# CSE 326: Data Structures Disjoint Set Union/Find (part 2)

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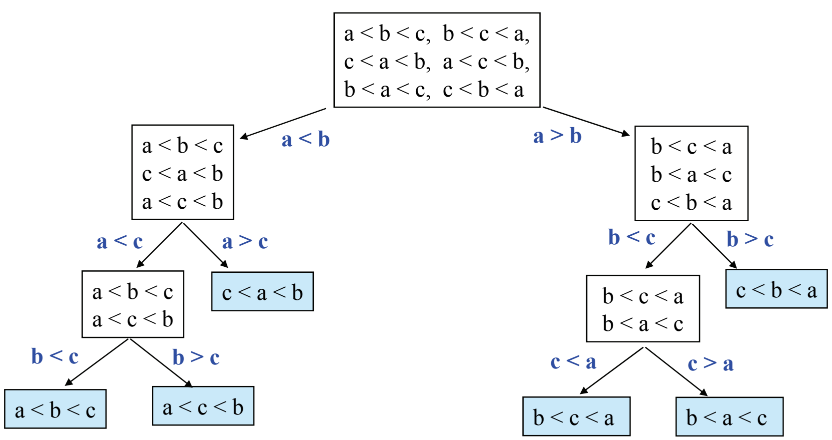
# Announcements (5/21/08)

- Homework due beginning of class on Friday.
- Reading for this lecture: Chapter 8.



# Decision Tree




The leaves contain all the possible orderings of a, b, c.



# Alternate Explanation of the Comparison-based Sorting Bound

At each decision point, one child has ≤ ½ of the options remaining, the other has ≥ ½ remaining.

Worst case: we always end up with ≥ ½ remaining.

Best outcome, in the worst case: we always end up with *exactly* ½ remaining.

Thus, in the worst case, the best we can hope for is halving the space *d* times (with *d* comparisons), until we have an answer, i.e., until the space is reduced to size = 1.

The space starts at *N*! in size, and halving *d* times means multiplying by $1/2^d$, giving us a lower bound on the worst case:

$$\frac{N!}{2^d} = 1 \quad \Rightarrow \quad N! = 2^d \quad \Rightarrow \quad d = \log_2(N!) \in \Omega(N \log N)$$

# Implementation: Take 1

**Approach**:

- Each set is a doubly-linked list (with pointer to last element).
- Store set name with object.

**Find**: get set name of object

- Worst case complexity?

**Union**: put one list on the end of the other, update set names of objects to be all the same

- Worst case complexity?



# Implementation: Take 2

**Approach**:

- Each set is a doubly-linked list (with pointer to last element).
- Front of list is set identifier.

**Find**: traverse linked list until reaching the front

- Worst case complexity?

**Union**: put one list on the end of the other

- Worst case complexity?



# Union/Find Trade-off

- Known result:
  - Find and Union cannot *both* be done in worst-case $O(1)$ time with any data structure.
- We will instead aim for good *amortized* complexity.
- For $m$ operations on $n$ elements:
  - Target complexity: $O(m)$ *i.e.* $O(1)$ amortized



# Tree-based Approach

We'll build on the "fast union" approach (linked list, with head node as set name, no set names explicitly stored in nodes).

Improvements:

- Instead of linked lists, use a forest of trees (one tree per set).
- Root of each tree is the set name.
- Allow large fanout. Why is this good?

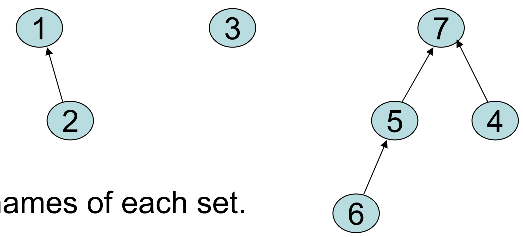
# Up-Tree for DS Union/Find

**Observation**: we will only traverse these trees upward from any given node to find the root.

**Idea**: *reverse* the pointers (make them point up from child to parent). The result is an **up-tree**.

Initial state


Intermediate state

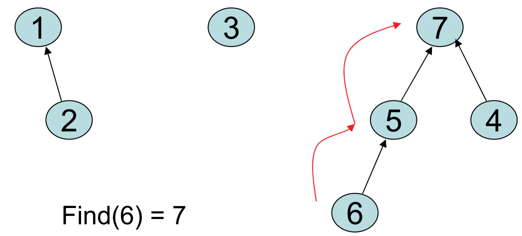



Roots are the names of each set.



# Find Operation

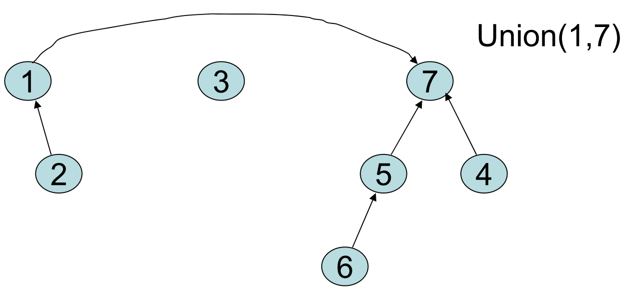
Find(x) follow x to the root and return the root.






# Union Operation

Union(i, j) - assuming i and j roots, point i to j.



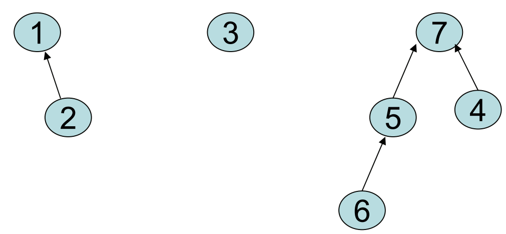



# Simple Implementation

- Array of indices

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| up | -1 | 1 | -1 | 7 | 7 | 5 | -1 |

Up[x] = -1 means x is a root.



# Implementation

```
void Union(int x, int y) {
  up[y] = x;
}
```

```
int Find(int x) {
  while(up[x] >= 0) {
    x = up[x];
  }
  return x;
}
```

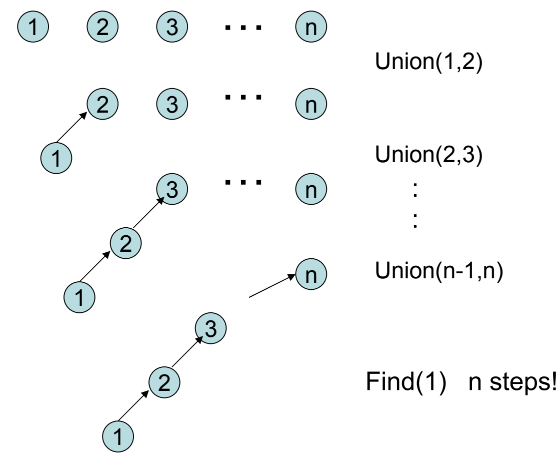
*runtime for Union*:

*runtime for Find*:

Amortized complexity is no better.



# A Bad Case






# Two Big Improvements

Can we do better? *Yes!*

## 1. Union-by-size

- Improve Union so that *Find* only takes worst case time of $\Theta(\log n)$.

## 2. Path compression

- Improve Find so that, with Union-by-size, Find takes amortized time of <u>almost</u> $\Theta(1)$.

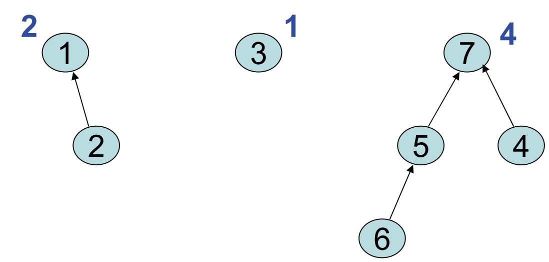


# Union-by-Size

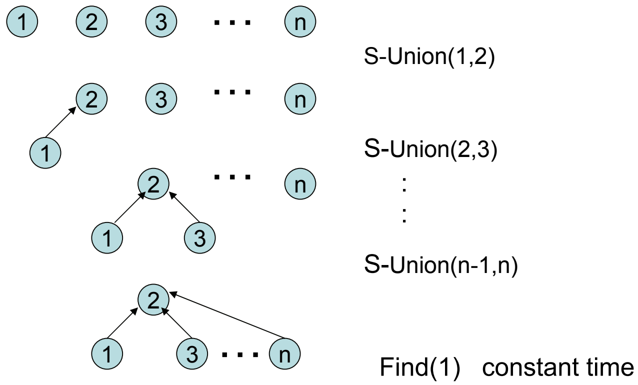
Union-by-size

- Always point the smaller tree to the root of the larger tree

S-Union(1,7)



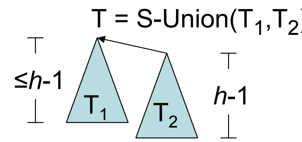
# Example Again






# Analysis of Union-by-Size

- Theorem: With union-by-size an up-tree of height $h$ has size at least $2^h$.
- Proof by induction
  - Base case: $h = 0$. The up-tree has one node, $2^0 = 1$
  - Inductive hypothesis: Assume true for $h$-1
  - Inductive step: Then true for $h$.
  - Observation: tree gets taller only as a result of a union.






# Analysis of Union-by-Size

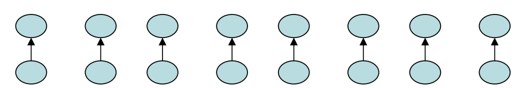
- What is worst case complexity of Find(x) in an up-tree forest of $n$ nodes?

- (Amortized complexity is no better.)

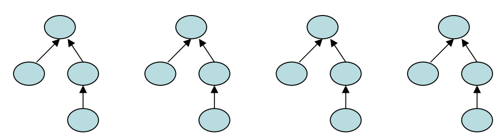


# Worst Case for Union-by-Size

n/2 Unions-by-size

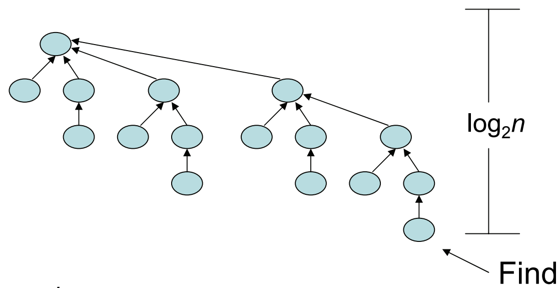


n/4 Unions-by-size



# Example of Worst Cast (cont')

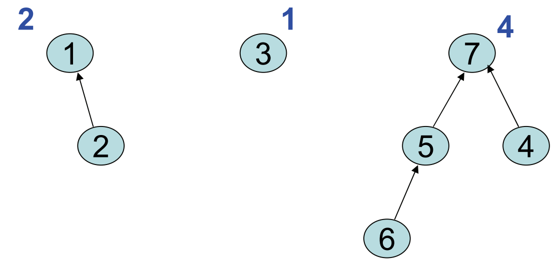
After $n$ -1 = $n$/2 + $n$/4 + …+ 1 Unions-by-size




If there are $n = 2^k$ nodes then the longest path from leaf to root has length $k$.



# Array Implementation

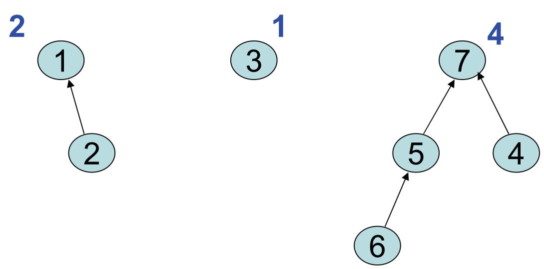



Can store separate size array:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| up | -1 | 1 | -1 | 7 | 7 | 5 | -1 |
| size | 2 | | 1 | | | | 4 |



# Elegant Array Implementation




Better, store sizes in the up array:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| up | -2 | 1 | -1 | 7 | 7 | 5 | -4 |

Negative up-values correspond to sizes of roots.



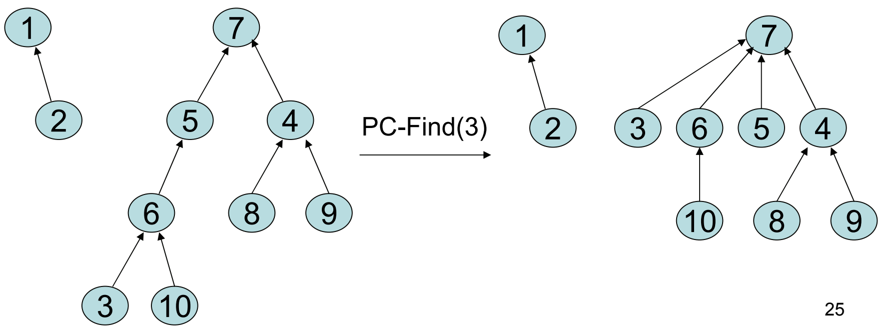
# Code for Union-by-Size

```
S-Union(i,j){
  // Collect sizes
  si = -up[i];
  sj = -up[j];

  // verify i and j are roots
  assert(si >=0 && sj >=0)
  // point smaller sized tree to
  // root of larger, update size
  if (si < sj) {
    up[i] = j;
    up[j] = -(si + sj);
  else {
    up[j] = i;
    up[i] = -(si + sj);
  }
}
```

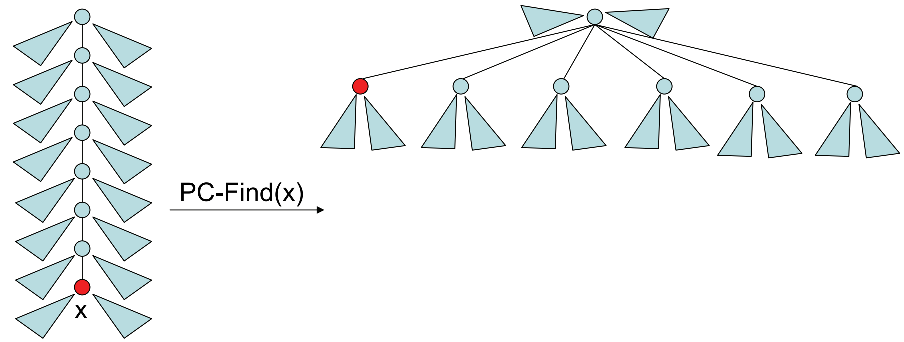
# Path Compression

- To improve the amortized complexity, we'll borrow an idea from splay trees:
  - When going up the tree, *improve nodes on the path*!
- On a Find operation point all the nodes on the search path directly to the root. This is called "path compression."





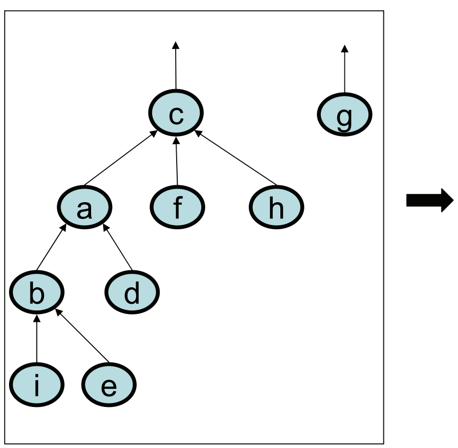

# Self-Adjustment Works






Your turn

# Draw the result of Find(e):






# Code for Path Compression Find

```
PC-Find(i) {
  j = i;

  //find root
  while (up[j] >= 0) {
    j = up[j];
  root = j;

  //compress path
  if (i != root) {
    parent = up[i];
    while (parent != root) {
      up[i] = root;
      i = parent;
      parent = up[parent];
    }
  }
  return(root)
}
```

## Complexity of Union-by-Size + Path Compression

- Worst case time complexity for…
  - …a single Union-by-size is:
  - …a single PC-Find is:

- Time complexity for $m \geq n$ operations on $n$ elements has been shown to be $O(m \log^* n)$.
  [See Weiss for proof.]
  - Amortized complexity is then $O(\log^* n)$
  - What is log* ?



## log* *n*

**log* *n* = number of times you need to apply log to bring value down to at most 1**

$\log^* 2 = 1$

$\log^* 4 = \log^* 2^2 = 2$

$\log^* 16 = \log^* 2^{2^2} = 3$ (log log log 16 = 1)

$\log^* 65536 = \log^* 2^{2^{2^2}} = 4$ (log log log log 65536 = 1)

$\log^* 2^{65536} = \ldots\ldots\ldots\ldots\ldots \approx \log^* (2 \times 10^{19,728}) = 5$

log * $n \leq 5$ for all reasonable $n$.



## The Tight Bound

In fact, Tarjan showed the time complexity for $m \geq n$ operations on $n$ elements is:

$$\Theta(m\,\alpha(m, n))$$

Amortized complexity is then $\Theta(\alpha(m, n))$ .

What is $\alpha(m, n)$?
- Inverse of Ackermann's function.
- For reasonable values of *m, n*, grows even slower than log * *n.* So, it's even "more constant."

Proof is beyond scope of this class. A simple algorithm can lead to incredibly hardcore analysis!