Divide and Conquer Sorting

CSE 326

Data Structures

Lecture 18

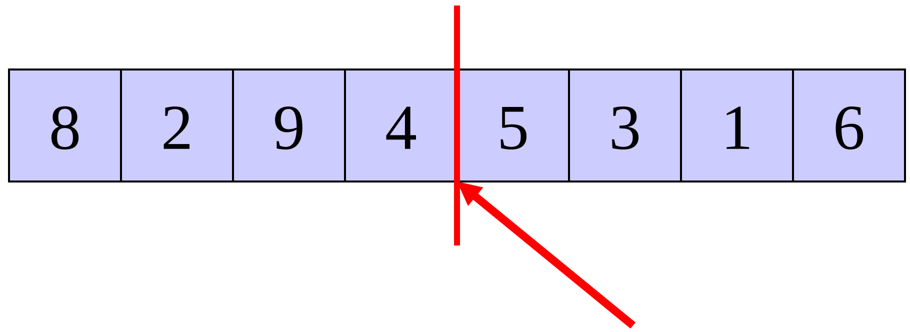
Insertion Sort

- What if first *k* elements of array are already sorted?
  - 4, 7, 12, 5, 19, 16
- We can shift the tail of the sorted elements list down and then *insert* next element into proper position and we get k+1 sorted elements
  - 4, 5, 7, 12, 19, 16

“Divide and Conquer”

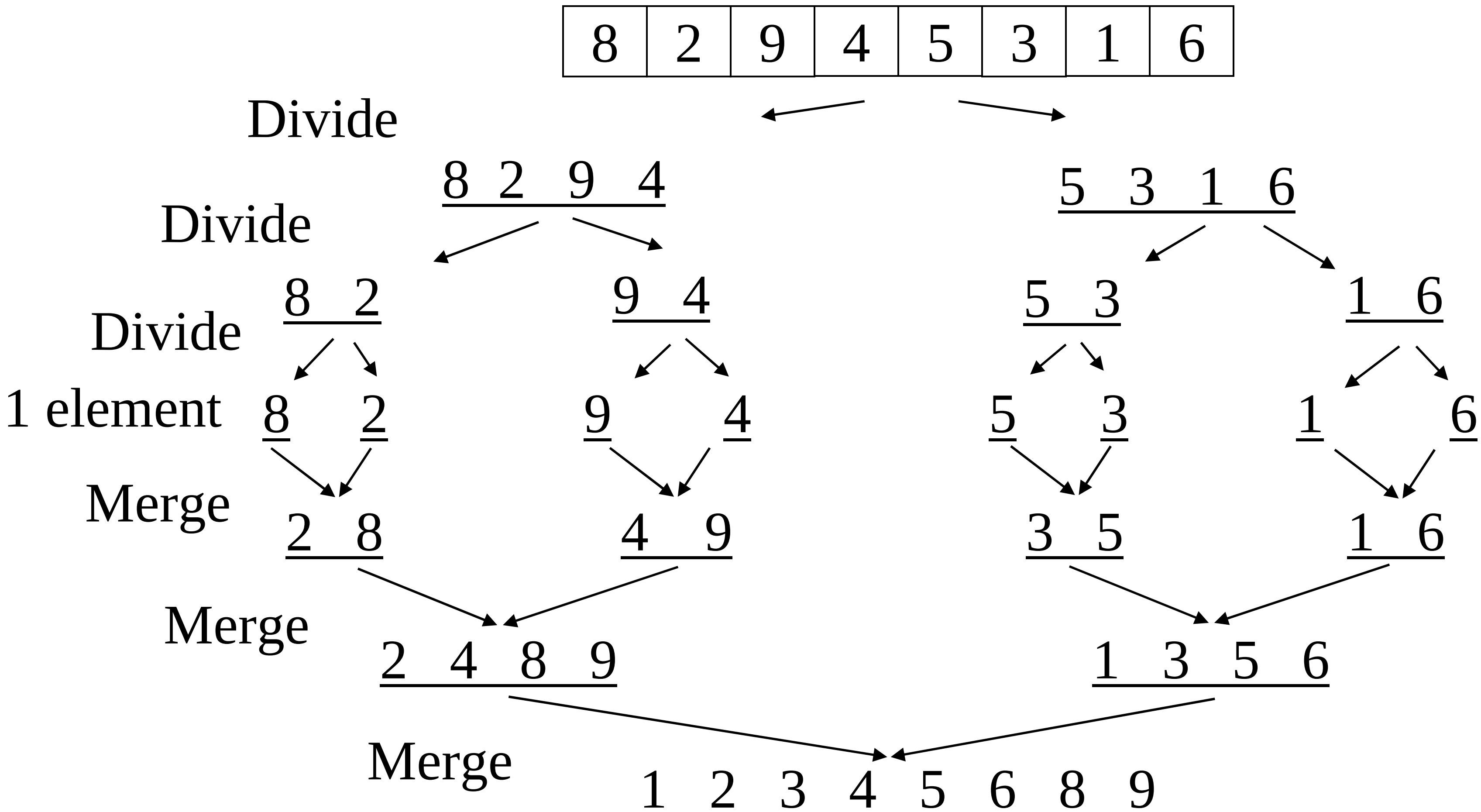
- Very important strategy in computer science:
  - Divide problem into smaller parts
  - Independently solve the parts
  - Combine these solutions to get overall solution
- **Idea 1**: Divide array into two halves, *recursively* sort left and right halves, then *merge* two halves → known as Mergesort
- **Idea 2 :** Partition array into small items and large items, then recursively sort the two sets → known as Quicksort

Mergesort



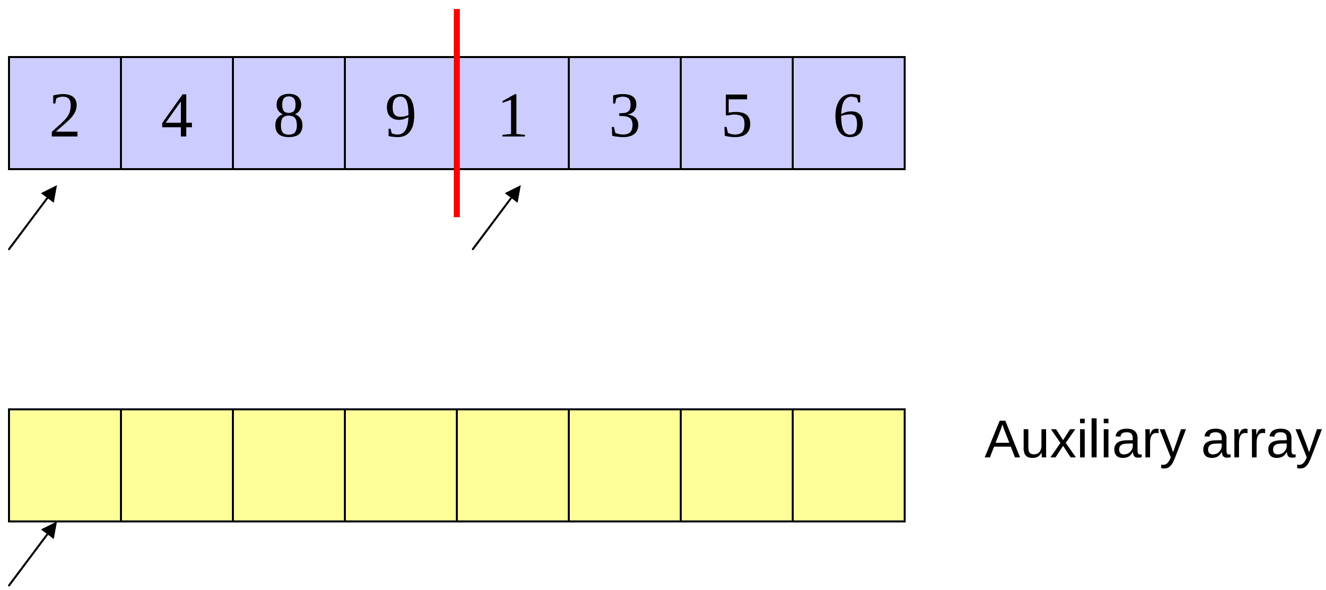
- Divide it in two at the midpoint
- Conquer each side in turn (by recursively sorting)
- Merge two halves together

Mergesort Example



Auxiliary Array

- The merging requires an auxiliary array.



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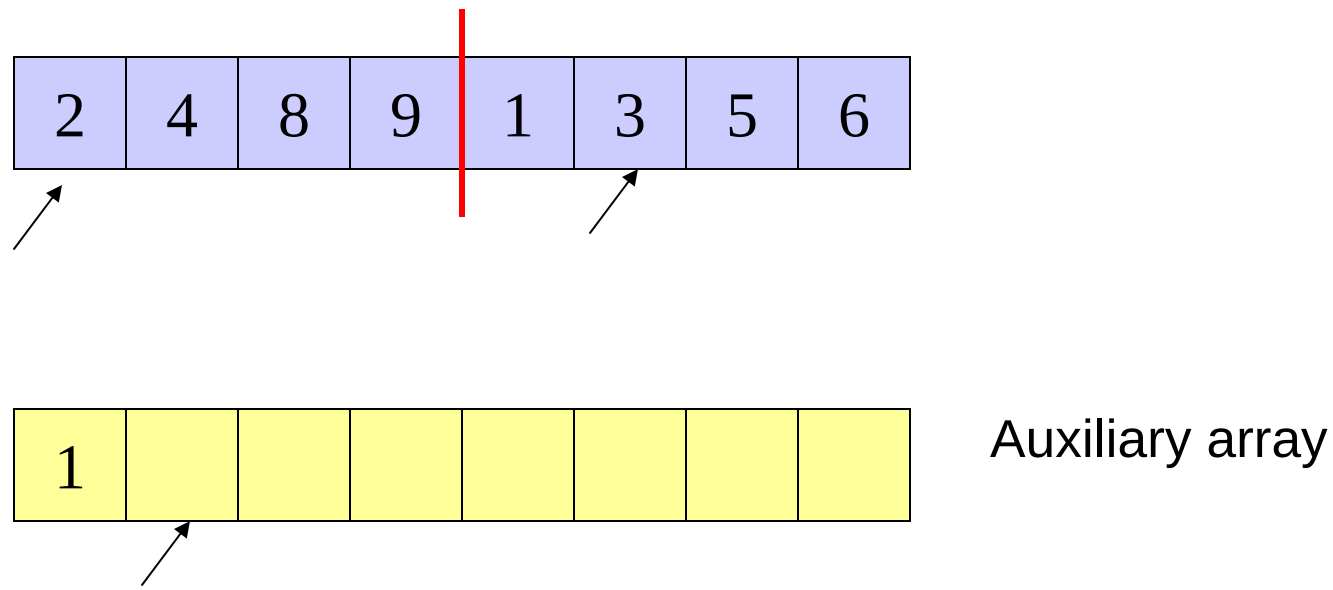

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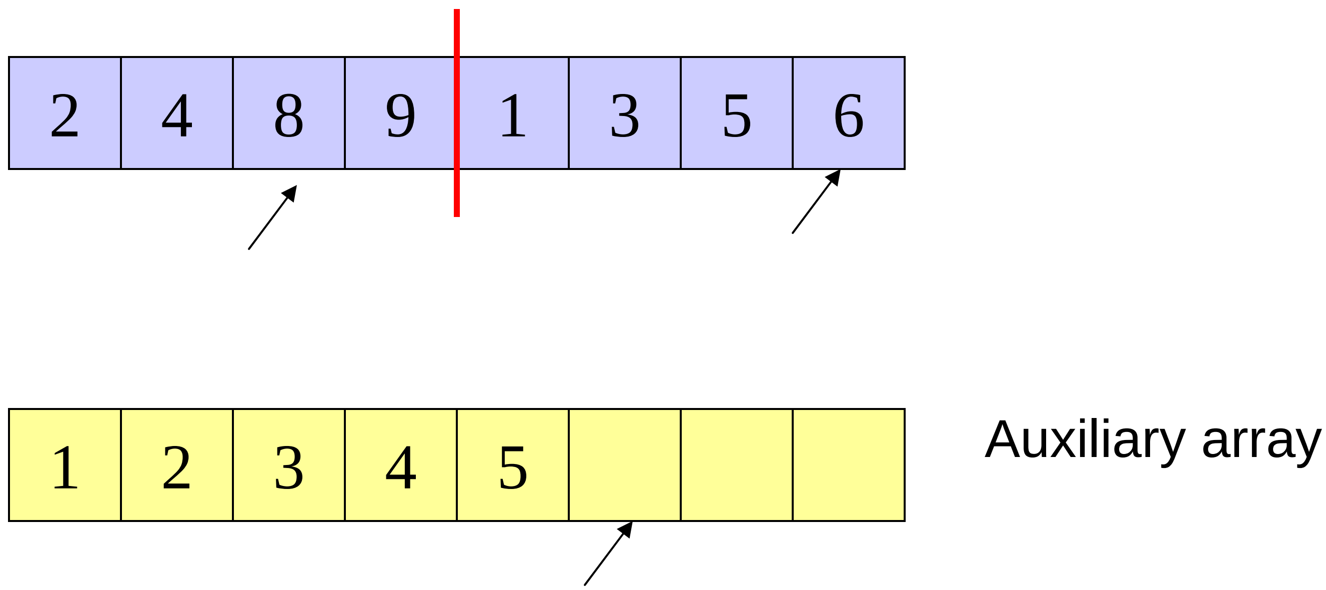
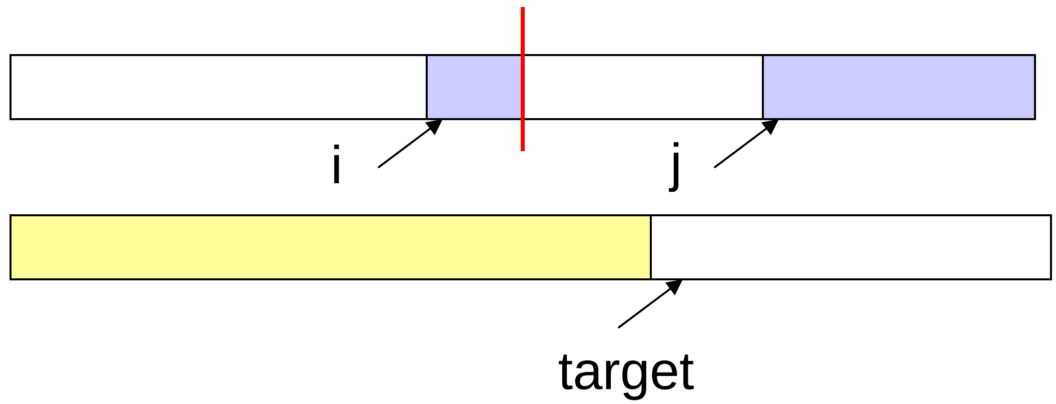
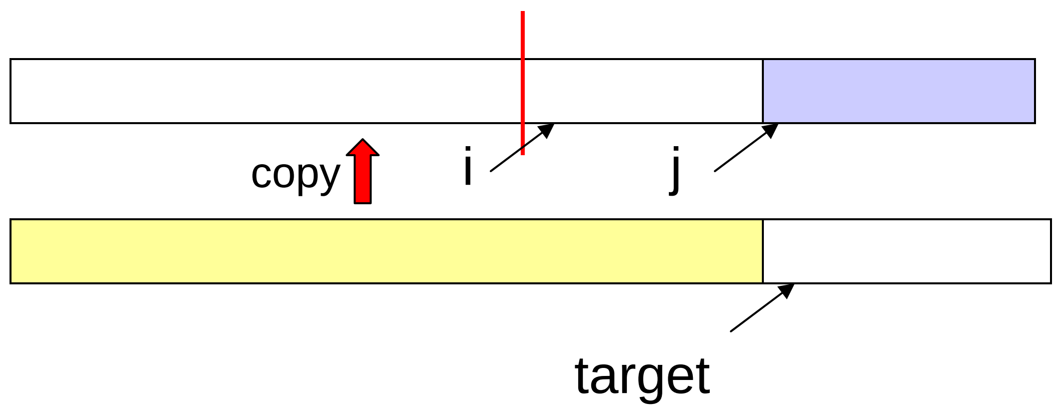


Merging

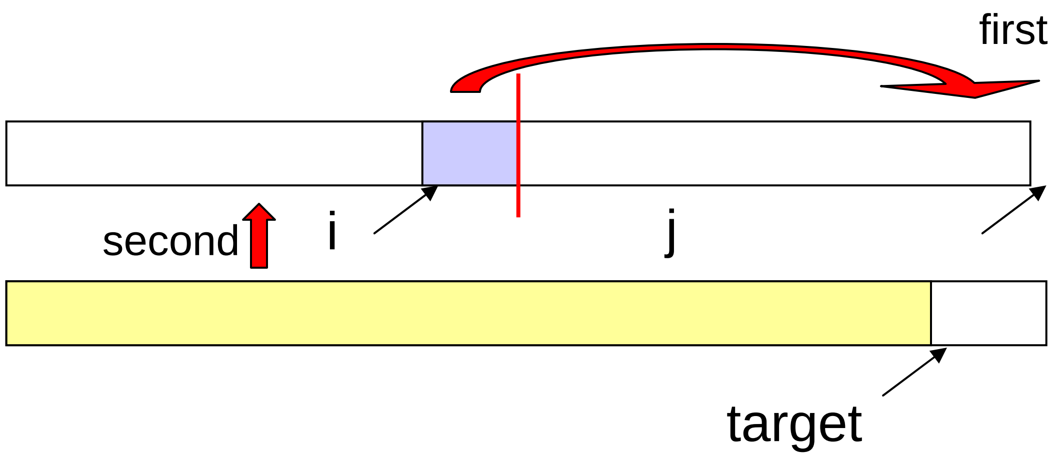


normal



Left completed first

Merging



Right completed first

Merging

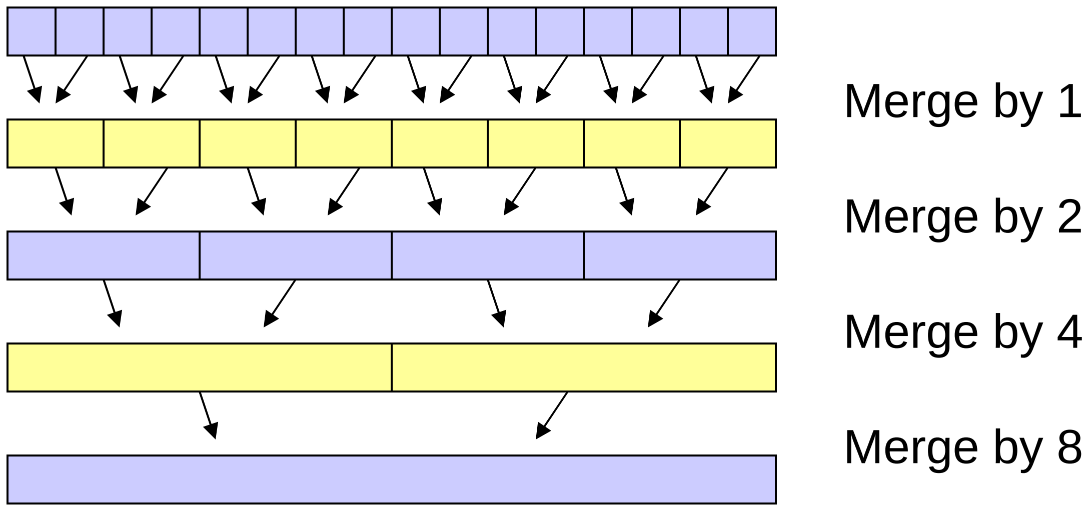
```
Merge(A[], T[] : integer array, left, right : integer) : {
  mid, i, j, k, l, target : integer;
  mid := (right + left)/2;
  i := left; j := mid + 1; target := left;
  while i ≤ mid and j ≤ right do
    if A[i] ≤ A[j] then T[target] := A[i] ; i:= i + 1;
      else T[target] := A[j]; j := j + 1;
    target := target + 1;
  if i > mid then //left completed//
    for k := left to target-1 do A[k] := T[k];
  if j > right then //right completed//
    k : = mid; l := right;
    while k ≥ i do A[l] := A[k]; k := k-1; l := l-1;
    for k := left to target-1 do A[k] := T[k];
}
```

Recursive Mergesort

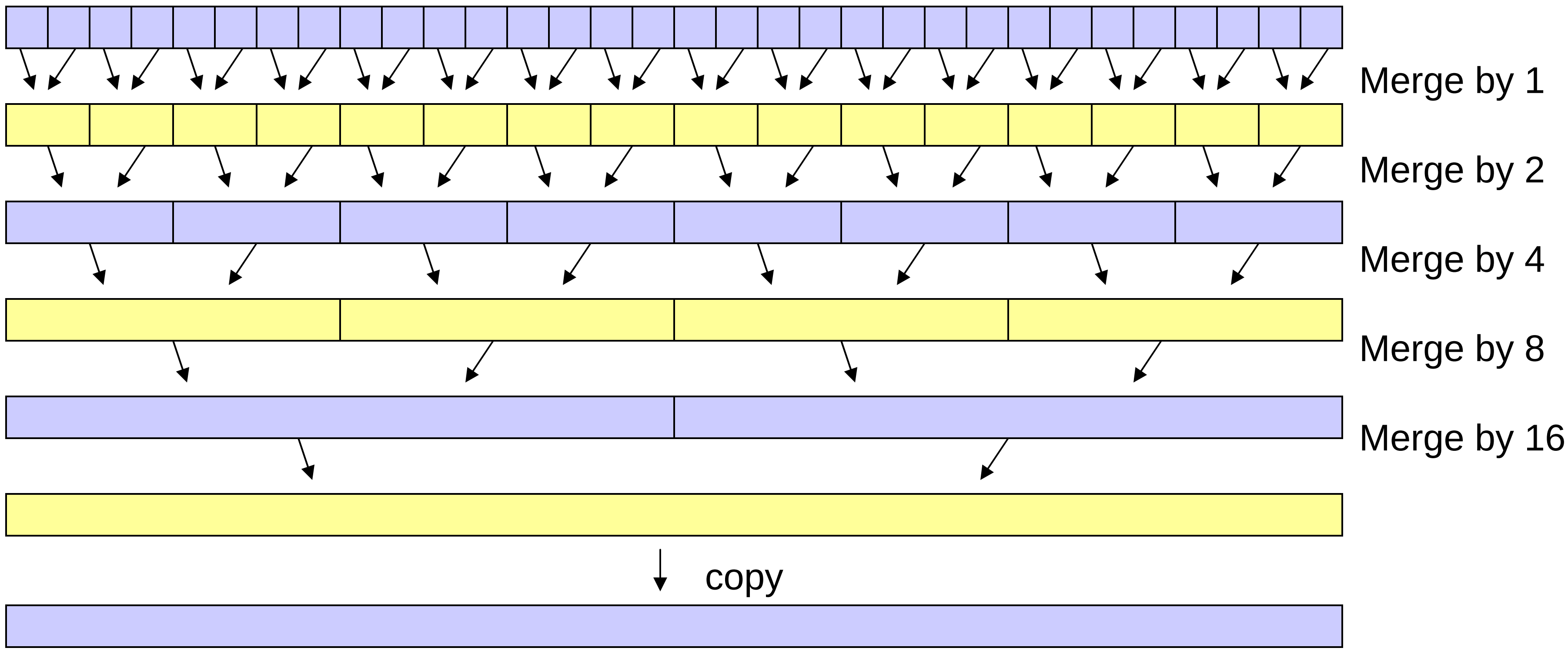
```
Mergesort(A[], T[] : integer array, left, right : integer) : {
  if left < right then
    mid := (left + right)/2;
    Mergesort(A,T,left,mid);
    Mergesort(A,T,mid+1,right);
    Merge(A,T,left,right);
}

MainMergesort(A[1..n]: integer array, n : integer) : {
  T[1..n]: integer array;
  Mergesort[A,T,1,n];
}
```

Iterative Mergesort



Iterative Mergesort



Iterative pseudocode

- Sort(array A of length N)
  - › Let m = 2, let B be temp array of length N
  - › While m<N
    - For i = 1…N in increments of m
      - – merge A[i…i+m/2] and A[i+m/2…i+m] into B[i…i+m]
    - Swap role of A and B
    - m=m*2
  - › If needed, copy B back to A

Mergesort Analysis

- Let T(N) be the running time for an array of N elements
- Mergesort divides array in half and calls itself on the two halves. After returning, it merges both halves using a temporary array
- Each recursive call takes T(N/2) and merging takes O(N)

Mergesort Recurrence Relation

- The recurrence relation for T(N) is:
  - › $T(1) \le c$
    - base case: 1 element array → constant time
  - › $T(N) \le 2T(N/2) + dN$
    - Sorting n elements takes
      - – the time to sort the left half
      - – plus the time to sort the right half
      - – plus an O(N) time to merge the two halves
- $T(N) = O(N \log N)$

Properties of Mergesort

- Not in-place
  - Requires an auxiliary array
- Very few comparisons
- Iterative Mergesort reduces copying.

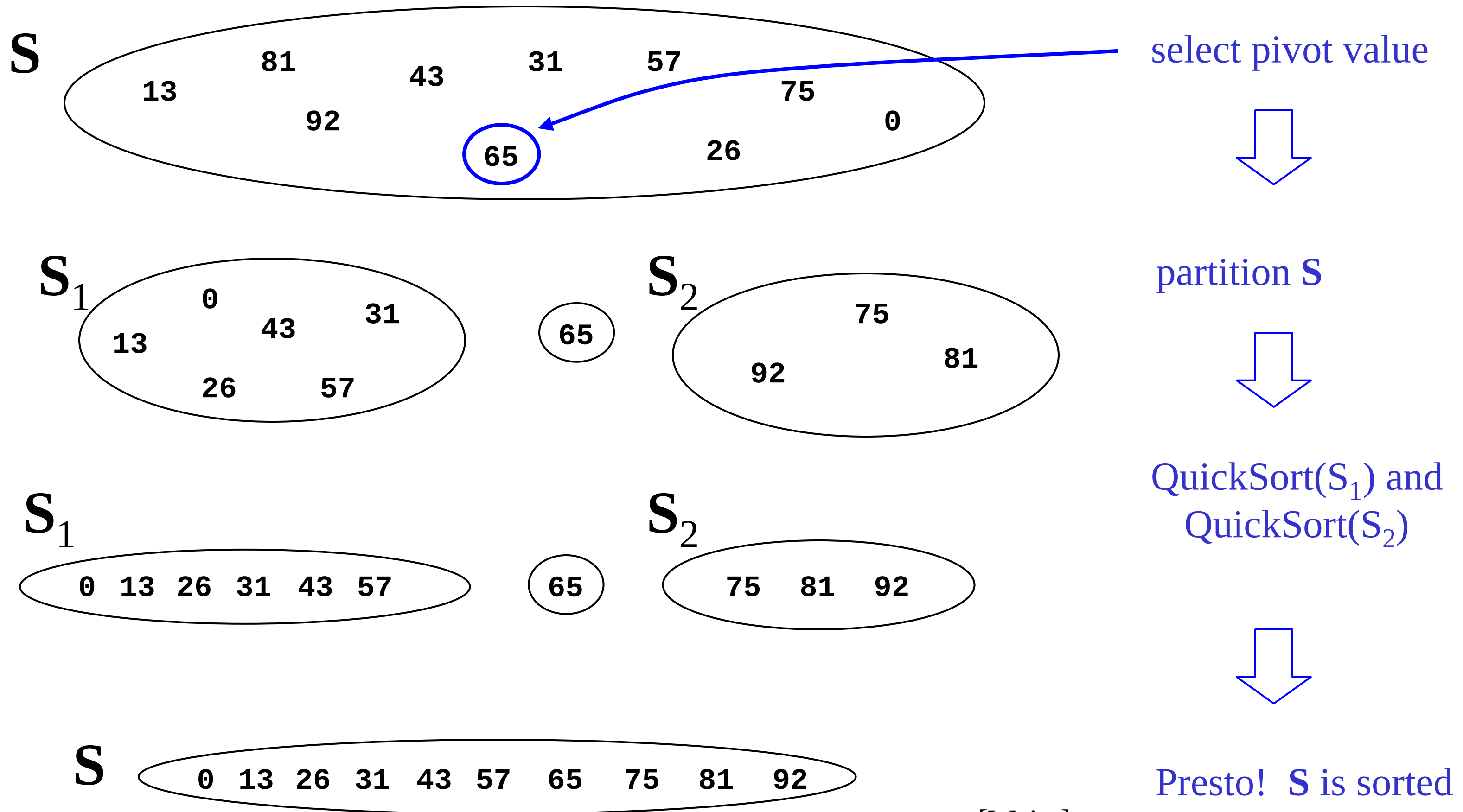
Quicksort

- Quicksort uses a divide and conquer strategy, but does not require the O(N) extra space that MergeSort does
  - › Partition array into left and right sub-arrays
    - the elements in left sub-array are all less than pivot
    - elements in right sub-array are all greater than pivot
  - › Recursively sort left and right sub-arrays
  - › Concatenate left and right sub-arrays in O(1) time

“Four easy steps”

- To sort an array **S**
  - › If the number of elements in **S** is 0 or 1, then return. The array is sorted.
  - › Pick an element *v* in **S**. This is the *pivot* value.
  - › Partition **S**-{*v*} into two disjoint subsets, $\mathbf{S}_1$ = {all values $x \leq v$}, and $\mathbf{S}_2$ = {all values $x \geq v$}.
  - › Return QuickSort($\mathbf{S}_1$), *v*, QuickSort($\mathbf{S}_2$)

The steps of QuickSort



[Weiss]

Details, details

- "The algorithm so far lacks quite a few of the details"
- Picking the pivot
  - want a value that will cause $|S_1|$ and $|S_2|$ to be non-zero, and close to equal in size if possible
- Implementing the actual partitioning
- Dealing with cases where the element equals the pivot

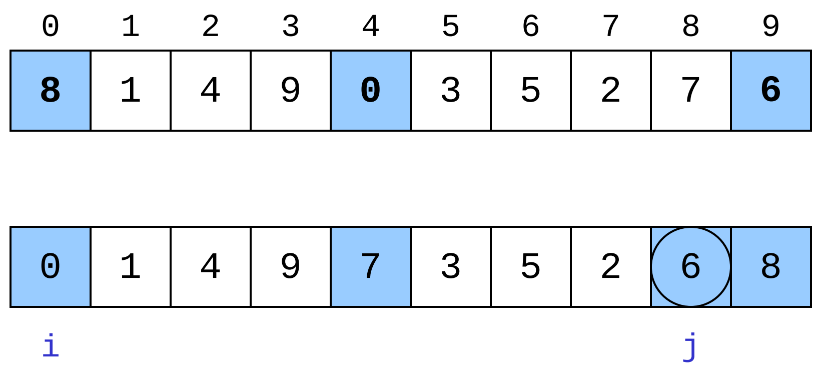
Alternative Pivot Rules

- Chose A[left]
  - Fast, but too biased, enables worst-case
- Chose A[random], left ≤ random ≤ right
  - Completely unbiased
  - Will cause relatively even split, but slow
- Median of three, A[left], A[right], A[(left+right)/2]
  - The standard, tends to be unbiased, and does a little sorting on the side.

Quicksort Partitioning

- Need to partition the array into left and right sub-arrays
  - the elements in left sub-array are ≤ pivot
  - elements in right sub-array are ≥ pivot
- How do the elements get to the correct partition?
  - Choose an element from the array as the pivot
  - Make one pass through the rest of the array and swap as needed to put elements in partitions

Example



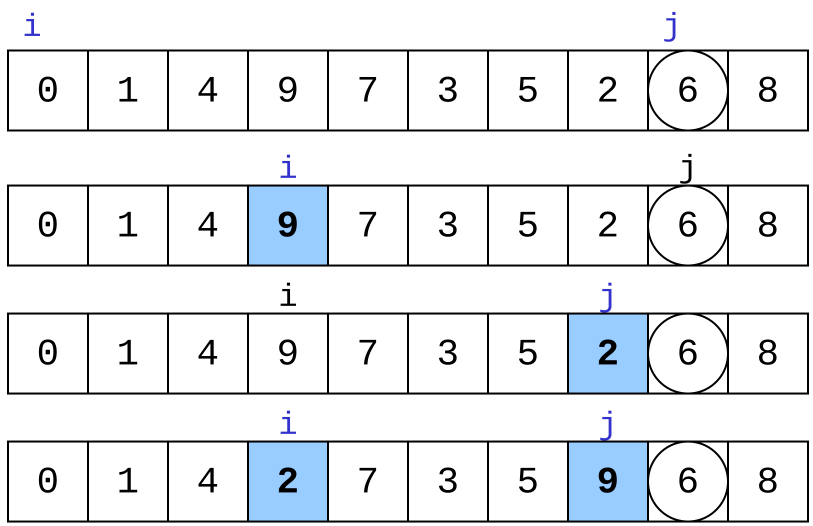
Choose the pivot as the median of three.

Place the pivot and the largest at the right
and the smallest at the left

Partitioning is done In-Place

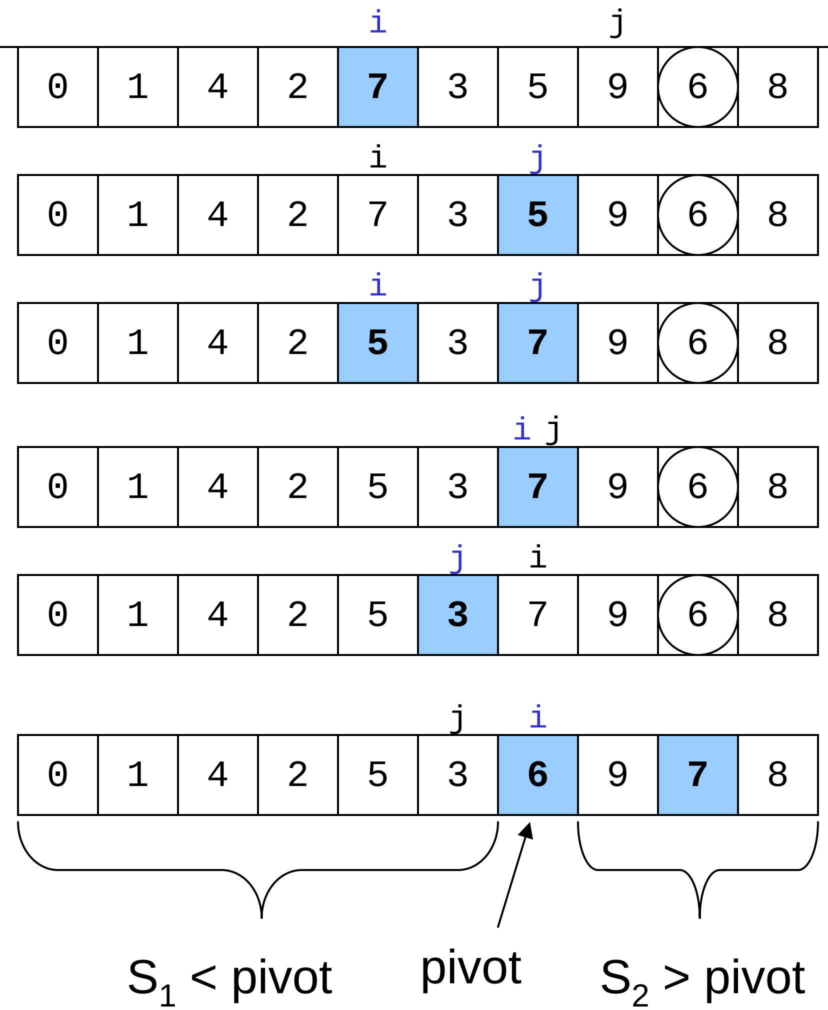
- One implementation (there are others)
  - median3 finds pivot and sorts left, center, right
  - Swap pivot with next to last element
  - Set pointers i and j to start and end of array
  - Increment i until you hit element A[i] > pivot
  - Decrement j until you hit element A[j] < pivot
  - Swap A[i] and A[j]
  - Repeat until i and j cross
  - Swap pivot (= A[N-2]) with A[i]

Example



Move i to the right to be larger than pivot.
Move j to the left to be smaller than pivot.
Swap

Example


i j
0 1 4 2 7 3 5 9 6 8
i j
0 1 4 2 7 3 5 9 6 8
i j
0 1 4 2 5 3 7 9 6 8
i j
0 1 4 2 5 3 7 9 6 8
j i
0 1 4 2 5 3 7 9 6 8
j i
0 1 4 2 5 3 6 9 7 8
S1 < pivot
pivot
S2 > pivot

Recursive Quicksort

```
Quicksort(A[]: integer array, left,right : integer): {
pivotindex : integer;
if left + CUTOFF ≤ right then
  pivot := median3(A,left,right);
  pivotindex := Partition(A,left,right-1,pivot);
  Quicksort(A, left, pivotindex – 1);
  Quicksort(A, pivotindex + 1, right);
else
  Insertionsort(A,left,right);
}
```

Don’t use quicksort for small arrays.
CUTOFF = 10 is reasonable.

Quicksort Best Case Performance

- Algorithm always chooses best pivot and splits sub-arrays in half at each recursion
  - › $T(0) = T(1) = O(1)$
    - constant time if 0 or 1 element
  - › For N > 1, 2 recursive calls plus linear time for partitioning
  - › $T(N) = 2T(N/2) + O(N)$
    - Same recurrence relation as Mergesort
  - › $T(N) = O(N \log N)$

Quicksort Worst Case Performance

- Algorithm always chooses the worst pivot – one sub-array is empty at each recursion
  - $T(N) \leq a$ for $N \leq C$
  - $T(N) \leq T(N-1) + bN$
  - $\leq T(N-2) + b(N-1) + bN$
  - $\leq T(C) + b(C+1) + \ldots + bN$
  - $\leq a + b(C + C+1 + C+2 + \ldots + N)$
  - $T(N) = O(N^2)$
- Fortunately, *average case performance* is O(N log N) (see text for proof)

Properties of Quicksort

- No iterative version (without using a stack).
- Pure quicksort not good for small arrays.
- “In-place”, but uses auxiliary storage because of recursive calls.
- $O(n \log n)$ average case performance, but $O(n^2)$ worst case performance.

Folklore

- "Quicksort is the best in-memory sorting algorithm."

- Mergesort and Quicksort make different tradeoffs regarding the cost of comparison and the cost of a swap