Loops and invariants

CSE 331
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Reasoning about loops

- A loop represents an unknown number of paths
  - Case analysis is problematic
  - Recursion presents the same problem as loops
- Cannot enumerate all paths
  - This is what makes testing and reasoning hard

Reasoning about loops: values and termination

```
// assert x ≥ 0 & y = 0
while (x != y) {
    y = y + 1;
}
// assert x = y
```

Does "x=y" hold after this loop?

Does this loop terminate?

1) Pre-assertion guarantees that $x \geq y$

2) Every time through loop

   $x \geq y$ holds at the test – and if body is entered, $x > y$

   y is incremented by 1

   x is unchanged

   Therefore, y is closer to x (but $x \geq y$ still holds)

3) Since there are only a finite number of integers between x and y, y will eventually equal x

4) Execution exits the loop as soon as $x = y$ (but $x \geq y$ still holds)

Understanding loops by induction

- We just made an inductive argument
  - Inducting over the *number of iterations*
- Computation induction
  - Show that conjecture holds if zero iterations
  - Show that it holds after n+1 iterations
    - (assuming that it holds after n iterations)
- Two things to prove
  - Some property is preserved (known as "partial correctness"), if the code terminates
    - Loop invariant is preserved by each iteration, if the iteration completes
  - The loop completes (known as "termination")
    - The "decrementing function" is reduced by each iteration and cannot be reduced forever

How to choose a loop invariant, LI

```
{ P }
while (b) S;
{ Q }
```

Find an invariant, LI, such that

1. $P \Rightarrow LI$ // true initially
2. $\{ LI \;\&\; b \}$ **S** $\{ LI \}$ // true if the loop executes once
3. $(LI \;\&\; \neg b) \Rightarrow Q$ // establishes the postcondition

It is sufficient to know that if loop terminates, Q will hold.

Finding the invariant is the key to reasoning about loops.

Inductive assertions is a "complete method of proof":

- If a loop satisfies pre/post conditions, then there exists an invariant sufficient to prove it

Loop invariant for the example

// assert $x \geq 0$ & $y = 0$

```
while (x != y) {
    y = y + 1;
}
```

// assert $x = y$

A suitable invariant:

LI $= x \geq y$

1. $x \geq 0$ & $y = 0 \Rightarrow$ LI // true initially
2. { LI & $x \neq y$ } `y = y+1;` { LI } // true if the loop executes once
3. (LI & $\neg(x \neq y)$) $\Rightarrow$ $x = y$ // establishes the postcondition

Total correctness via well-ordered sets

Total correctness = partial correctness + termination

We have not established that the loop terminates

Suppose that the loop always reduces some variable's value. Does the loop terminate if the variable is a

- Natural number?
- Integer?
- Non-negative real number?
- Boolean?
- ArrayList?

The loop terminates if the variable values are (a subset of) a well-ordered set

- Ordered set
- Every non-empty subset has least element

Decrementing function

Decrementing function D(X)

- Maps state (program variables) to some well-ordered set
- Tip: always use the natural numbers

This greatly simplifies reasoning about termination

Consider: `while (b) S;`

We seek D(X), where X is the state, such that

1. An execution of the loop reduces the function's value:
   { LI & b } **S** { $D(X_{post}) < D(X_{pre})$ }
2. If the function's value is minimal, the loop terminates:
   (LI & D(X) = minVal) $\Rightarrow \neg$b

Proving termination

// assert $x \geq 0$ & $y = 0$
// Loop invariant: $x \geq y$
// Loop decrements: (x-y)

```
while (x != y) {
    y = y + 1;
}
```

// assert $x = y$

Is this a good decrementing function?

1. Does the loop reduce the decrementing function's value?
   // assert $(y \neq x)$; let $d_{pre} = (x-y)$
   `y = y + 1;`
   // assert $(x_{post} - y_{post}) < d_{pre}$
2. If the function has minimum value, does the loop exit?
   $(x \geq y \;\&\; x - y = 0) \Rightarrow (x = y)$

Choosing loop invariants

For straight-line code, the wp (weakest precondition) function gives us the appropriate property

For loops, you have to guess:

- The loop invariant
- The decrementing function

Then, use reasoning techniques to prove the goal property

If the proof doesn't work:

- Maybe you chose a bad invariant or decrementing function
  - Choose another and try again
- Maybe the loop is incorrect
  - Fix the code

Automatically choosing loop invariants is a research topic

When to use code proofs for loops

Most of your loops need no proofs

```
for (String name : friends) { ... }
```

Write loop invariants and decrementing functions when you are unsure about a loop

If a loop is not working:

- Add invariant and decrementing function if missing
- Write code to check them
- Understand why the code doesn't work
- Reason to ensure that no similar bugs remain

Example: Factorial

{ $n \geq 0 \land t=n$ }

```
r=1;
```

$n \geq 0 \land t=n \land r=1$

```
while (n≠0) {
```

$r=t!/n! \land t \geq n \geq 0$

```
  r=r*n;
```

$r=t!/(n-1)! \land t \geq n > 0$

```
  n=n-1;
}
```

{ $r=t!$ }

Example: Quotient and remainder

```
r := x;
q := 0;
while (y <= r) {
  r := r - y;
  q := 1 + q;
}
```

$\{ x = r + y \times q \text{ and } y > r \}$

$x = x + y \times 0$

$x = r + y \times 0$

$x = r + y \times q$

Example: Greatest common divisor

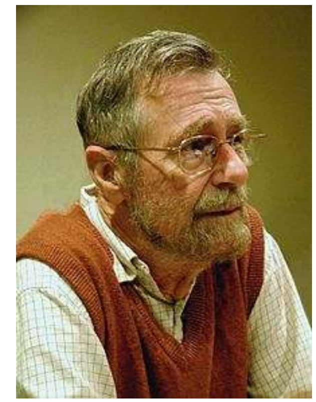
```
{ x1>0 ∧ x2>0 }
y1:=x1;
y2:=x2;
while ¬(y1=y2) do
    if y1>y2 then y1:=y1-y2
              else y2:=y2-y1 fi
od
{ y1=gcd(x1,x2) }
```

Recall: if y1,y2 are both positive integers, then:

- If y1>y2 then gcd(y1,y2)=gcd(y1-y2,y2)
- If y2>y1 then gcd(y1,y2)=gcd(y1,y2-y1)
- If y1-y2 then gcd(y1,y2)=y1=y2

Dutch National Flag

- Given an array containing balls of three colors, arrange them with like colors together and in the right order



- Precondition:

Mixed colors: red, white, blue

- Postcondition:

Red	White	Blue

- Loop invariant:

Red	White	Blue	Mixed

Red	White	Mixed	Blue