Today’s Outline

• Announcements
  – Last Homework! Written Homework #8 due Fri March 1

• Today’s Topics:
  – Graphs
    • All-Pairs Shortest Paths

Graphs

• Representations
• Topological Sort
• Finding paths
  – DFS
  – BFS
  – Dijkstra’s
• MST
  – Prim’s
  – Kruskal’s

Single-Source Shortest Path

• Given a graph \( G = (V, E) \) and a single distinguished vertex \( s \), find the shortest weighted path from \( s \) to every other vertex in \( G \).

All-Pairs Shortest Path:
• Find the shortest paths between all pairs of vertices in the graph.
• How?

Analysis

• Total running time for Dijkstra’s:
  \( O(|V|^2 + |E|) \) (linear scan)
  \( O(|V| \log |V| + |E| \log |V|) \) (heaps)

What if we want to find the shortest path from each point to ALL other points?

Dynamic Programming

Algorithmic technique that systematically records the answers to sub-problems in a table and re-uses those recorded results (rather than re-computing them).

Simple Example: Calculating the Nth Fibonacci number.

\[ \text{Fib}(N) = \text{Fib}(N-1) + \text{Fib}(N-2) \]
Floyd-Warshall

for (int k = 1; k <= V; k++)
for (int i = 1; i <= V; i++)
for (int j = 1; j <= V; j++)
    if ( (M[i][k] + M[k][j] < M[i][j])
        M[i][j] = M[i][k] + M[k][j]

Invariant: After the kth iteration, the matrix includes the shortest paths for all pairs of vertices (i, j) containing only vertices 1...k as intermediate vertices

Initial state of the matrix:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0</td>
<td>2</td>
<td>-4</td>
<td>-4</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>-</td>
<td>0</td>
<td>1</td>
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<td>0</td>
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<tr>
<td>c</td>
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<td>d</td>
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<td>4</td>
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<td>e</td>
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<td>-</td>
<td>0</td>
</tr>
</tbody>
</table>

M[i][j] = min(M[i][j], M[i][k] + M[k][j])

Final Matrix Contents

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
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</thead>
<tbody>
<tr>
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<tr>
<td>b</td>
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<tr>
<td>c</td>
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</tbody>
</table>
Transitive Closure

The transitive closure of a graph $G=(V,E)$
Is the graph $G^* = (V, E^*)$ where

$E^* = \{ (i,j) : \text{there is a path from vertex } i \text{ to vertex } j \text{ in } G \}$

“All-pairs reachability”