CSE 326: Data Structures Graphs, Paths & Dijkstra's Algorithm

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Lectures 22-23

Today's Outline

Shortest path algorithms

1. Unweighted graphs: BFS
2. Weighted graphs without negative cost edges: Dijkstra's Algorithm
3. Negative cost edges but no negative cost cycles

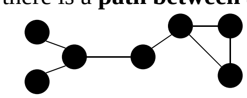
Reading: Weiss, Ch. 9

Graph Traversals

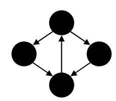
- Breadth-first search (and depth-first search) work for arbitrary (directed or undirected) graphs - not just mazes!
  - Must **mark** visited vertices so you do not go into an infinite loop!
- Either can be used to determine **connectivity**:
  - Is there a path between two given vertices?
  - Is the graph (weakly) connected?
- Which one:
  - Uses a queue?
  - Uses a stack?
  - Always finds the shortest path (for unweighted graphs)?

Graph Connectivity

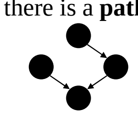
Undirected graphs are *connected* if there is a **path between any two vertices**



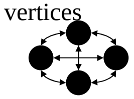
Directed graphs are *strongly connected* if there is a **path from any one vertex to any other**



Directed graphs are *weakly connected* if there is a **path between any two vertices, *ignoring direction***



A *complete* graph has an **edge** between every pair of vertices



The Shortest Path Problem

Given a graph G, edge costs $c_{i,j}$, and vertices s and t in G, find the shortest path from s to t.

For a path $p = v_0\ v_1\ v_2\ \ldots\ v_k$

- *unweighted length* of path $p = k$ (a.k.a. *length*)
- *weighted length* of path $p = \sum_{i=0..k-1} c_{i,i+1}$ (a.k.a *cost*)

Path length equals path cost when ?

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Single Source Shortest Paths (SSSP)

Given a graph G, edge costs $c_{i,j}$, and vertex s, find the shortest paths from s to all vertices in G.

- Is this harder or easier than the previous problem?

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All Pairs Shortest Paths (APSP)

Given a graph G and edge costs $c_{i,j}$, find the shortest paths between all pairs of vertices in G.

- Is this harder or easier than SSSP?
- Could we use SSSP as a subroutine to solve this?

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Variations of SSSP

- Weighted vs. unweighted
- Directed vs undirected
- Cyclic vs. acyclic
- Positive weights only vs. negative weights allowed
- Shortest path vs. longest path
- …

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Applications

– Network routing
– Driving directions
– Cheap flight tickets
– Critical paths in project management (see textbook)
– …

SSSP: Unweighted Version

Ideas?

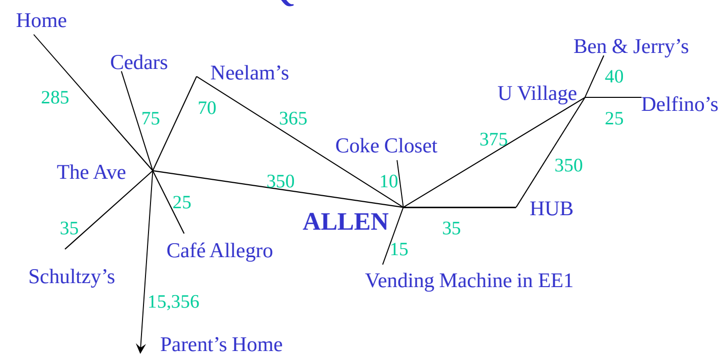

```
void Graph::unweighted (Vertex s){
  // initialize x.dist=INFINITY for every vertex
  Queue q(NUM_VERTICES);
  Vertex v, w;
  q.enqueue(s);
  s.dist = 0;

  while (!q.isEmpty()){
    v = q.dequeue();
    for each w adjacent to v
      if (w.dist == INFINITY){
        w.dist = v.dist + 1;
        w.path = v;
        q.enqueue(w);
      }
  }
}
```

each edge examined at most once – if adjacency lists are used

each vertex enqueued at most once

total running time: O(   )

Weighted SSSP: The Quest For Food



Can we calculate shortest distance to all nodes from Allen Center?

Dijkstra, Edsger Wybe

Legendary figure in computer science; was a professor at Eindhoven, then later at University of Texas.

Supported teaching introductory computer courses without computers (pencil and paper programming)

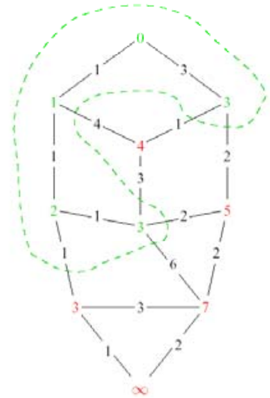
Supposedly wouldn't (until very late in life) read his e-mail; so, his staff had to print out messages and put them in his box.

E.W. Dijkstra (1930-2002)

EWD memos online at utexas - fascinating

1972 Turning Award Winner,
Programming Languages, semaphores, and …

Dijkstra's Algorithm: Idea

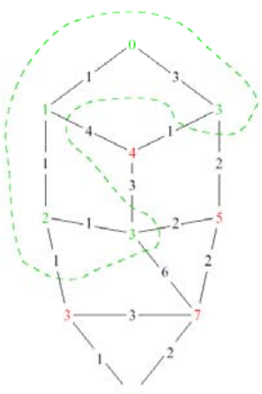


Adapt BFS to handle weighted graphs

Two kinds of vertices:

- Finished or known vertices
  - Shortest distance has been computed
- Unknown vertices
  - Have tentative distance

Dijkstra's Algorithm: Idea



At each step:

1) Pick closest unknown vertex
2) Add it to known vertices
3) Update distances

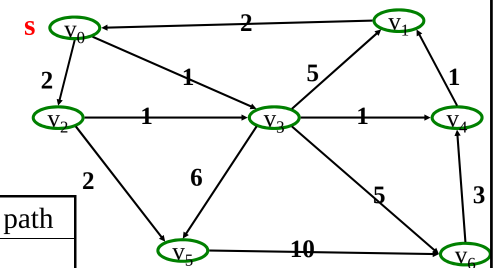
Dijkstra's Algorithm: Pseudocode

Initialize the cost of each node to ∞

Initialize the cost of the source to 0

While there are unknown nodes left in the graph
  Select an unknown node *b* with the lowest cost
  Mark *b* as known
  For each node *a* adjacent to *b*
    a's cost = min(*a*'s old cost, *b*'s cost + cost of (*b*, *a*))

```
void Graph::dijkstra(Vertex s){
  Vertex v,w;

   Initialize s.dist = 0 and set dist of all other
   vertices to infinity

  while (there exist unknown vertices, find the
   one b with the smallest distance)
     b.known = true;

     for each a adjacent to b
       if (!a.known)
         if (b.dist + Cost_ba < a.dist){
           decrease(a.dist to= b.dist + Cost_ba);
           a.path = b;
         }
  }
}
```

V	Known	Dist	path
v0			
v1			
v2			
v3			
v4			
v5			
v6			

Dijkstra's Alg: Implementation

Initialize the cost of each node to ∞

Initialize the cost of the source to 0

While there are unknown nodes left in the graph

- Select the unknown node *b* with the lowest cost
- Mark *b* as known
- For each node *a* adjacent to *b*
  - *a*'s cost = min(*a*'s old cost, *b*'s cost + cost of (*b*, *a*))

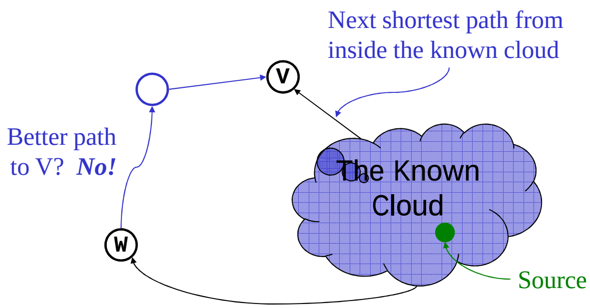
What data structures should we use?

Running time?

Dijkstra's Algorithm: Summary

- Classic algorithm for solving SSSP in weighted graphs *without negative weights*
- A *greedy* algorithm (irrevocably makes decisions without considering future consequences)
- Intuition for correctness:
  - shortest path from source vertex to itself is 0
  - cost of going to adjacent nodes is at most edge weights
  - cheapest of these must be shortest path to that node
  - update paths for new node and continue picking cheapest path

Correctness: The Cloud Proof



How does Dijkstra's decide which vertex to add to the Known set next?

- If path to **V** is shortest, path to **W** must be *at least as long*
  (or else we would have picked **W** *as the next vertex)*
- So the path through **W** to **V** cannot be any shorter!

Correctness: Inside the Cloud

Prove by induction on # of nodes in the cloud:

Initial cloud is just the source with shortest path 0

Assume: Everything inside the cloud has the correct shortest path

Inductive step: Only when we prove the shortest path to some node ***v*** (which is *not* in the cloud) is correct, we add it to the cloud

When does Dijkstra's algorithm not work?

Dijkstra's vs BFS

At each step:

1) Pick closest unknown vertex
2) Add it to finished vertices
3) Update distances

Dijkstra's Algorithm

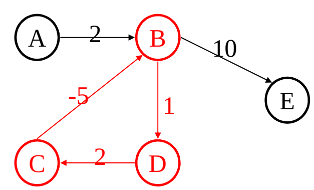
At each step:

1) Pick vertex from queue
2) Add it to visited vertices
3) Update queue with neighbors

Breadth-first Search

Some Similarities:

The Trouble with Negative Weight Cycles



What's the shortest path from A to E?

Problem?