CSE 326: Data Structures
Graphs – Topological Sort

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Lectures 22-23

Agenda

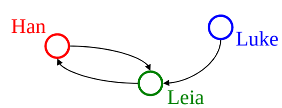
- Basic graph terminology
- Graph representations
- Topological sort

- Reference: Weiss, Ch. 9

Graph… ADT?

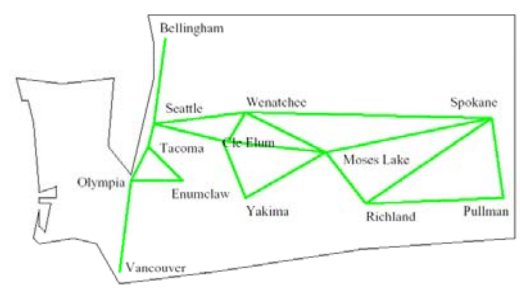
- Not quite an ADT…
  - operations not clear
  - the internal data matters
- A formalism for representing relationships between objects

  Graph `G = (V,E)`
  - *Set of vertices*:
    $V = \{v_1, v_2, \ldots, v_n\}$
  - *Set of edges*:
    $E = \{e_1, e_2, \ldots, e_m\}$
    where each e_i connects two vertices (v_{i1}, v_{i2})




```
V = {Han, Leia, Luke}
E = {(Luke, Leia),
     (Han, Leia),
     (Leia, Han)}
```

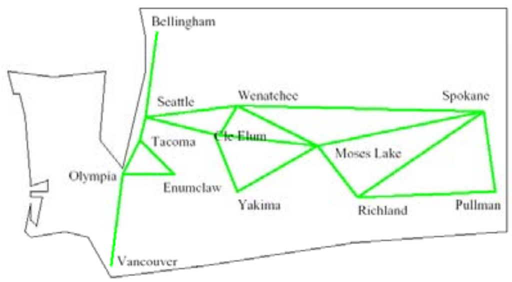

Some Applications: Moving Around Washington



What's the *shortest way* to get from Seattle to Pullman?

Edge labels:

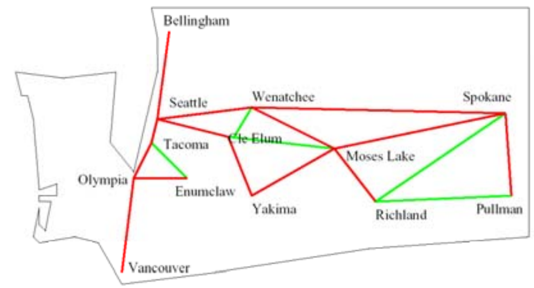
Some Applications:
Moving Around Washington



What's the *fastest way* to get from Seattle to Pullman?

Edge labels:

Some Applications:
Reliability of Communication



If Wenatchee's phone exchange *goes down*,
can Seattle still talk to Pullman?

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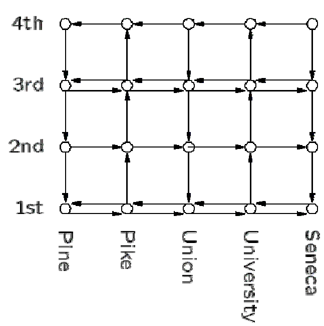
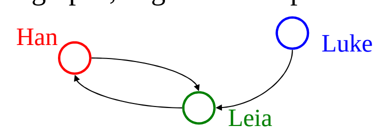
Some Applications:
Bus Routes in Downtown Seattle



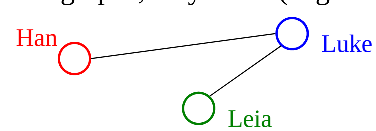
If we're at 3rd and Pine, how can we get to
1st and University using Metro?

Graph Definitions

In *directed* graphs, edges have a specific direction:



In *undirected* graphs, they don't (edges are two-way):



v is *adjacent* to **u** if **(u,v)** $\in$ **E**

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More Definitions: Simple Paths and Cycles

A *simple path* repeats no vertices (except that the first can be the last):

p = {Seattle, Salt Lake City, San Francisco, Dallas}
p = {Seattle, Salt Lake City, Dallas, San Francisco, Seattle}

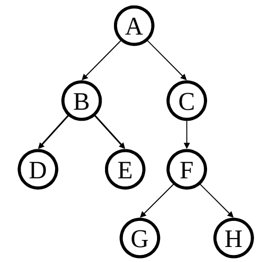
A *cycle* is a path that starts and ends at the same node:

p = {Seattle, Salt Lake City, Dallas, San Francisco, Seattle}
p = {Seattle, Salt Lake City, Seattle, San Francisco, Seattle}

A *simple cycle* is a cycle that repeats no vertices except that the first vertex is also the last (in undirected graphs, no edge can be repeated)

Trees as Graphs

- Every tree is a graph!
- Not all graphs are trees!

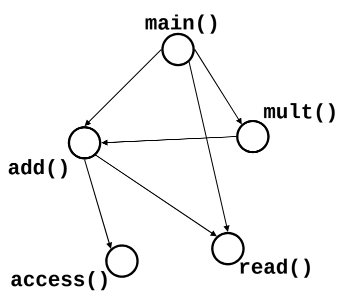


A graph is a tree if

- There are *no cycles* (directed or undirected)
- There is a *path* from the root *to every node*

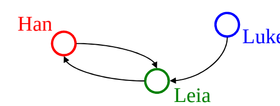
Directed Acyclic Graphs (DAGs)

DAGs are directed graphs with no (directed) cycles.



Aside: *If program call-graph is a DAG, then all procedure calls can be in-lined*

Graph Representations



0. List of vertices + list of edges
1. 2-D matrix of vertices (marking edges in the cells)
   "adjacency matrix"
2. List of vertices each with a list of adjacent vertices
   "adjacency list"

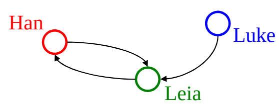
Vertices and edges may be labeled

Things we might want to do:

- iterate over vertices
- iterate over edges
- iterate over vertices adj. to a vertex
- check whether an edge exists

Representation 1: Adjacency Matrix

A **|V| x |V|** array in which an element **(u,v)** is true if and only if there is an edge from **u** to **v**



	Han	Luke	Leia
Han			
Luke			
Leia			

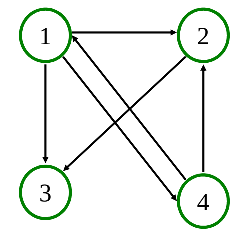
space requirements: *runtime:*

Weighted Edges

- adjacency **matrix**:

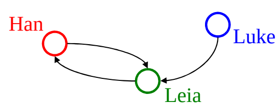
$$A[u][v] = \begin{cases} \text{weight} & \text{, if } (u, v) \in E \\ 0 & \text{, if } (u, v) \notin E \end{cases}$$

	1	**2**	**3**	**4**
1				
2				
3				
4				



Representation 2: Adjacency List

A **|V|**-ary list (array) in which each entry stores a list (linked list) of all adjacent vertices

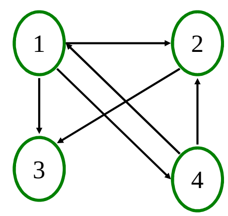


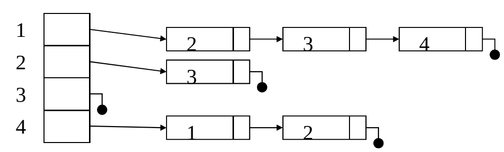
Han	
Luke	
Leia	

space requirements: *runtime:*

Representation

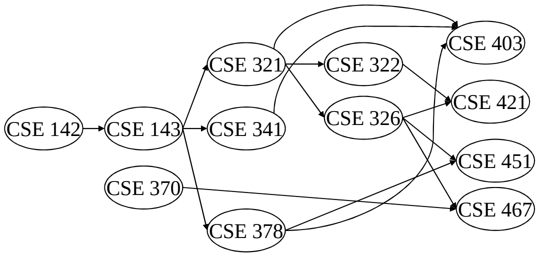
- adjacency **list:**





Application: Topological Sort

Given a directed graph, **`G = (V,E)`**, output all the vertices in **`V`** such that no vertex is output before any other vertex with an edge to it.



Is the output unique?

Topological Sort: Take One

1. Label each vertex with its *in-degree* (# of inbound edges)
2. **While** there are vertices remaining:
   a. Choose a vertex *v* of *in-degree zero*; output *v*
   b. Reduce the in-degree of all vertices adjacent to *v*
   c. Remove *v* from the list of vertices

Runtime:

```
void Graph::topsort(){
  Vertex v, w;

  labelEachVertexWithItsIn-degree();

  for (int counter=0; counter < NUM_VERTICES;
                                  counter++){
    v = findNewVertexOfDegreeZero();

    v.topologicalNum = counter;
    for each w adjacent to v
      w.indegree--;
  }
}
```

Topological Sort: Take Two

1. Label each vertex with its in-degree
2. Initialize a queue *Q* to contain all in-degree zero vertices
3. While *Q* not empty
   a. *v* = *Q*.dequeue; output *v*
   b. Reduce the in-degree of all vertices adjacent to *v*
   c. If new in-degree of any such vertex *u* is zero *Q*.enqueue(*u*)

Note: could use a stack, list, set, box, … instead of a queue

Runtime:

```
void Graph::topsort(){
  Queue q(NUM_VERTICES);  int counter = 0; Vertex v, w;
  labelEachVertexWithItsIn-degree();

  q.makeEmpty();
  for each vertex v
    if (v.indegree == 0)
      q.enqueue(v);

  while (!q.isEmpty()){
    v = q.dequeue();
    v.topologicalNum = ++counter;
    for each w adjacent to v
      if (--w.indegree == 0)
        q.enqueue(w);
  }
}
```

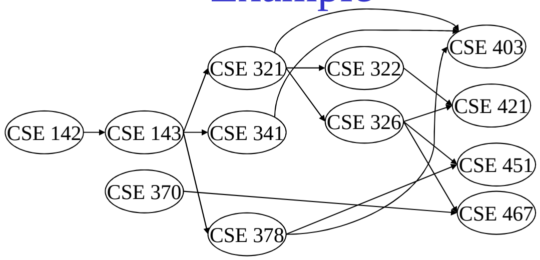
intialize the queue

get a vertex with indegree 0

insert new eligible vertices

Runtime:

Example



Q:

Output: