CSE 326 Data Structures

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B-Trees
• Midterms graded, get back in section Thurs

• Project 2 due Wednesday at 11:59pm

• Homework 4 due Friday in class
  (Problem 2 Weiss 4.27 not 4.71)

• Reading: Finish Chapter 4
  → Chapt. 5.
CPU (has registers)

Cache

SRAM
8KB - 4MB

DRAM
up to 10GB

Main Memory

Disk

Time to access:
1 ns per instruction

Cache
2-10 ns

Main Memory
40-100 ns

Disk
a few milliseconds (5-10 Million ns)
Dictionary AOT

Trees so far

N = 10 Million

- BST
  Worst case $O(N)$
  10 mil disk accesses
  $10^6 \cdot 10^{-3} s = 10^3 \text{ sec}$

- AVL
  Worst case $O(\log_2 N)$
  $\log_2 10^7 \approx 23$
  $23 \cdot 10^{-3} = 0.23$

- Splay
  Amortized
  $O(\log_2 N)$
  3 disk accesses
  $\frac{2}{4}$
Chapter 4 in Weiss
M-ary Search Tree

- Maximum branching factor of $M$
- Complete tree has height $= O(\log_{M} n)$

# disk accesses for find:
- Best: $O(\log_{M} n)$
- Worst: $O(n)$

Runtime of find:
$O(\log_{M} n \log_2 M) = O(\log_2 n)$
Solution: B-Trees

- specialized $M$-ary search trees

  \[ M \text{ edges for each node} \]

- Each node has (up to) $M-1$ keys:
  - subtree between two keys $x$ and $y$ contains leaves with values $v$ such that $x \leq v < y$

- Pick branching factor $M$ such that each node takes one full \{page, block\} of memory
B-Trees

What makes them disk-friendly?

1. Many keys stored in a node
   - All brought to memory/cache in one access!

2. Internal nodes contain only keys;
   Only leaf nodes contain keys and actual data
   - The tree structure can be loaded into memory irrespective of data object size
   - Data actually resides in disk
B-Tree: Example

B-Tree with $M = 4$ (# pointers in internal node) and $L = 4$ (# data items in leaf)

Data objects, that I’ll ignore in slides

Note: All leaves at the same depth!
B-Tree Properties

- Data is stored at the leaves
- All leaves are at the same depth and contains between $\lfloor L/2 \rfloor$ and $L$ data items
- Internal nodes store up to $M-1$ keys
- Internal nodes have between $\lfloor M/2 \rfloor$ and $M$ children
- Root (special case) has between 2 and $M$ children (or root could be a leaf)

†These are technically B+-Trees
Example, Again

B-Tree with $M = 4$
and $L = 4$

24 entries $\rightarrow$ 2 deep
BST $\rightarrow$ 4 deep

(Only showing keys, but leaves also have data!)
B-trees vs. AVL trees

Suppose we have 100 million items (100,000,000):

- Depth of AVL Tree
  \[ \log_2 10^8 = 26.6 \]
- Depth of B+ Tree with \( M = 128 \), \( L = 64 \)
  \[ \log_{128} 10^8 = 3.8 \]
Building a B-Tree

The empty B-Tree

$M = 3 \quad L = 2$

Insert(3)

Insert(14)

Now, Insert(1)?
$M = 3 \quad L = 2$

Splitting the Root

Too many keys in a leaf!

Insert(1)

So, split the leaf.

And create a new root
Overflowing leaves

$M = 3 \quad L = 2$

Too many keys in a leaf!

So, split the leaf.

Parent have room?

And add a new child.
Propagating Splits

\[ M = 3 \quad \underline{L = 2} \]

Insert(5)

Split the leaf, but no space in parent!

Create a new root

So, split the node.
Insertion Algorithm

1. Insert the key in its leaf
2. If the leaf ends up with \( L+1 \) items, **overflow**!
   - Split the leaf into two nodes:
     - original with \( \left\lceil (L+1)/2 \right\rceil \) items
     - new one with \( \left\lfloor (L+1)/2 \right\rfloor \) items
   - Add the new child to the parent
   - If the parent ends up with \( M+1 \) items, **overflow**!

   **This makes the tree deeper!**

3. If an internal node ends up with \( M+1 \) items, **overflow**!
   - Split the node into two nodes:
     - original with \( \left\lceil (M+1)/2 \right\rceil \) items
     - new one with \( \left\lfloor (M+1)/2 \right\rfloor \) items
   - Add the new child to the parent
   - If the parent ends up with \( M+1 \) items, **overflow**!

4. Split an overflowed root in two and hang the new nodes under a new root
After More Routine Inserts

How to delete?
**Deletion**

1. Delete item from leaf
2. Update keys of ancestors if necessary

What could go wrong?

might cause leaf $\leq \frac{L}{2}$ children
Deletion and Adoption

$M = 3 \quad L = 2$

A leaf has too few keys!

So, borrow from a sibling

What if not enough to borrow from?
Does Adoption Always Work?

- What if the sibling doesn’t have enough for you to borrow from?

  e.g. you have $\left\lfloor L/2 \right\rfloor - 1$ and sibling has $\left\lceil L/2 \right\rceil$?

*Merge Together*
Deletion and Merging

A leaf has too few keys!

And no sibling with surplus!

But now an internal node has too few subtrees!

So, delete the leaf
Deletion with Propagation (More Adoption)

$M = 3 \quad L = 2$
A Bit More Adoption

\( M = 3 \quad L = 2 \)

Delete(1) (adopt a sibling)
Pulling out the Root

A leaf has too few keys!
And no sibling with surplus!

So, delete the leaf; merge

But now the root has just one subtree!

A node has too few subtrees and no neighbor with surplus!

Delete the node
Pulling out the Root (continued)

The root has just one subtree!

Simply make the one child the new root!
Deletion Algorithm

1. Remove the key from its leaf

2. If the leaf ends up with fewer than $\lceil L/2 \rceil$ items, underflow!
   - Adopt data from a sibling; update the parent
   - If adopting won’t work, delete node and merge with neighbor
   - If the parent ends up with fewer than $\lceil M/2 \rceil$ items, underflow!
Deletion Slide Two

3. If an internal node ends up with fewer than $\lceil M/2 \rceil$ items, underflow!
   - Adopt from a neighbor; update the parent
   - If adoption won’t work, merge with neighbor
   - If the parent ends up with fewer than $\lceil M/2 \rceil$ items, underflow!

4. If the root ends up with only one child, make the child the new root of the tree.

This reduces the height of the tree!
Thinking about B-Trees

- B-Tree insertion can cause (expensive) splitting and propagation
- B-Tree deletion can cause (cheap) adoption or (expensive) deletion, merging and propagation
- Propagation is rare if $M$ and $L$ are large *(Why?)*
- If $M = L = 128$, then a B-Tree of height 4 will store at least 30,000,000 items
Tree Names You Might Encounter

FYI:
- B-Trees with $M = 3$, $L = x$ are called 2-3 trees
  - Nodes can have 2 or 3 keys
- B-Trees with $M = 4$, $L = x$ are called 2-3-4 trees
  - Nodes can have 2, 3, or 4 keys