CSE 326: Data Structures
Binary Search Trees

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Lecture 10

Today's Outline

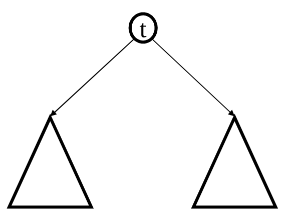
- Quick Tree Review
- Binary Trees
- **Dictionary ADT / Search ADT**
- **Binary Search Trees**

- **Reading: Weiss ch. 4**

Tree Calculations

Recall: height is max number of edges from root to a leaf

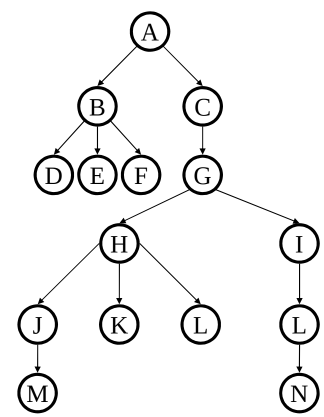
Find the height of the tree...



runtime:

Tree Calculations Example

How high is this tree?

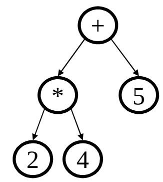


More Recursive Tree Calculations: Tree Traversals

A *traversal* is an order for visiting all the nodes of a tree

Three types:

- Pre-order: Root, left subtree, right subtree
- In-order: Left subtree, root, right subtree
- Post-order: Left subtree, right subtree, root



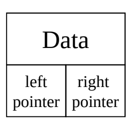
(an expression tree)

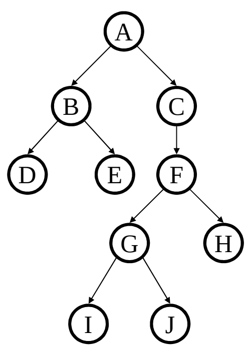
Inorder Traversal

```
void traverse(BNode t){
  if (t != NULL)
    traverse (t.left);
    process t.element;
    traverse (t.right);
  }
}
```

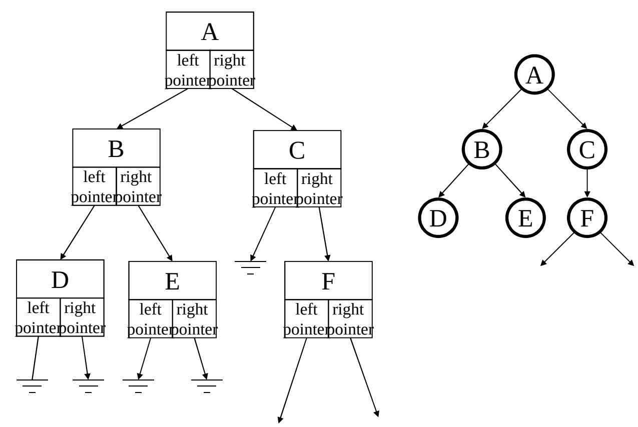

Binary Trees

- Binary tree is
  - a root
  - left subtree *(maybe empty)*
  - right subtree *(maybe empty)*
- Representation:

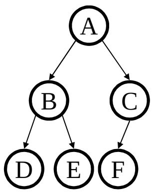




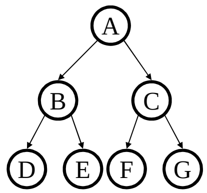
Binary Tree: Representation



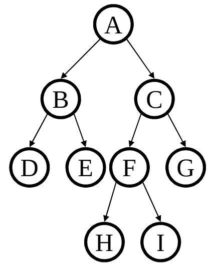
Binary Tree: Special Cases



Complete Tree



Perfect Tree



Full Tree

Binary Tree: Some Numbers!

For binary tree of height *h*:

- max # of leaves:
- max # of nodes:
- min # of leaves:
- min # of nodes:

ADTs Seen So Far

- Stack
  - Push
  - Pop
- Queue
  - Enqueue
  - Dequeue
- Priority Queue
  - Insert
  - DeleteMin

Then there is decreaseKey…

The Dictionary ADT

- Data:
  - a set of (key, value) pairs
- Operations:
  - Insert (key, value)
  - Find (key)
  - Remove (key)

insert(perkins, ….)

find(tuite)

- tuite
  Kathleen Tuite, …

- perkins
  Hal Perkins
  OH: MW 3:40
  CSE 360
- tuite
  Kathleen Tuite,
  OH: W 10:30, F 12:00
  CSE 220
- ray
  Ray Smith,
  OH: T 2:30, F 12:00
  CSE 220

*The Dictionary ADT is also called the "**Map ADT**"*

A Modest Few Uses

- Sets
- Dictionaries
- Networks : Router tables
- Operating systems : Page tables
- Compilers : Symbol tables

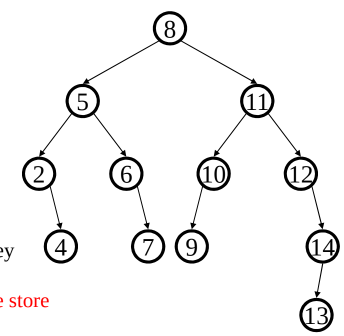
Probably the most widely used ADT!

Implementations

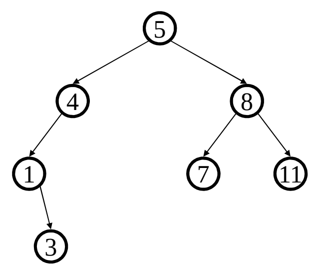
	insert	find	delete
• Unsorted Linked-list			
• Unsorted array			
• Sorted array			

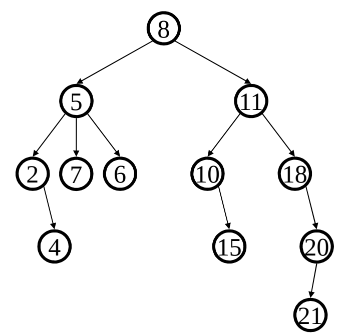
Binary Search Tree Data Structure

- Structural property
  - each node has ≤ 2 children
  - result:
    - storage is small
    - operations are simple
    - average depth is small
- Order property
  - all keys in left subtree smaller than root's key
  - all keys in right subtree larger than root's key
  - result: easy to find any given key
- What must we know about what we store in the tree??



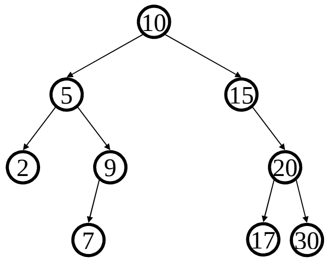
Example and Counter-Example





BINARY SEARCH TREES?

Find in BST, Recursive

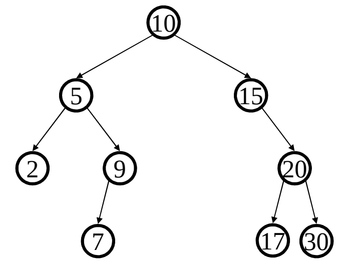



```
Node Find(Object key,
            Node root) {
  if (root == NULL)
    return NULL;

  if (key < root.key)
    return Find(key,
                root.left);
  else if (key > root.key)
    return Find(key,
                root.right);
  else
    return root;
}
```

Runtime:

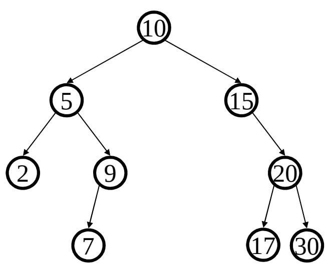
Find in BST, Iterative

```
Node Find(Object key,
            Node root) {

  while (root != NULL &&
         root.key != key) {
    if (key < root.key)
      root = root.left;
    else
      root = root.right;
  }

  return root;
}
```



Runtime:

Insert in BST



Insert(13)
Insert(8)
Insert(31)

Insertions happen only at the leaves – easy!

Runtime:

BuildTree for BST

- Suppose keys 1, 2, 3, 4, 5, 6, 7, 8, 9 are inserted into an initially empty BST.

  Runtime depends on the order!

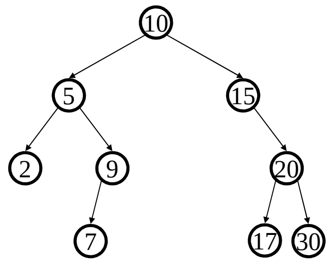
  – in given order

  – in reverse order

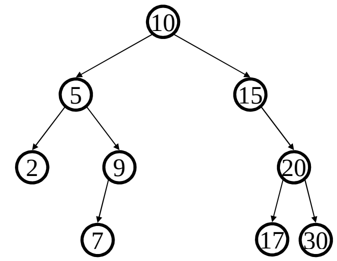
  – median first, then left median, right median, etc.

Bonus: FindMin/FindMax

- Find minimum
- Find maximum



Deletion in BST



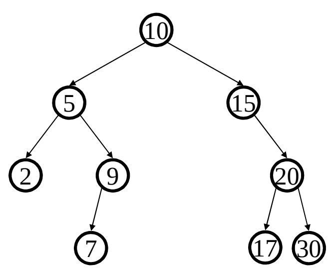
Why might deletion be harder than insertion?

Lazy Deletion

Instead of physically deleting nodes, just mark them as deleted

+ simpler
+ physical deletions done in batches
+ some adds just flip deleted flag

– extra memory for “deleted” flag
– many lazy deletions = slow finds
– some operations may have to be modified (e.g., min and max)

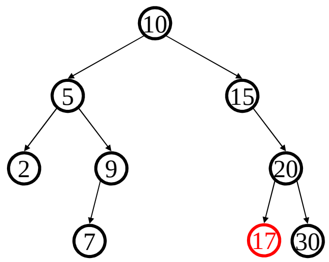


Non-lazy Deletion

- Removing an item disrupts the tree structure.
- Basic idea: find the node that is to be removed. Then “fix” the tree so that it is still a binary search tree.
- Three cases:
  – node has no children (leaf node)
  – node has one child
  – node has two children

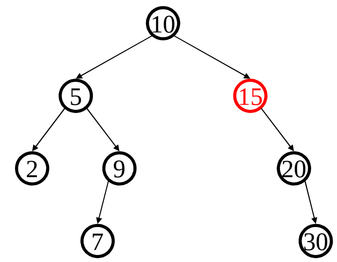
Non-lazy Deletion – The Leaf Case

Delete(17)



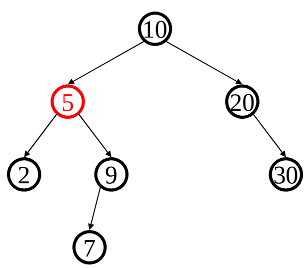
Deletion – The One Child Case

Delete(15)



Deletion – The Two Child Case

Delete(5)



What can we replace 5 with?

Deletion – The Two Child Case

Idea: Replace the deleted node with a value guaranteed to be between the two child subtrees

Options:

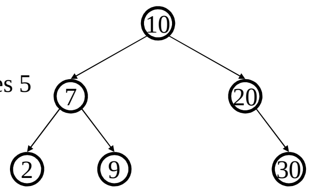
- *succ* from right subtree: findMin(t.right)
- *pred* from left subtree : findMax(t.left)

Now delete the original node containing *succ* or *pred*

- Leaf or one child case – easy!

Finally…

7 replaces 5



Original node containing
7 gets deleted

Balanced BST

Observation
- BST: the shallower the better!
- For a BST with *n* nodes
  - Average height is O(log *n*)
  - Worst case height is O(*n*)
- Simple cases such as insert(1, 2, 3, ..., n) lead to the worst case scenario

Solution: Require a **Balance Condition** that
1. ensures depth is O(log *n*) – strong enough!
2. is easy to maintain – not too strong!

Potential Balance Conditions

1. Left and right subtrees of the root have equal number of nodes

2. Left and right subtrees of the root have equal *height*

Potential Balance Conditions

3. Left and right subtrees of *every node* have equal number of nodes

4. Left and right subtrees of *every node* have equal *height*

The AVL Balance Condition

Left and right subtrees of *every node*
have equal *heights* **differing by at most 1**

Define: **balance**(x) = height(x.left) – height(x.right)

AVL property: **–1 ≤ balance(x) ≤ 1, for every node x**

- Ensures small depth
  - Will prove this by showing that an AVL tree of height h must have a lot of (i.e. $O(2^h)$) nodes
- Easy to maintain
  - Using single and double rotations

The AVL Tree Data Structure

Structural properties

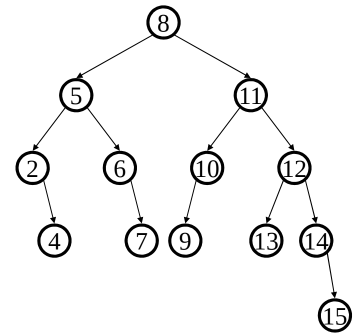
1. Binary tree property
2. Balance property: balance of every node is between -1 and 1

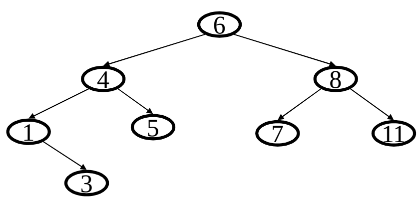
Result:

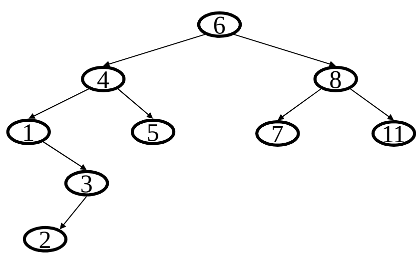
Worst case depth is $O(\log n)$

Ordering property

– Same as for BST







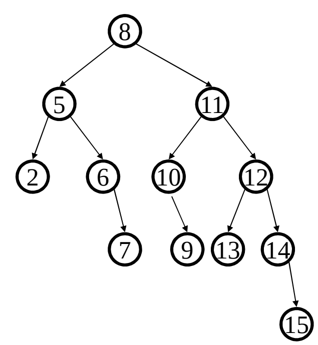
Proving Shallowness Bound

Let **S**(h) be the min # of nodes in an AVL tree of height h

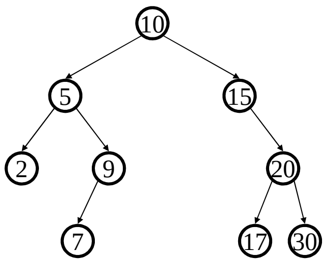
Claim: **S**(h) = **S**(h-1) + **S**(h-2) + 1

Solution of recurrence: **S**(h) = $O(2^h)$
(like Fibonacci numbers)

AVL tree of height h=4
with the min # of nodes



Testing the Balance Property



We need to be able to:

1.

2.

3.

`NULL`s have height **`-1`**

An AVL Tree