

CSE 326: Data Structures

AVL Trees

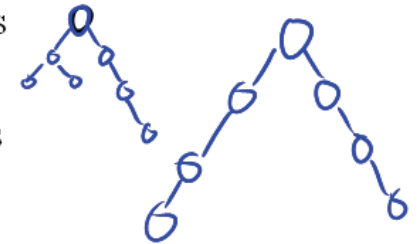
Brian Curless
Spring 2008

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Potential Balance Conditions

What are some candidate balance conditions on BST's?

1. Left and right subtrees of the root have equal number of nodes



2. Left and right subtrees of the root have equal *height*

conditions too weak!

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Potential Balance Conditions

3. Left and right subtrees of *every node* have equal number of nodes

perfect trees

4. Left and right subtrees of *every node* have equal *height*

conditions too strong!

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Balancing Trees

- Many algorithms exist for keeping trees balanced
 - Adelson-Velskii and Landis (AVL) trees
 - Splay trees and other *self-adjusting* trees
 - B-trees and other multiway search trees (for very large trees)
- Today we will talk about **AVL trees**...

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The AVL Tree Data Structure

Ordering property

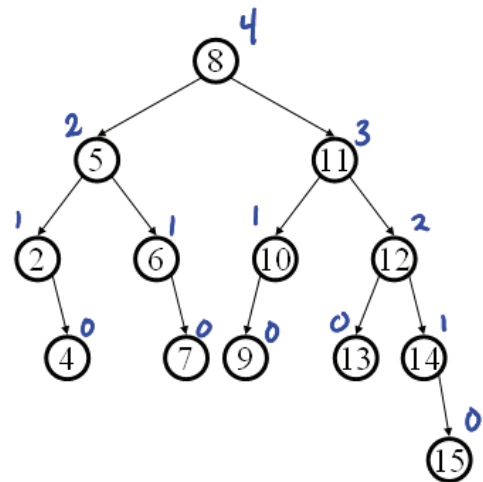
- Same as for BST

Structural properties

1. Binary tree property (0,1, or 2 children)
2. Heights of left and right subtrees of *every node* differ by at most 1

Result:

Worst case depth of any node is: $O(\log n)$



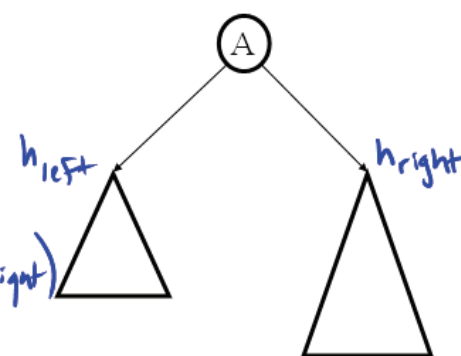
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Recursive Height Calculation

Recall: height is max number of edges from root to a leaf

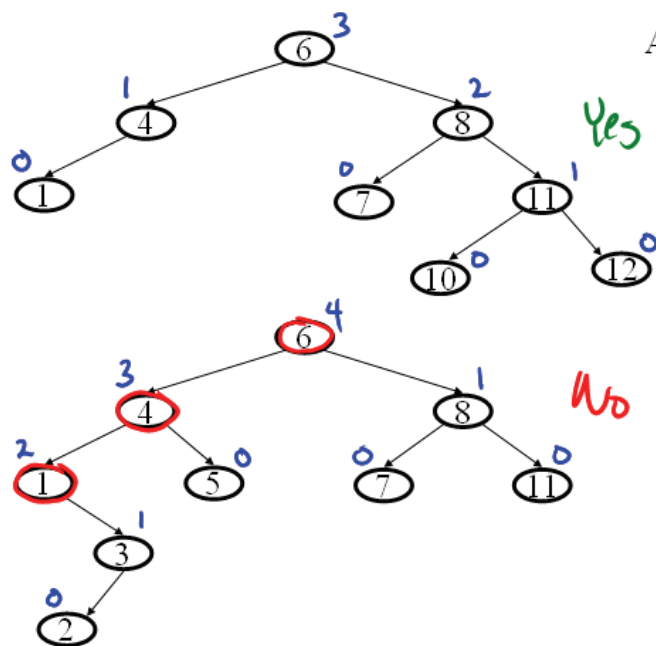
What is the height at A?

$$\text{height}(A) = 1 + \max(h_{\text{left}}, h_{\text{right}})$$



Note: $\text{height}(\text{null}) = -1$

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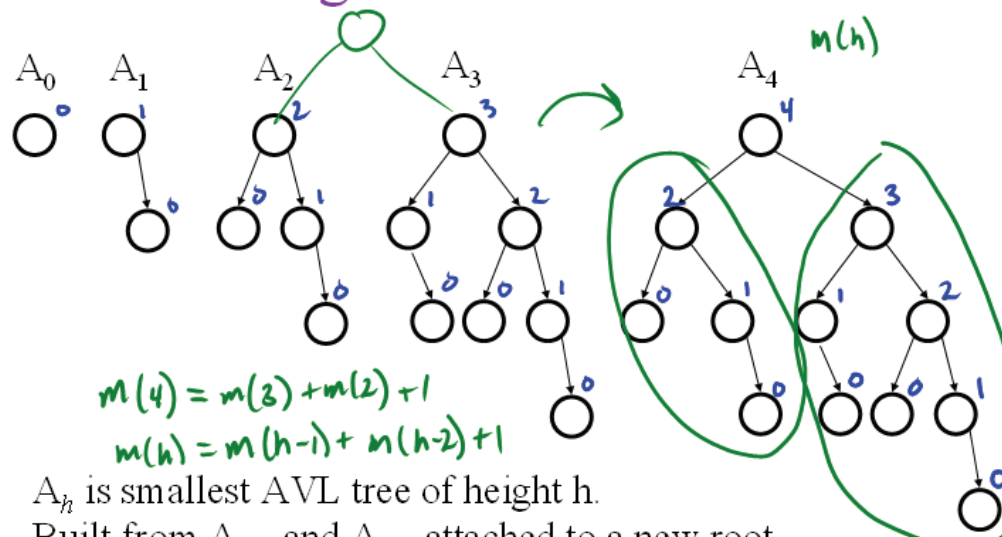
AVL trees or not?

Yes

No

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Proving Shallowness Bound



$$m(4) = m(2) + m(2) + 1$$

$$m(h) = m(h-1) + m(h-2) + 1$$

A_h is smallest AVL tree of height h .

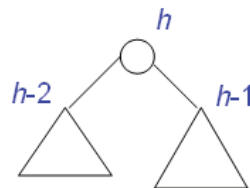
Built from A_{h-1} and A_{h-2} attached to a new root.

(Note: these A_h 's are not unique; e.g., could swap children.)

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Minimum Size of an AVL Tree

- $m(h)$ = minimum number of nodes in an AVL tree of height h .
- Base cases
 - $m(0) = 1, m(1) = 2$
- Induction
 - $m(h) = m(h-1) + m(h-2) + 1$
- Bound solution
 - $m(h) > \phi^h - 1$
 - ϕ is the golden ratio, $(1+\sqrt{5})/2$



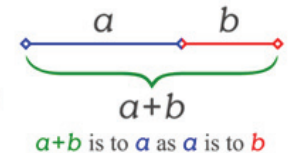
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The Golden Ratio

Since the Renaissance, many artists and architects have proportioned their work (e.g., length:height) to approximate the golden ratio:

$$\frac{a+b}{a} = \frac{a}{b} = \phi$$

The golden section:



Set $a=1$, solve for positive b , compute ratio: $\phi = \frac{1+\sqrt{5}}{2} \approx 1.62$

Note: $\phi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2} + \frac{2}{2}$

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Proof that $m(h) > \phi^h - 1$

- Base cases $h=0,1$:

$$m(0) = 1 > \phi^0 - 1 = 0$$

$$m(1) = 2 > \phi^1 - 1 \approx 0.62$$

- Assume true for $h-2$ and $h-1$:

$$m(h-2) > \phi^{h-2} - 1$$

$$m(h-1) > \phi^{h-1} - 1$$

- Induction step:

$$m(h) = m(h-1) + m(h-2) + 1 > (\phi^{h-1} - 1) + (\phi^{h-2} - 1) + 1$$

$$\begin{aligned} (\phi^{h-1} - 1) + (\phi^{h-2} - 1) + 1 &= \phi^{h-2} (\phi + 1) - 1 \\ &= \phi^{h-2} (\phi^2) - 1 \\ &= \phi^h - 1 \end{aligned}$$

$$\rightarrow m(h) > \phi^h - 1$$

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Maximum Height of an AVL Tree

Suppose we have n nodes in an AVL tree of height h .

We can now say:

$$n \geq m(h) > \phi^h - 1$$

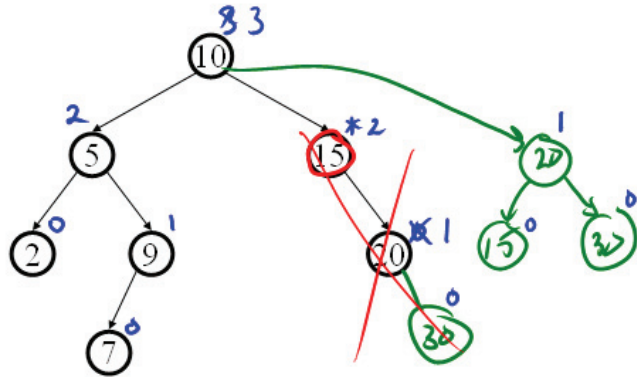
What does this say about the complexity of h ?

$$\begin{aligned} n+1 &> \phi^h \\ \log_{\phi}(n+1) &> h \\ h &< \log_{\phi}(n+1) \\ h &\in O(\log n) \end{aligned}$$

$$\begin{aligned} n &\leq 2^{h+1} - 1 \\ (n+1) &\leq 2^{h+1} \\ \log(n+1) &\leq h+1 \\ h &\geq \log(n+1) - 1 \\ h &\in \Omega(\log n) \end{aligned}$$

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Testing the Balance Property



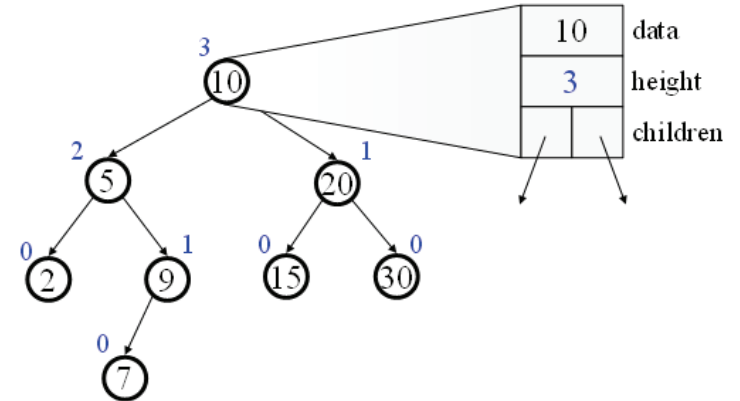
We need to be able to:

1. Track Balance
2. Detect Imbalance
3. Restore Balance

What if we insert(30)?

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An AVL Tree



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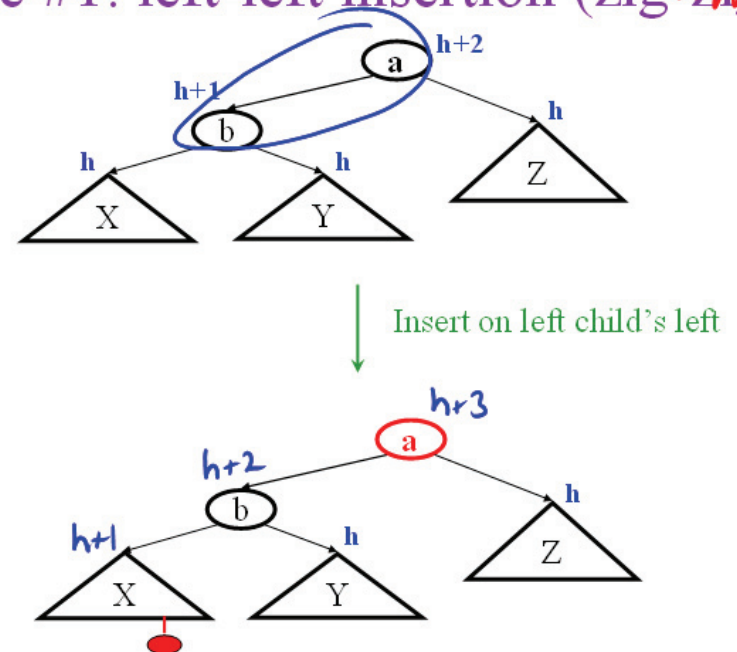
AVL trees: find, insert

- **AVL find:**
 - same as BST find.
- **AVL insert:**
 - same as BST insert, *except* may need to “fix” the AVL tree after inserting new value.

We will consider the 4 fundamental insertion cases...

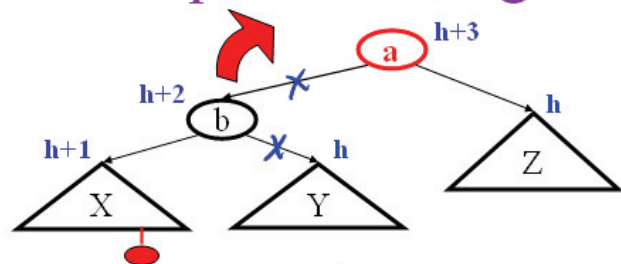
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Case #1: left-left insertion (zig-zag)



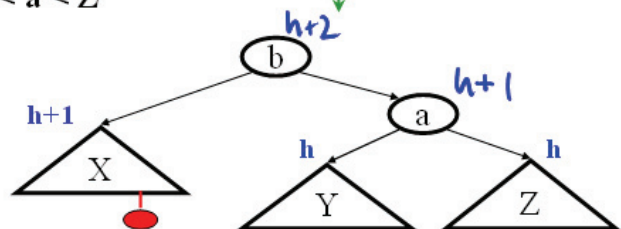
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Case #1: repair with single rotation



$X < b < Y < a < Z$

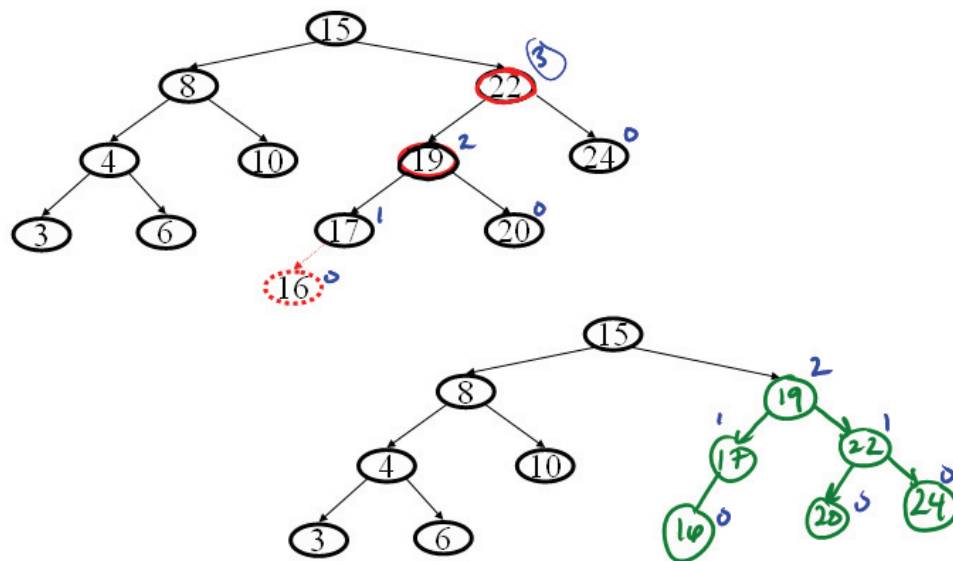
Single rotation



Height of tree before/after? $h+2 \rightarrow$ no change
Effect on Ancestors? None
Cost? $O(1)$

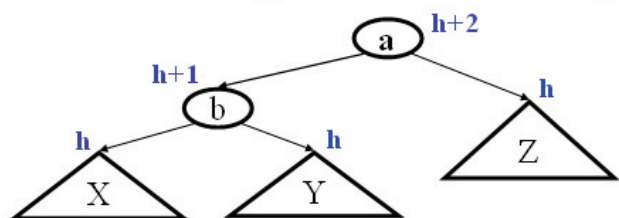
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Single rotation example

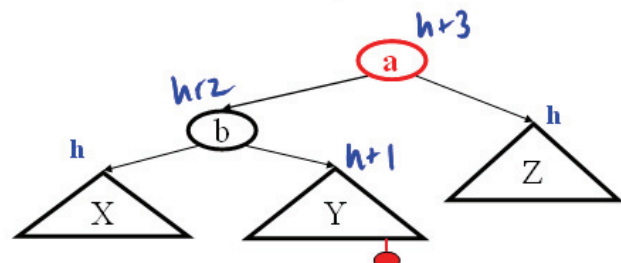


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Case #2: left-right insertion (zig-zag)

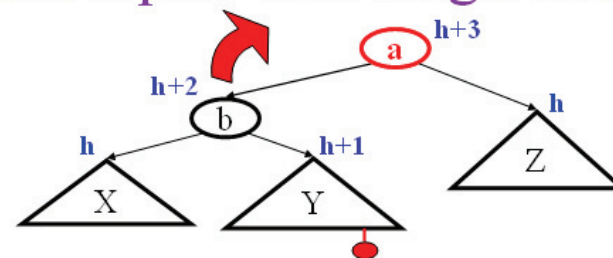


Insert on left child's right



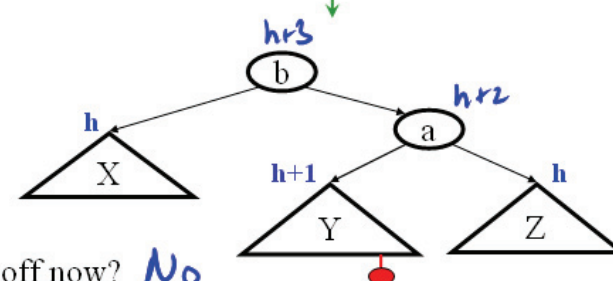
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Case #2: repair with single rotation



$X < b < Y < a < Z$

Single rotation

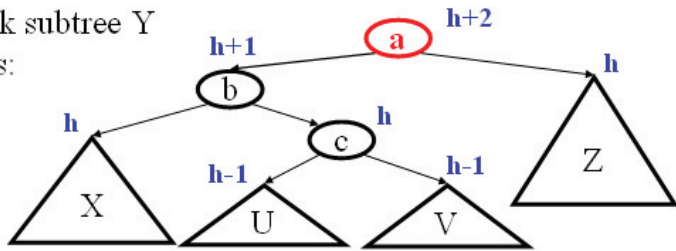


Are we better off now? No

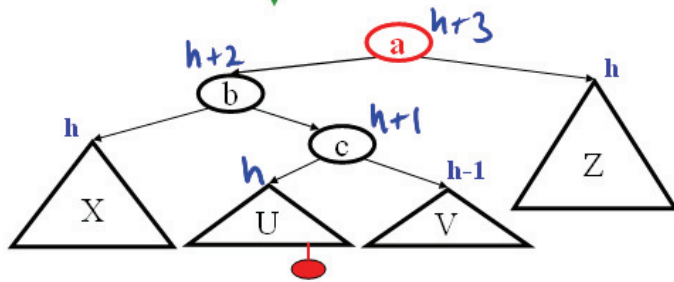
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Case #2: trying again

Let's break subtree Y into pieces:

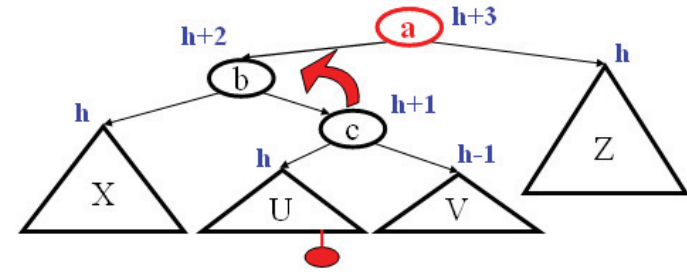


Insert on left child's right (at U or V)

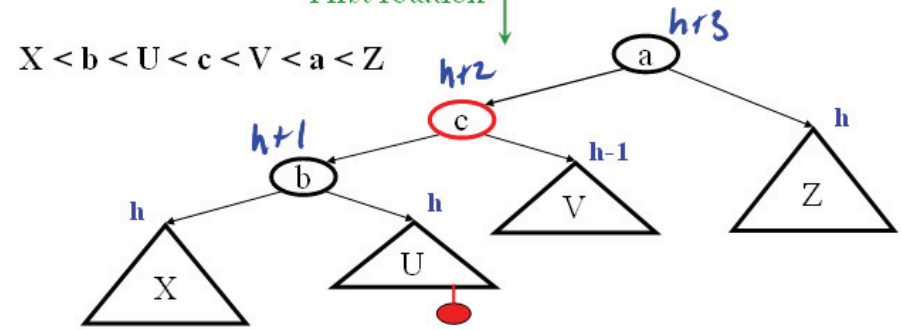


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Case #2: first rotation

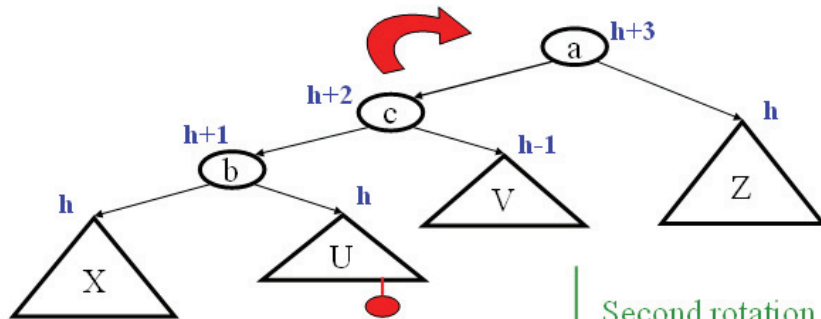


First rotation

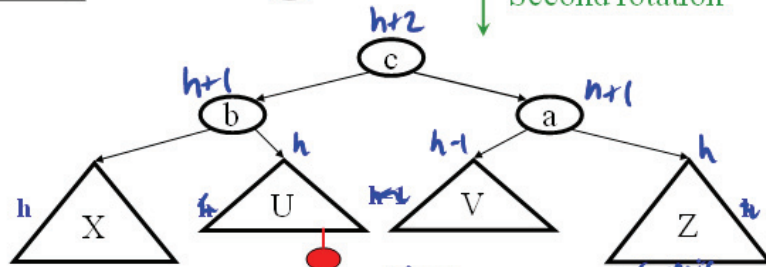


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Case #2: second rotation



Second rotation



No change

Height of tree before/after?

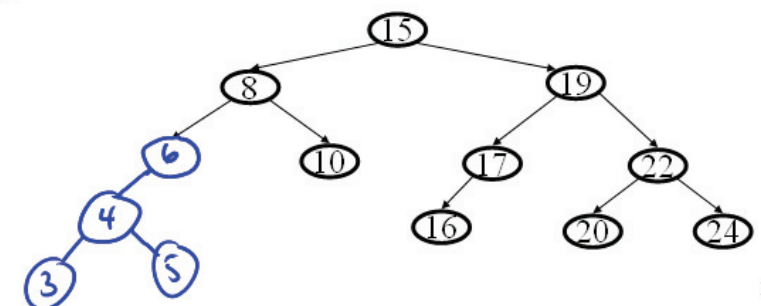
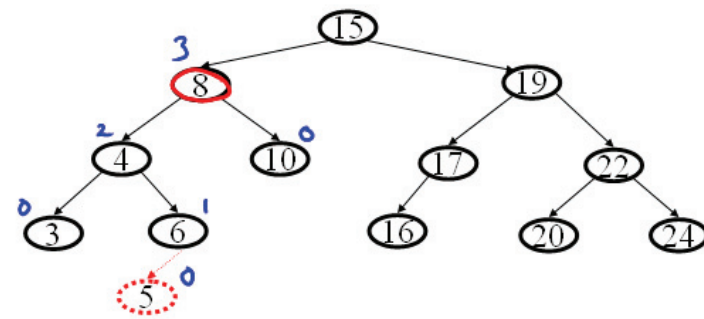
None

Effect on Ancestors?

Cost? $O(1)$

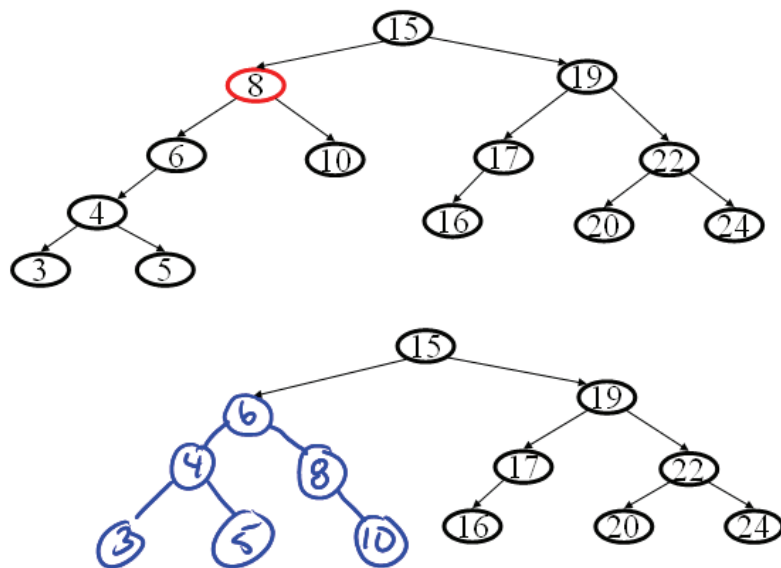
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Double rotation, step 1



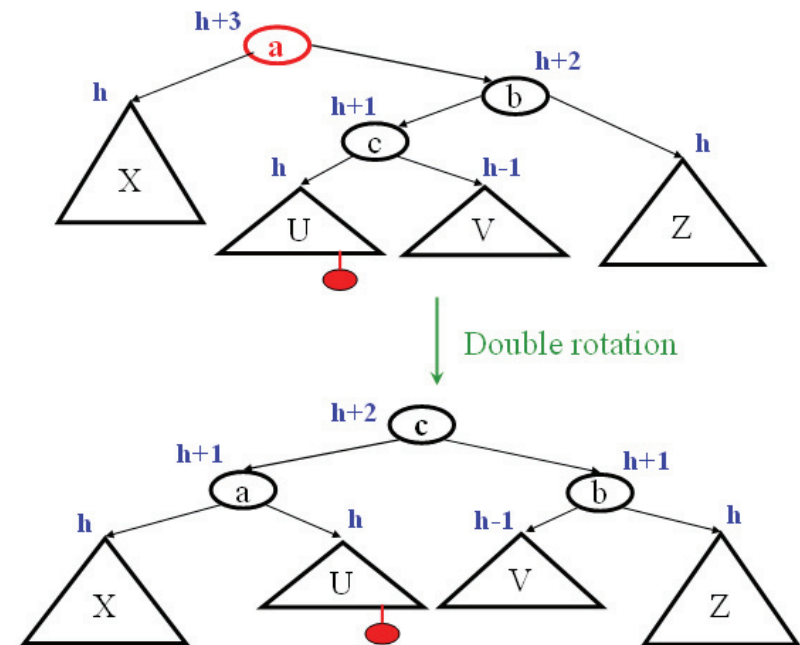
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Double rotation, step 2



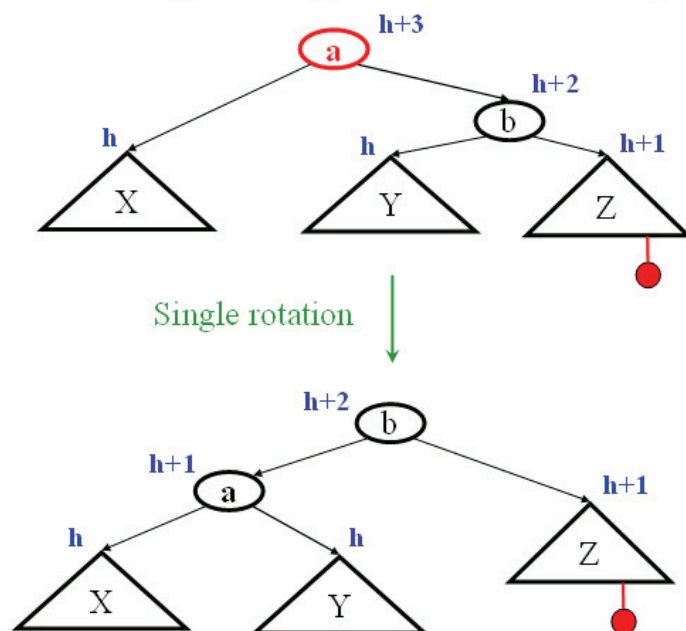
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Case #3: right-left insertion (zig-zag)



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Case #4: right-right insertion (zig-~~zag~~)



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AVL tree case summary

Let a be the node where an imbalance occurs.

Four cases to consider. The insertion below a is in the

1. left child's left subtree. (zig-~~zag~~)
2. left child's right subtree. (zig-zag)
3. right child's left subtree. (zig-zag)
4. right child's right subtree. (zig-~~zag~~)

Cases 1 & 4 are solved by a **single rotation**:

1. Rotate between a and child

Cases 2 & 3 are solved by a **double rotation**:

1. Rotate between a 's child and grandchild
2. Rotate between a and a 's new child

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Single and Double Rotations:

Inserting what integer values would cause the tree to need a:

1..14 not in tree

1. single rotation?

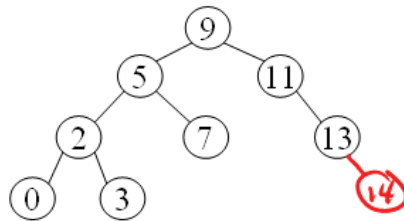
1, 14

2. double rotation?

4, 12

3. no rotation?

6, 8, 10



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Insertion procedure

1. Find spot for new key
2. Hang new node there with this key
3. Search back up the path for imbalance
4. If there is an imbalance:



cases #1,#4: Perform single rotation and exit

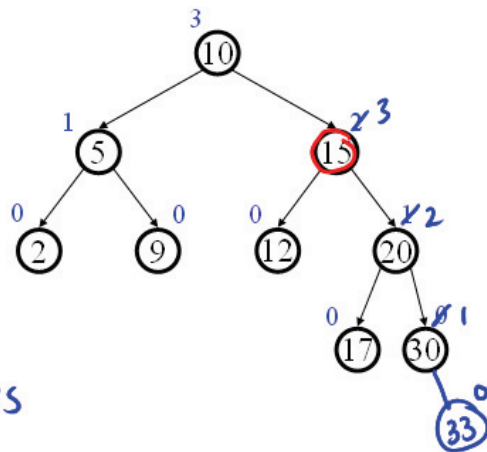
cases #2,#3: Perform double rotation and exit

Both rotations keep the subtree height unchanged.
Hence only one rotation is sufficient!

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More insert examples

Insert(33)

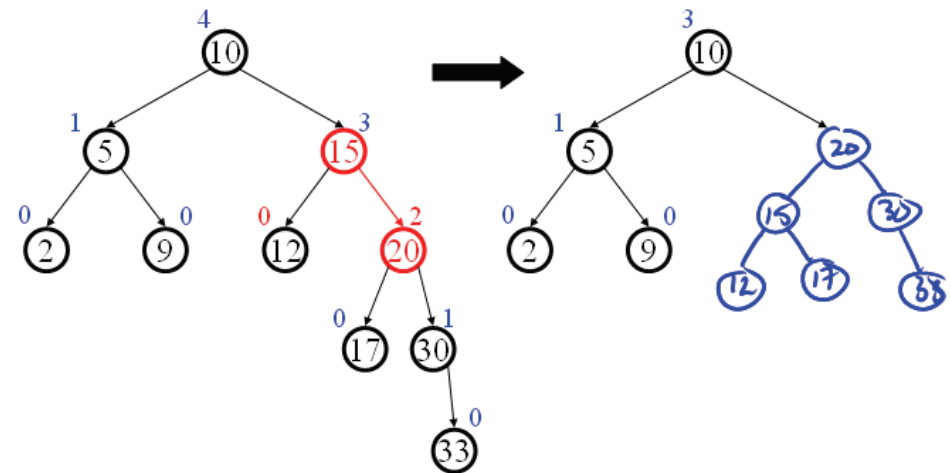


Unbalanced? *Yes*

How to fix?

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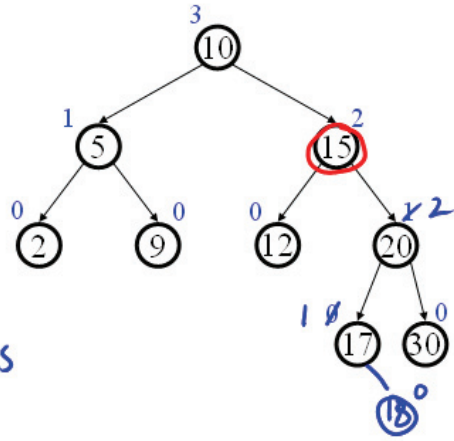
Single Rotation



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More insert examples

Insert(18)

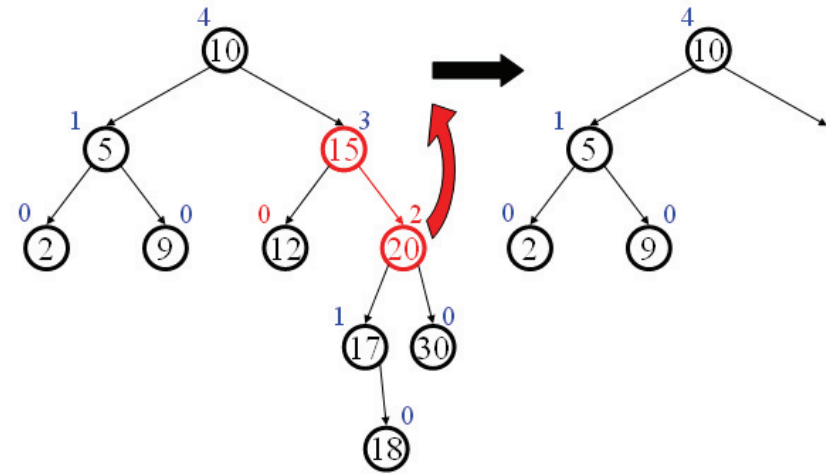


Unbalanced? **Yes**

How to fix?

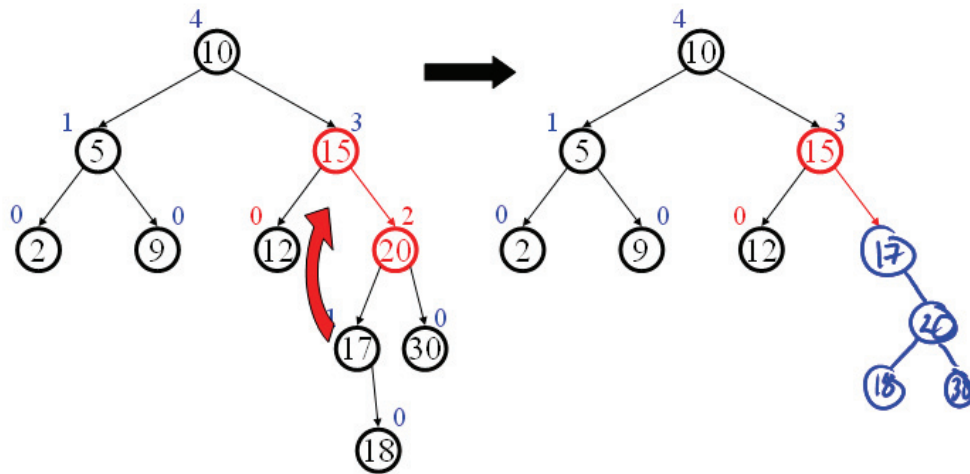
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Single Rotation (oops!)



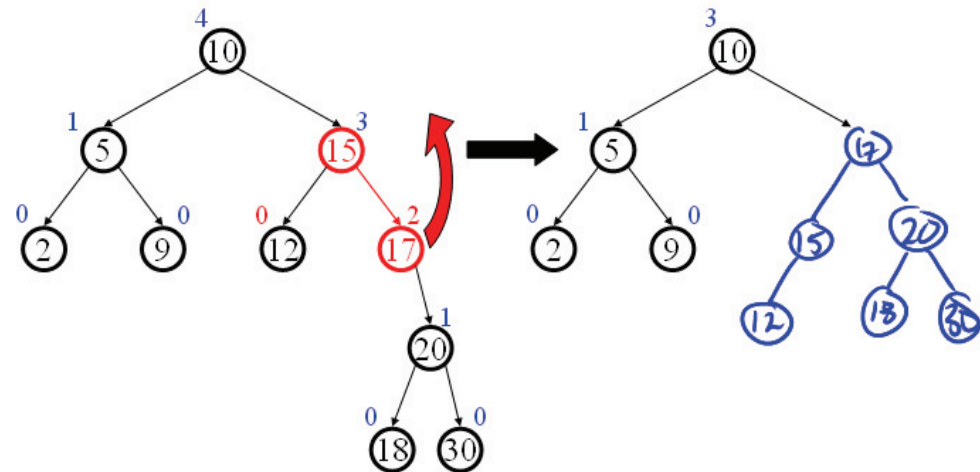
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Double Rotation (Step #1)



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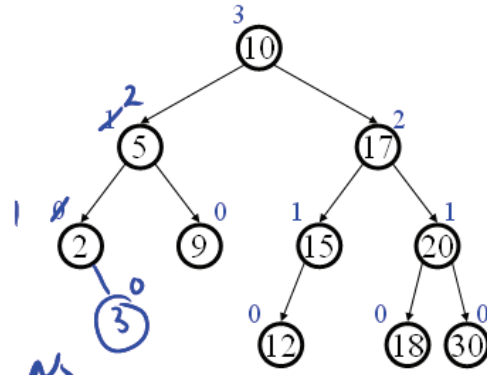
Double Rotation (Step #2)



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More insert examples

Insert(3)



Unbalanced?

No

~~How to fix?~~

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Insert into an AVL tree: 5, 8, 9, 4, 2, 7, 3, 1

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AVL complexity

What is the worst case complexity of an insert?

$O(\log n)$

What is the worst case complexity of a find?

$O(\log n)$

What is the worst case complexity of buildTree?

$O(n \log n)$

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