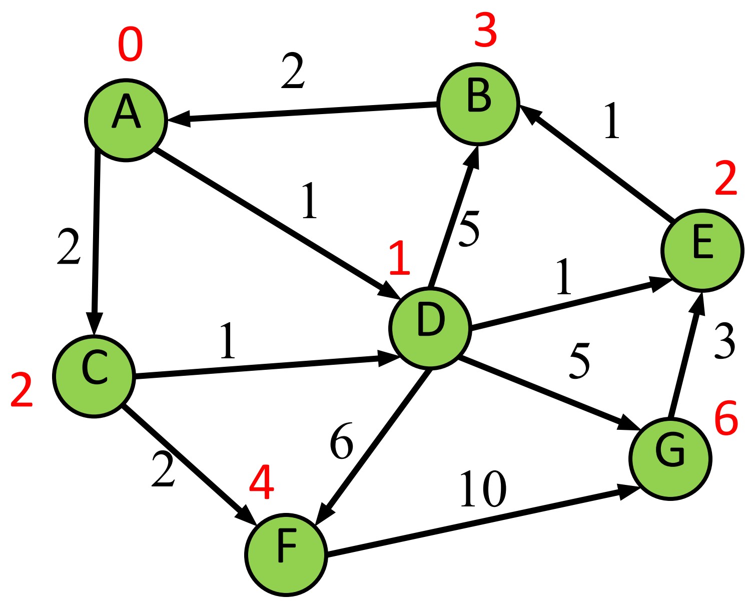
Example #2



Order Added to Known Set:

vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	

Example #2



Order Added to Known Set:

A, D, C, E, B, F, G

vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G	Y	6	D

Efficiency, first approach

Use pseudocode to determine asymptotic run-time

Notice each edge is processed only once

```
dijkstra(Graph G, Node start) {
  for each node: x.cost=infinity, x.known=false
  start.cost = 0
  while(not all nodes are known) {
    b = dequeue
    b.known = true
    for each edge (b,a) in G
     if(!a.known)
       if(b.cost + weight((b,a)) < a.cost){
         a.cost = b.cost + weight((b,a))
         a.path = b
       }
}
```

$O(|V|)$

$O(|V|^2)$

$O(|E|)$

$O(|V|^2)$

Priority Queue

- Increase efficiency by considering lowest cost unknown vertex with sorting instead of looking at all vertices
- PriorityQueue is like a queue, but returns elements by **lowest value** instead of insertion time

Priority Queue

Two different ways to define 'lowest value' for a priority queue:

1. Inserted elements must implement the java Comparable interface.
   a. class Node implements Comparable<Node>
   b. public int compareTo(other)
2. Define a Comparator object and hand it to your priority queue on construction.
   a. class NodeComparator extends Comparator<Node>
   b. new PriorityQueue(new NodeComparator())

Efficiency, second approach

Use pseudo code to determine asymptotic run-time

```
dijkstra(Graph G, Node start) {
  for each node: x.cost=infinity, x.known=false
  start.cost = 0
  build-heap with all nodes
  while(heap is not empty) {
    b = deleteMin()
    if (b.known) continue;
    b.known = true
    for each edge (b,a) in G
     if(!a.known) {
       add(b.cost + weight((b,a)) )
     }
}
```

O(|V|)

O(|V|log|V|)

O(|E|log|V|)

O(|E|log|V|)

Correctness: Intuition

Rough intuition:

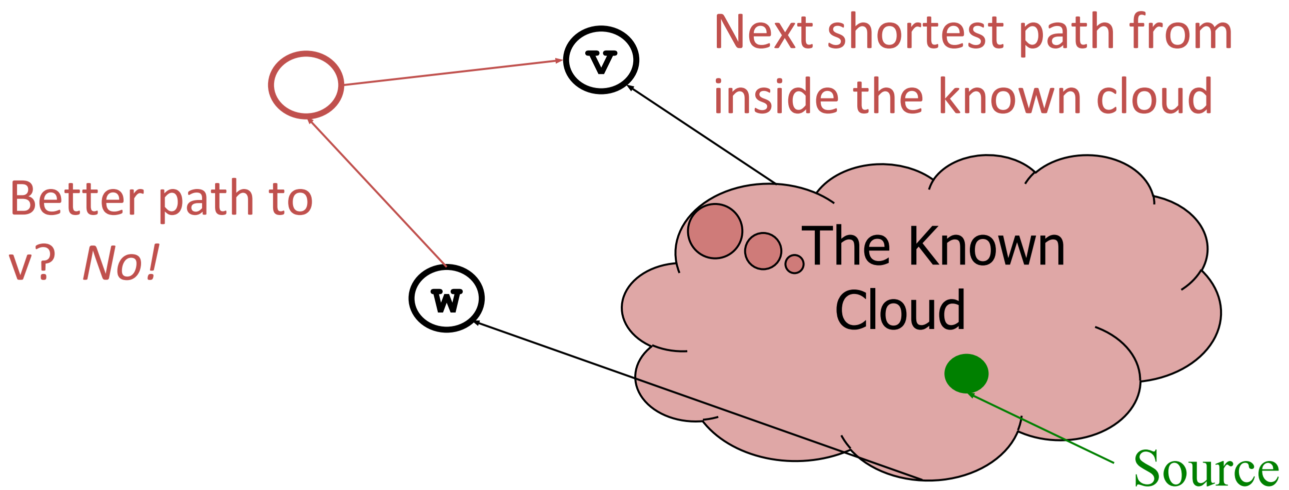
All the “known” vertices have the correct shortest path

- True initially: shortest path to start node has cost 0
- If it stays true every time we mark a node “known”, then by induction this holds and eventually everything is “known”

Key fact we need: When we mark a vertex “known” we won’t discover a shorter path later!

- This holds only because Dijkstra’s algorithm picks the node with the next shortest path-so-far
- The proof is by contradiction…

Correctness: The Cloud (Rough Sketch)



Suppose v is the next node to be marked known ("added to the cloud")

- The best-known path to v must have only nodes "in the cloud"
  - Else we would have picked a node closer to the cloud than v
- Suppose the actual shortest path to v is different
  - It won't use only cloud nodes, or we would know about it
  - So it must use non-cloud nodes. Let w be the *first* non-cloud node on this path. The part of the path up to w is already known and must be shorter than the best-known path to v. So v would not have been picked. Contradiction.

Use in HW

- Will use in HW7 to find paths between characters, weighted so characters that commonly appear together have short paths between them

- Will use in HW8/9 to map distances across campus