Reasoning about code

CSE 331

University of Washington

Reasoning about code

Determine what facts are true during execution

- `x > 0`
- for all nodes **`n`**: **`n.next.previous == n`**
- array **`a`** is sorted
- `x + y == z`
- if **`x != null`**, then **`x.a > x.b`**

Applications:

- Ensure code is correct (via reasoning or testing)
- Understand why code is incorrect

Forward reasoning

You know what is true before running the code

What is true after running the code?

Given a precondition, what is the postcondition?

Example:

```
// precondition:  x is even
x = x + 3;
y = 2x;
x = 5;
// postcondition:  ??
```

Application:

Rep invariant holds before running code

Does it still hold after running code?

Backward reasoning

- You know what you want to be true after running the code
  - What must be true beforehand in order to ensure that?
- Given a postcondition, what is the corresponding precondition?
- Example:
  - // precondition: ??
  - x = x + 3;
  - y = 2x;
  - x = 5;
  - // postcondition: y > x
- Application:
  - (Re-)establish rep invariant at method exit: what <u>requires</u>?
  - Reproduce a bug: what must the input have been?
  - Exploit a bug

SQL injection attack

Server code bug: SQL query constructed using unfiltered user input

```
query = "SELECT * FROM users "
        + "WHERE name='" + userInput + "';";
```

User inputs: **a' or '1'='1**

Result:

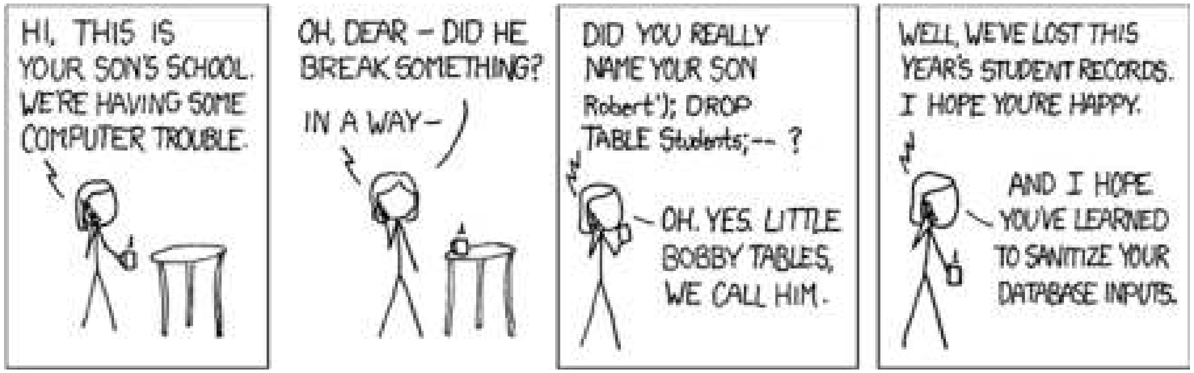
```
query ⇒ SELECT * FROM users
         WHERE name='a' or '1'='1';
```

Query returns information about all users

Program logic is supposed to scrub user inputs

Does it?

http://xkcd.com/327/



Forward vs. backward reasoning

- Forward reasoning is more intuitive for most people
  - Helps you understand what will happen (simulates the code)
  - Introduces facts that may be irrelevant to goal
    - Set of current facts may get large
  - Takes longer to realize that the task is hopeless
- Backward reasoning is usually more helpful
  - Helps you understand what should happen
  - Given a specific goal, indicates how to achieve it
    - Given an error, gives a test case that exposes it

Reasoning about code statements

Goal: Convert assertions about programs into logic

General plan

- Eliminate code a statement at a time
- Rely on knowledge of logic and types

There is a (forward and backward) rule for each statement in the programming language

- Loops have no rule: you have to *guess a loop invariant*

Jargon: P {`code`} Q

- P and Q are logical statements (about program values)
- `code` is Java code
- "P {`code`} Q" means "if P is true and you execute `code`, then Q is true afterward"
- Is this notation good for forward or for backward reasoning?

Reasoning: putting together statements

```
assert x >= 0;
                                    // x ≥ 0
z = 0;
                                    // x ≥ 0  &  z = 0
if (i != 0) {
                                    // x > 0  &  z = 0
   z = x;
                                    // x > 0  &  z = x
} else {
                                    // x = 0  &  z = 0
   z = z + 1;
                                    // x = 0  &  z = 1
}
                                    // x ≥ 0  &  z > 0
assert z > 0;
```

Forward reasoning with a loop

```
assert x >= 0;
                        // x ≥ 0
i = x;
                        // x ≥ 0 & i = x
z = 0;
                        // x ≥ 0 & i = x & z = 0
while (i != 0) {
                        // ???
    z = z + 1;
    i = i - 1;
                        // ???
}
                        // x ≥ 0 & i = 0 & z = x
assert x == z;
```

Infinite number of paths through this code
How do you know that the overall conclusion is correct?
Induction on the length of the computation

Backward reasoning: Assignment

```
// precondition: ??
x = e;
// postcondition: Q
```

Precondition = Q with all (free) occurrences of x replaced by e

Examples:

```
// assert: ??
y = x + 1;
// assert y > 0
```

Precondition = (x+1) > 0

```
// assert: ??
z = z + 1;
// assert z > 0
```

Precondition = (z+1) > 0

Notation: wp for “weakest precondition”

wp(“**x=e**;”, Q) = Q with x replaced by e

Method calls

// precondition: ??
```
x = foo();
```
// postcondition: Q

If the method has no side effects: just like ordinary assignment

// precondition: ?? (y = 22 or y = -22) and (x = anything)
```
x = Math.abs(y);
```
// postcondition: x = 22

If it has side effects: an assignment to every var in <u>modifies</u>

Use the method specification to determine the new value

// precondition: ?? z+1 = 22

incrementZ(); // spec: $z_{post} = z_{pre} + 1$

// postcondition: z = 22

Composition (statement sequences; blocks)

```
// precondition: ??
S1;          // some statement
S2;          // another statement
// postcondition:  Q
```

Work from back to front

Precondition = wp("`S1; S2;`", Q) = wp("`S1;`", wp("`S2;`", Q))

Example:

```
// precondition: ??
x = 0;
y = x+1;
// postcondition: y > 0
```

If statements

// precondition: ??

```
if (b) S1 else S2
```

// postcondition: Q

Do case analysis:

wp(“`if (b) S1 else S2`”, Q)

$= (\quad b \Rightarrow wp(\text{“}\texttt{s1}\text{”}, Q)$

$\quad \wedge\ \neg b \Rightarrow wp(\text{“}\texttt{s2}\text{”}, Q)\ )$

$= (\quad b \wedge wp(\text{“}\texttt{s1}\text{”}, Q)$

$\quad \vee\ \neg b \wedge wp(\text{“}\texttt{s2}\text{”}, Q)$

(Note no substitution in the condition.)

If statement example

```
// precondition: ??
if (x < 5) {
     x = x*x;
} else {
     x = x+1;
}
// postcondition:  x ≥ 9
```

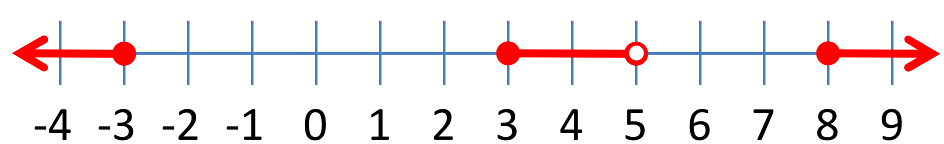
Precondition

$= wp(\text{“}\texttt{if (x<5) \{x = x*x;\} else \{x = x+1\}}\text{”}, x \geq 9)$

$= (x < 5 \land wp(\text{“}\texttt{x=x*x}\text{”}, x \geq 9)) \quad \lor \quad (x \geq 5 \land wp(\text{“}\texttt{x=x+1}\text{”}, x \geq 9))$

$= (\cancel{x} < 5 \land x*x \geq 9) \qquad \lor \quad (x \geq 5 \land x+1 \geq 9)$

$= (x \leq -3) \lor (x \geq 3 \land x < 5) \qquad \lor \quad (x \geq 8)$



Reasoning about loops

- A loop represents an unknown number of paths
  - Case analysis is problematic
  - Recursion presents the same problem as loops
- Cannot enumerate all paths
  - This is what makes testing and reasoning hard

Reasoning about loops: values and termination

```
// assert x ≥ 0 & y = 0
while (x != y) {
    y = y + 1;
}
// assert x = y
```

Does this loop terminate?

1) Pre-assertion guarantees that $x \geq y$
2) Every time through loop
   - $x \geq y$ holds – and if body is entered, $x > y$
   - y is incremented by 1
   - x is unchanged
   - Therefore, y is closer to x (but $x \geq y$ still holds)
3) Since there are only a finite number of integers between x and y, y will eventually equal x
4) Execution exits the loop as soon as $x = y$

Understanding loops by induction

We just made an inductive argument

- Inducting over the number of iterations

Computation induction

- Show that conjecture holds if zero iterations
- Show that it holds after n+1 iterations
  - (assuming that it holds after n iterations)

Two things to prove

- Some property is preserved (known as "partial correctness")
  - Loop invariant is preserved by each iteration
- The loop completes (known as "termination")
  - The "decrementing function" is reduced by each iteration
  - and cannot be reduced forever

How to choose a loop invariant, LI

// assert P
```
while (b) S;
```
// assert Q

Find an invariant, LI, such that

1. $P \Rightarrow LI$ // true initially
2. LI & b {**S**} LI // true if the loop executes once
3. $(LI \,\&\, \neg b) \Rightarrow Q$ // establishes the postcondition

It is sufficient to know that if loop terminates, Q will hold.

Finding the invariant is the key to reasoning about loops.

Inductive assertions is a complete method of proof:

- If a loop satisfies pre/post conditions, then there exists an invariant sufficient to prove it

Loop invariant for the example

```
//assert x ≥ 0 & y = 0
while (x != y) {
    y = y + 1;
}
//assert x = y
```

A suitable invariant:

LI = $x \geq y$

1. $x \geq 0$ & $y = 0 \Rightarrow$ LI // true initially
2. LI & $x \neq y$ {`y = y+1;`} LI // true if the loop executes once
3. (LI & $\neg(x \neq y)$) $\Rightarrow$ $x = y$ // establishes the postcondition

Total correctness via well-ordered sets

- We have not established that the loop terminates
- Suppose that the loop always reduces some variable's value. Does the loop terminate if the variable is a
  - Natural number?
  - Integer?
  - Non-negative real number?
  - Boolean?
  - ArrayList?
- The loop terminates if the variable values are (a subset of) a well-ordered set
  - Ordered set
  - Every non-empty subset has least element

Decrementing function

Decrementing function D(X)

- Maps state (program variables) to some well-ordered set
- Tip: always use the natural numbers

This greatly simplifies reasoning about termination

Consider: `while (b) S;`

We seek D(X), where X is the state, such that

1. An execution of the loop reduces the function's value:
   LI & b {**S**} $D(X_{post}) < D(X_{pre})$
2. If the function's value is minimal, the loop terminates:
   (LI & D(X) = minVal) $\Rightarrow \neg b$

Proving termination

// assert $x \geq 0$ & $y = 0$
// Loop invariant: $x \geq y$
// Loop decrements: (x-y)

```
while (x != y) {
    y = y + 1;
}
```

// assert $x = y$

Is this a good decrementing function?

1. Does the loop reduce the decrementing function's value?
   // assert $(y \neq x)$; let $d_{pre} = (x-y)$
   `y = y + 1;`
   // assert $(x_{post} - y_{post}) < d_{pre}$
2. If the function has minimum value, does the loop exit?
   $(x \geq y \ \& \ x - y = 0) \Rightarrow (x = y)$

Choosing loop invariants

For straight-line code, the wp (weakest precondition) function gives us the appropriate property

For loops, you have to guess:

- The loop invariant
- The decrementing function

Then, use reasoning techniques to prove the goal property

If the proof doesn't work:

- Maybe you chose a bad invariant or decrementing function
  - Choose another and try again
- Maybe the loop is incorrect
  - Fix the code

Automatically choosing loop invariants is a research topic

When to use code proofs for loops

Most of your loops need no proofs

```
for (String name : friends) { ... }
```

Write loop invariants and decrementing functions when you are unsure about a loop

If a loop is not working:

- Add invariant and decrementing function if missing
- Write code to check them
- Understand why the code doesn't work
- Reason to ensure that no similar bugs remain