CSE 326: Data Structures
AVL Trees

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Lecture 9

Balanced BST

Observation

- BST: the shallower the better!
- For a BST with *n* nodes
  - Average height is O(log *n*)
  - Worst case height is O(*n*)
- Simple cases such as insert(1, 2, 3, ..., n) lead to the worst case scenario

Solution: Require a **Balance Condition** that

1. ensures depth is O(log *n*) – strong enough!
2. is easy to maintain – not too strong!

Potential Balance Conditions

1. Left and right subtrees of the root have equal number of nodes
2. Left and right subtrees of the root have equal *height*
3. Left and right subtrees of *every node* have equal number of nodes
4. Left and right subtrees of *every node* have equal *height*

The AVL Balance Condition

Adelson-Velskii and Landis

AVL balance property:

Left and right subtrees of *every node* have *heights* **differing by at most 1**

- Ensures small depth
  - Will prove this by showing that an AVL tree of height h must have a lot of (i.e. $O(2^h)$) nodes
- Easy to maintain
  - Using single and double rotations

The AVL Tree Data Structure

Structural properties

1. Binary tree property (0,1, or 2 children)
2. Heights of left and right subtrees of *every node* **differ by at most 1**

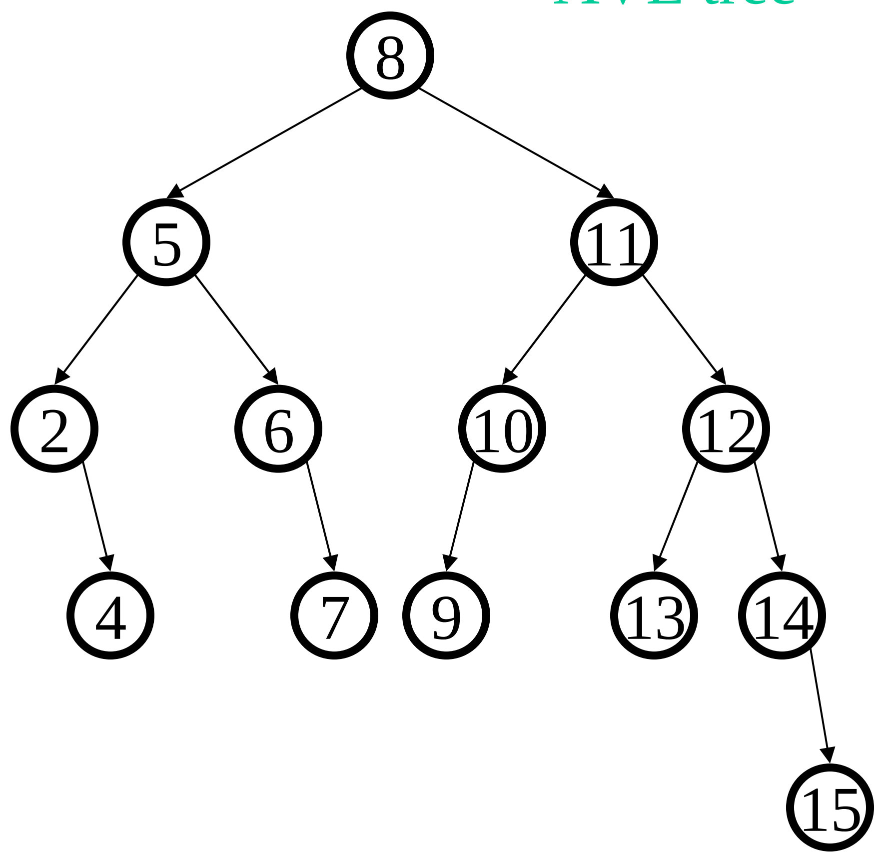
Result:

Worst case depth of any node is: O(log *n*)

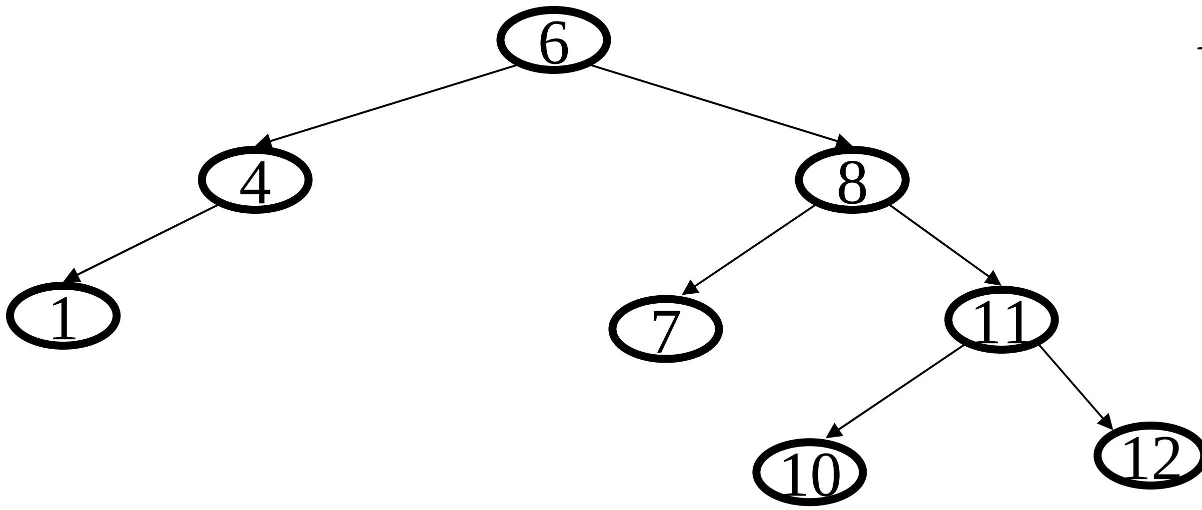
Ordering property

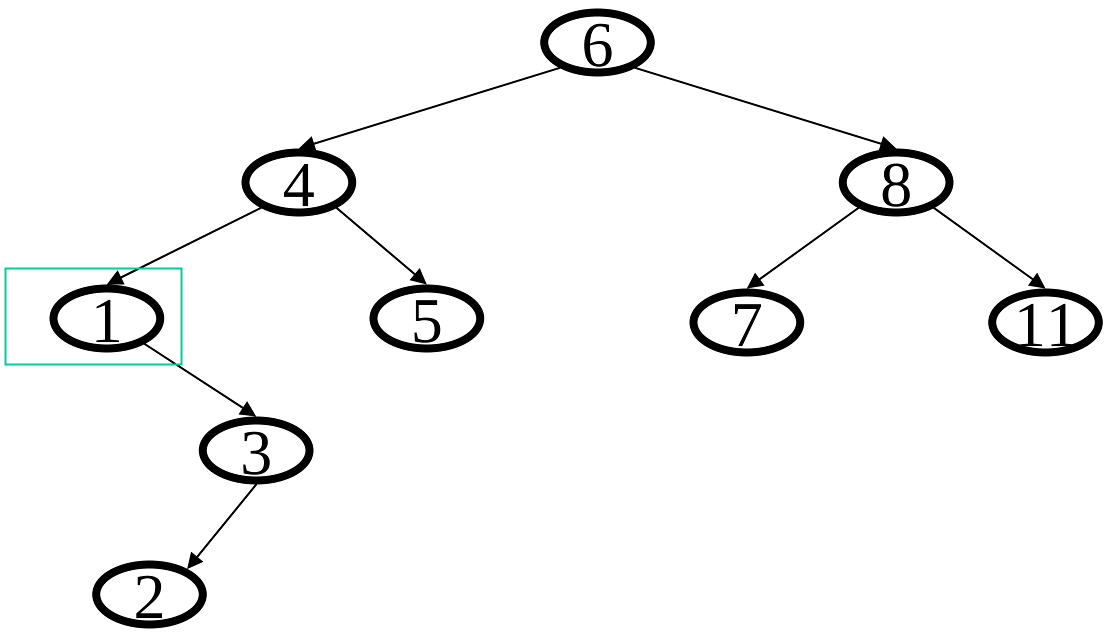
– Same as for BST

This is an AVL tree



AVL trees or not?





Proving Shallowness Bound

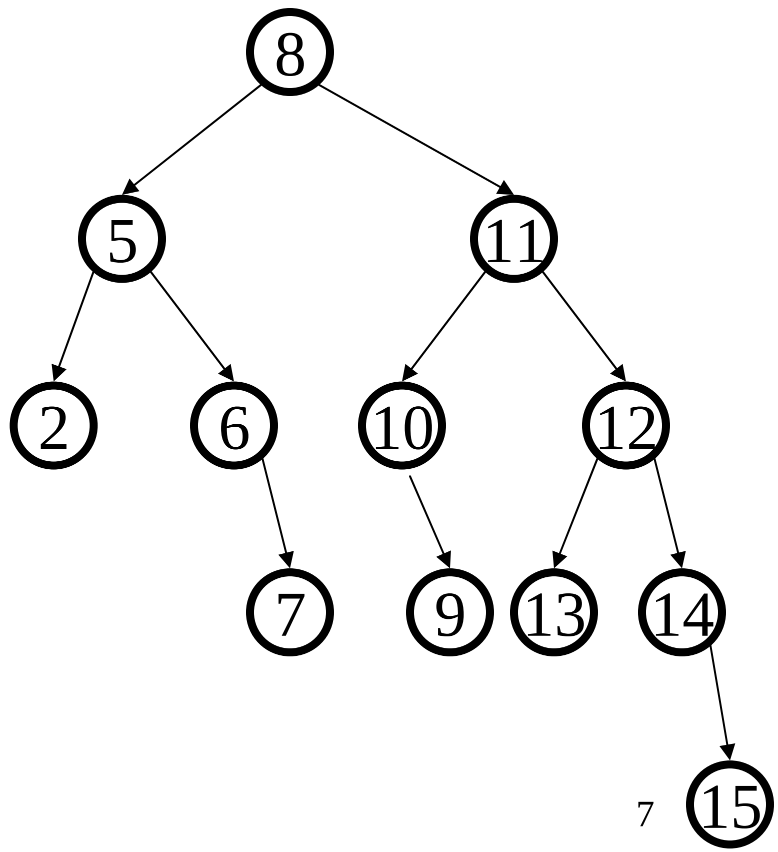
Let **S**(h) be the min # of nodes in an AVL tree of height h

Trees of height h = 1, 2, 3 ….

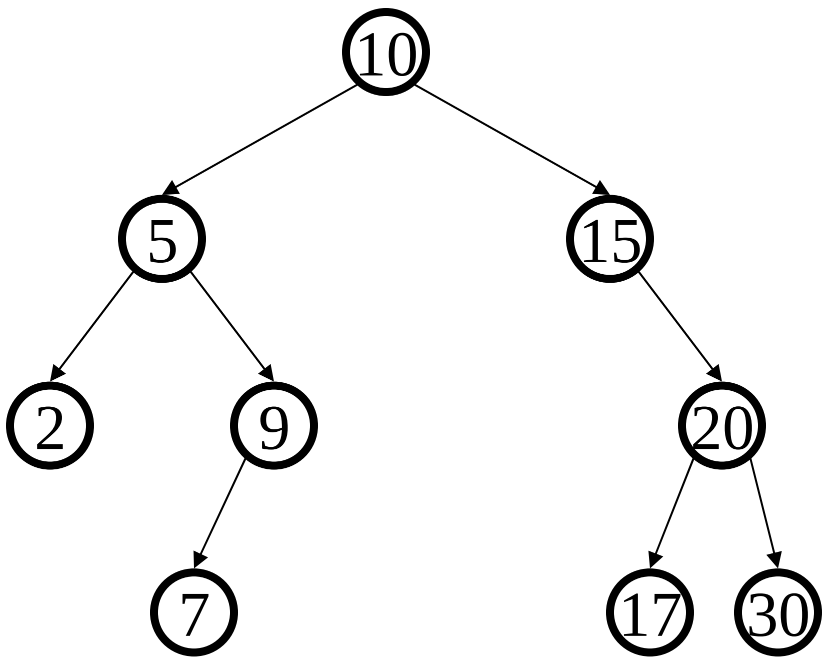
Claim: $\mathbf{S}(h) = \mathbf{S}(h\text{-}1) + \mathbf{S}(h\text{-}2) + 1$

Solution of recurrence: $\mathbf{S}(h) = O(2^h)$ (like Fibonacci numbers)

AVL tree of height h=4 with the min # of nodes (12)



Testing the Balance Property

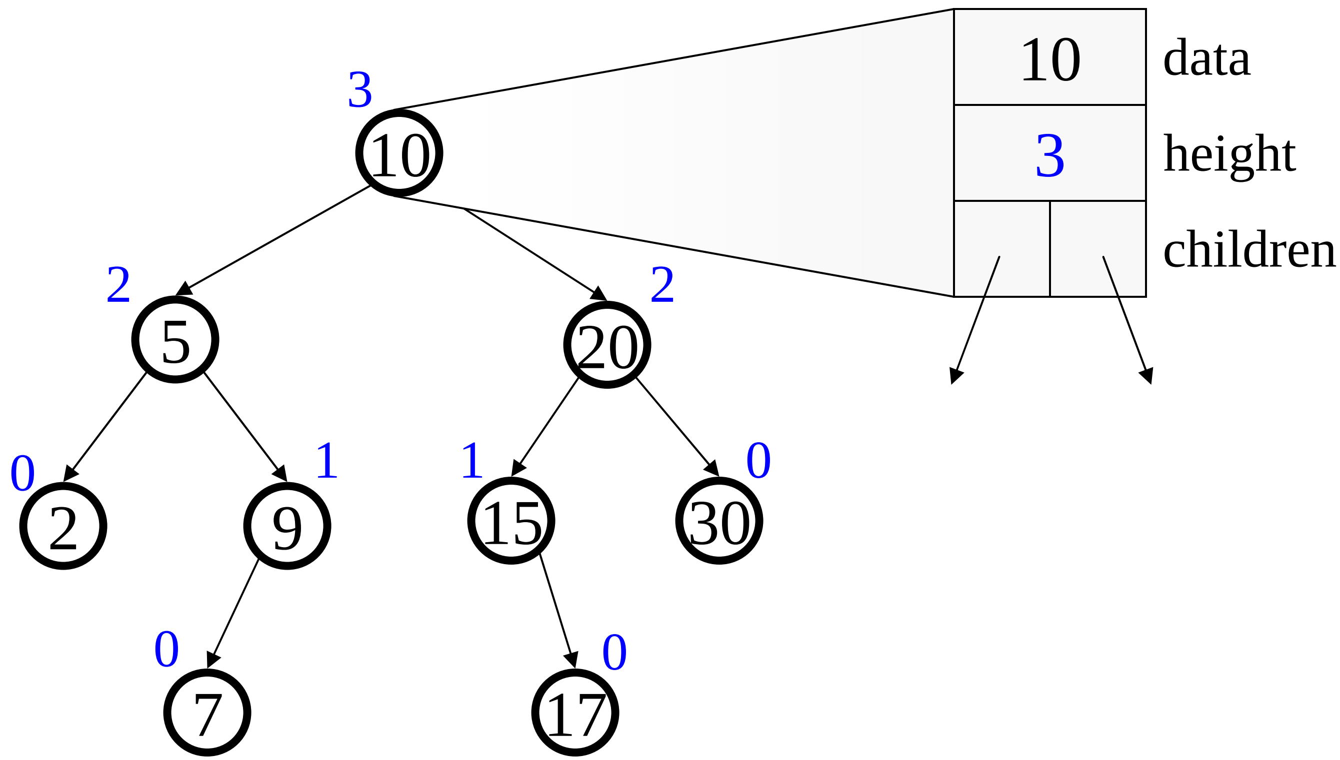


We need to be able to:

1. Track Balance
2. Detect Imbalance
3. Restore Balance

`NULL`s have height **`-1`**

An AVL Tree



Track height at all times. Why?

AVL trees: find, insert

- **AVL find**:
  - same as BST find.
- **AVL insert**:
  - same as BST insert, *except* may need to "fix" the AVL tree after inserting new value.

AVL tree insert

Let *x* be the node where an imbalance occurs.

Four cases to consider. The insertion is in the

1. left subtree of the left child of *x*.
2. right subtree of the left child of *x*.
3. left subtree of the right child of *x*.
4. right subtree of the right child of *x*.

Idea: Cases 1 & 4 are solved by a single rotation.
Cases 2 & 3 are solved by a double rotation.

Bad Case #1

Insert(6)

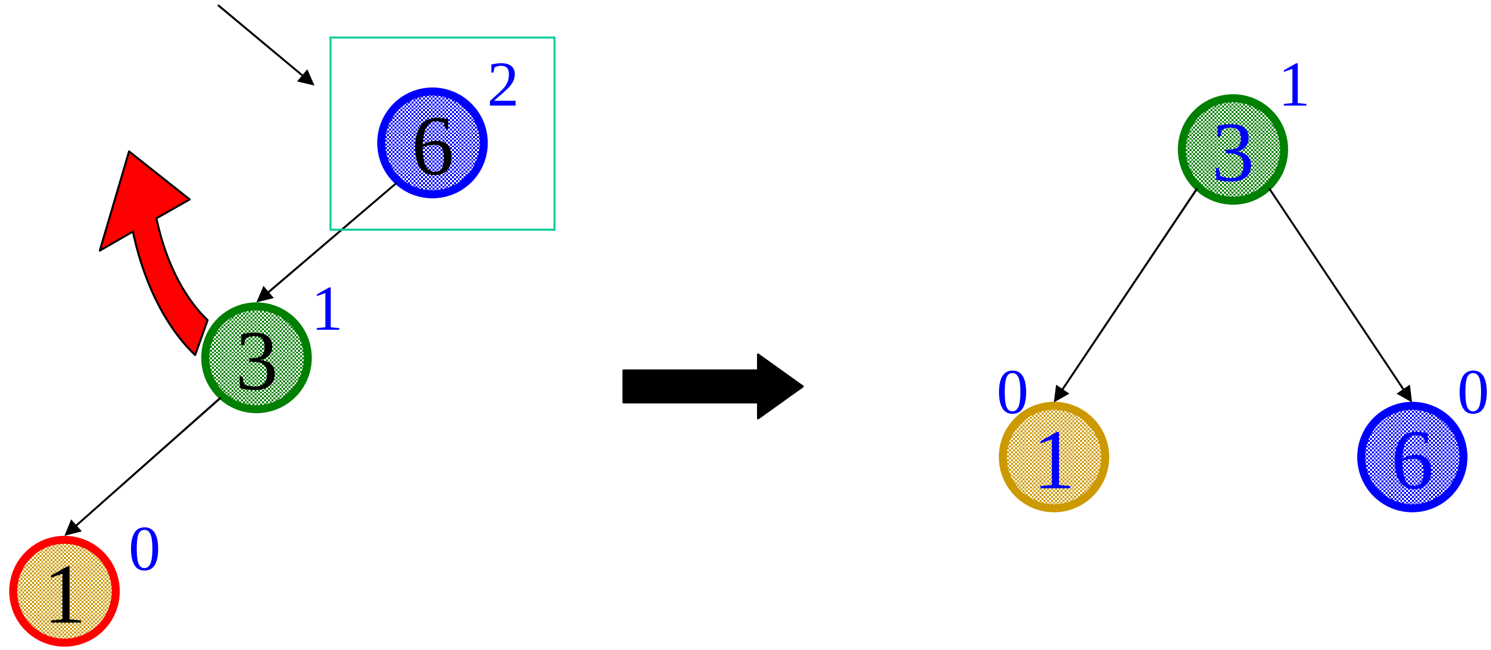
Insert(3)

Insert(1)

Where is AVL property violated?

Fix: Apply Single Rotation

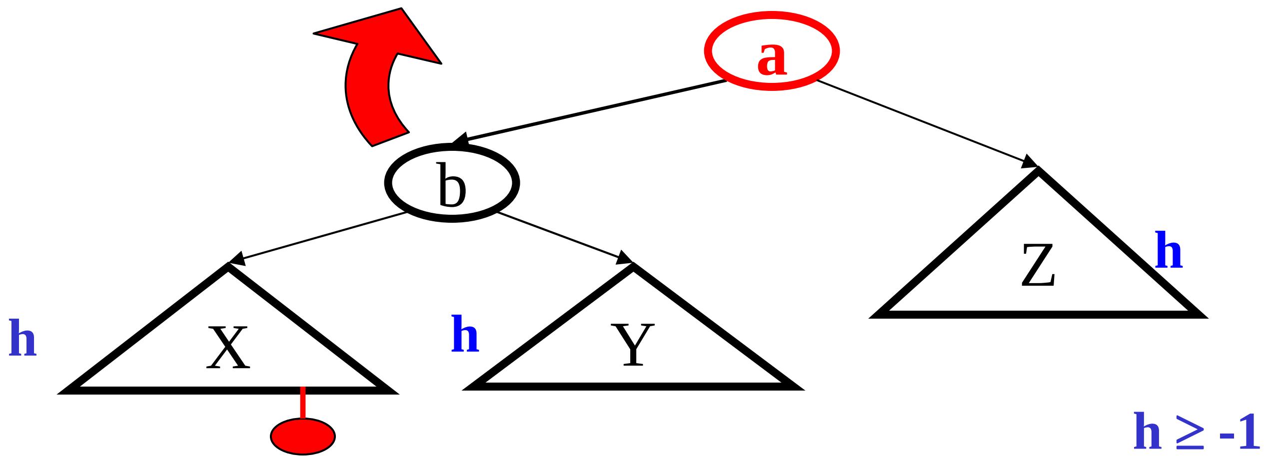
AVL Property violated at this node (x)



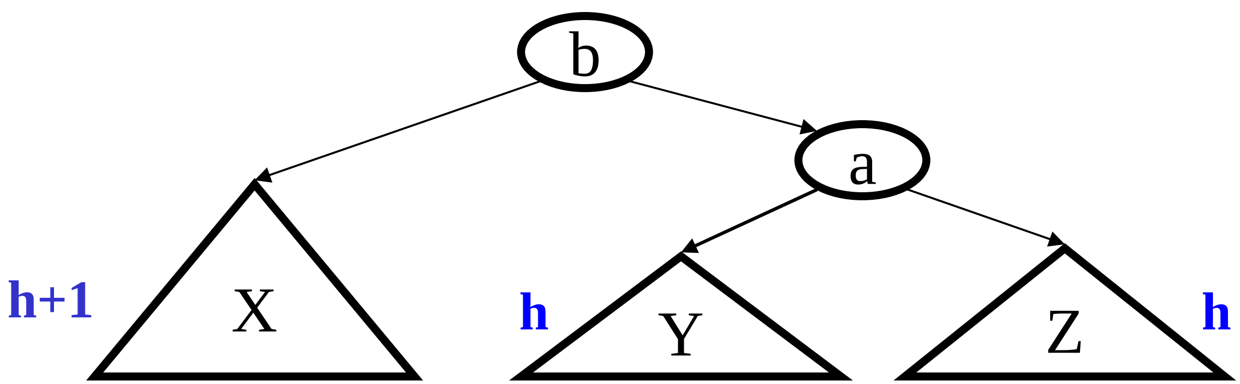
Single Rotation:

1. Rotate between x and child

Single rotation in general



X < b < Y < a < Z



Height of tree before?   Height of tree after? Effect on Ancestors?

Bad Case #2

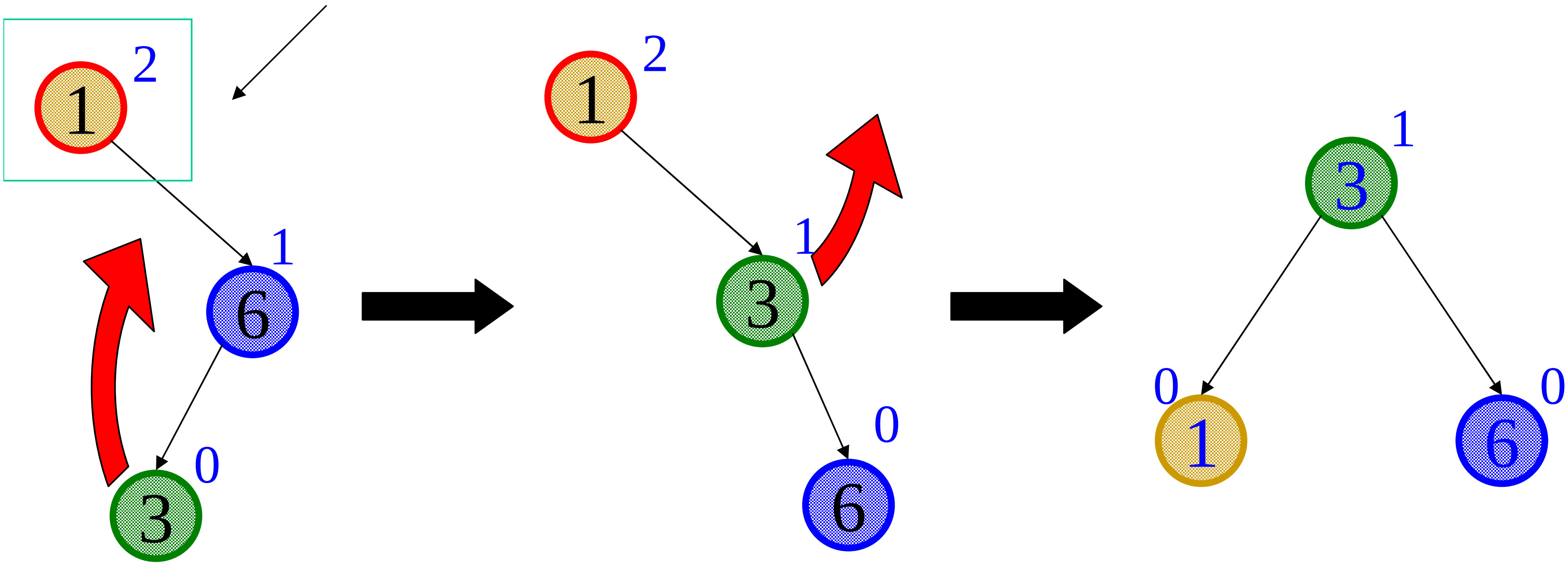
Insert(1)

Insert(6)

Insert(3)

Fix: Apply Double Rotation

AVL Property violated at this node (x)

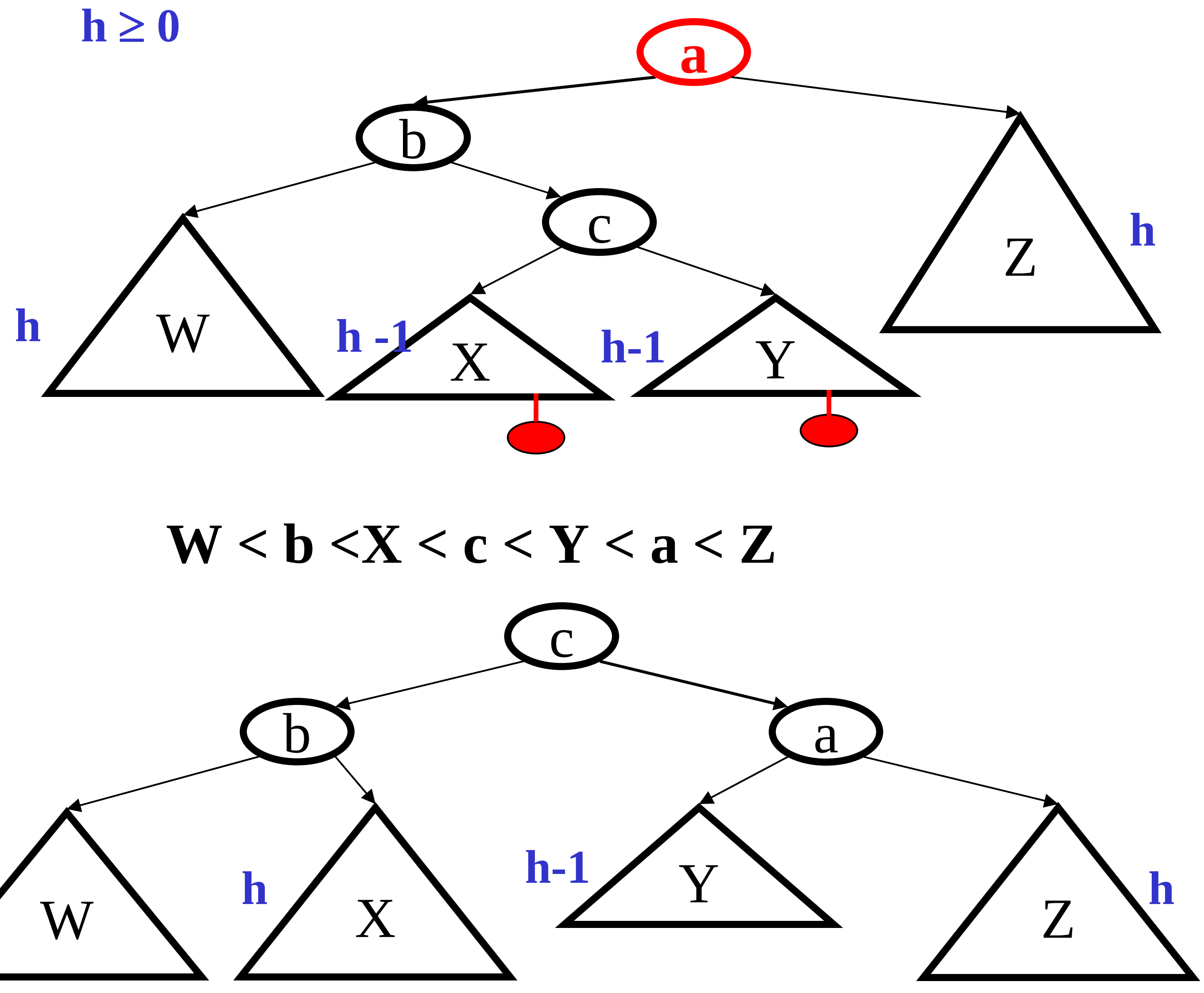


Intuition: 3 must become root

Double Rotation

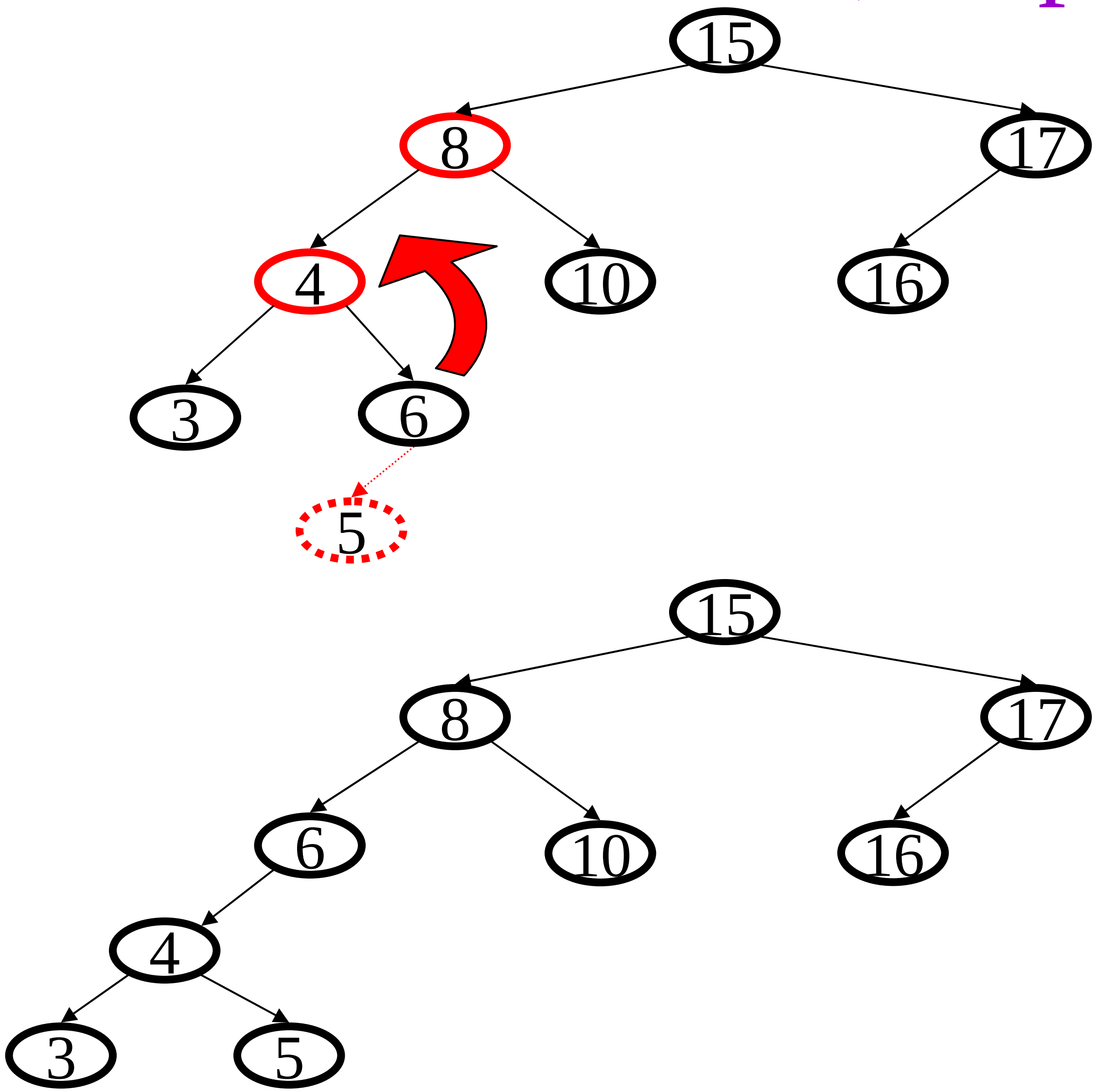
1. Rotate between x's child and grandchild
2. Rotate between x and x's new child

Double rotation in general

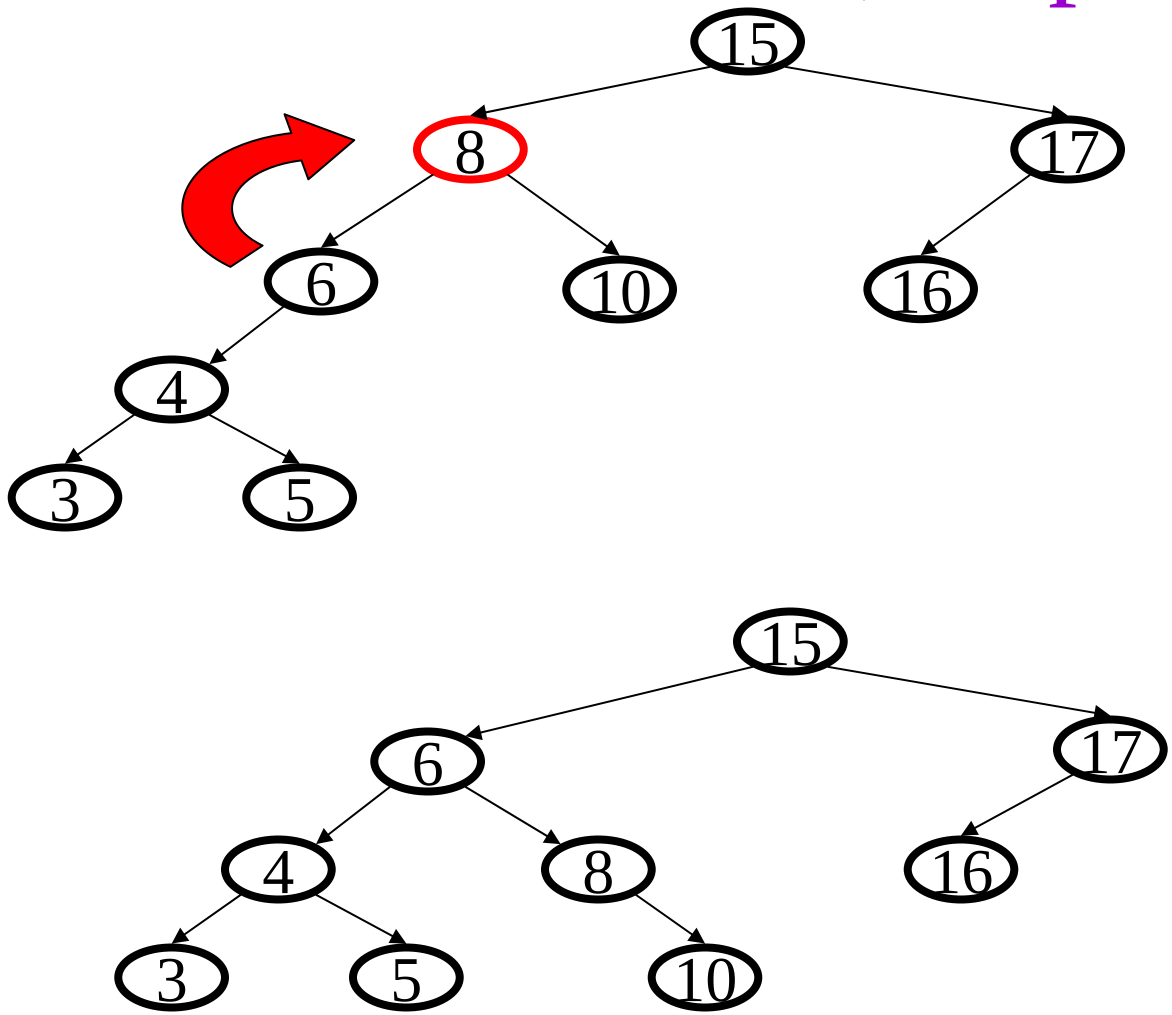


Height of tree before? Height of tree after? Effect on Ancestors?

Double rotation, step 1



Double rotation, step 2



Imbalance at node X

Single Rotation

1. Rotate between x and child

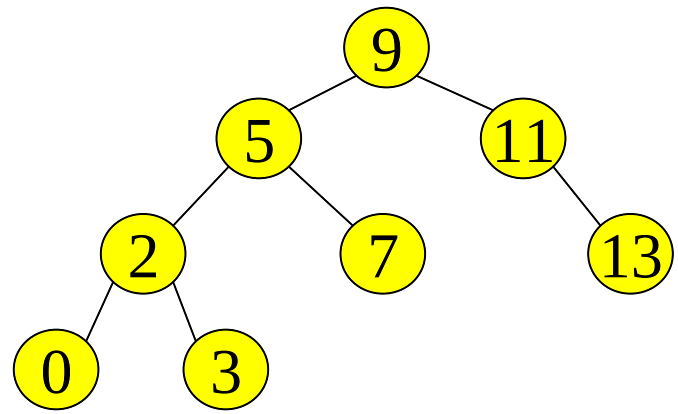
Double Rotation

1. Rotate between x's child and grandchild
2. Rotate between x and x's new child

Single and Double Rotations:

Inserting what integer values would cause the tree to need a:

1. single rotation?

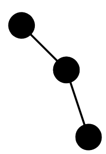


2. double rotation?

3. no rotation?

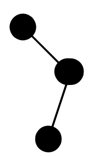
Insertion into AVL tree

1. Find spot for new key
2. Hang new node there with this key
3. Search back up the path for imbalance
4. If there is an imbalance:



case #1: Perform single rotation and exit

Zig-zig



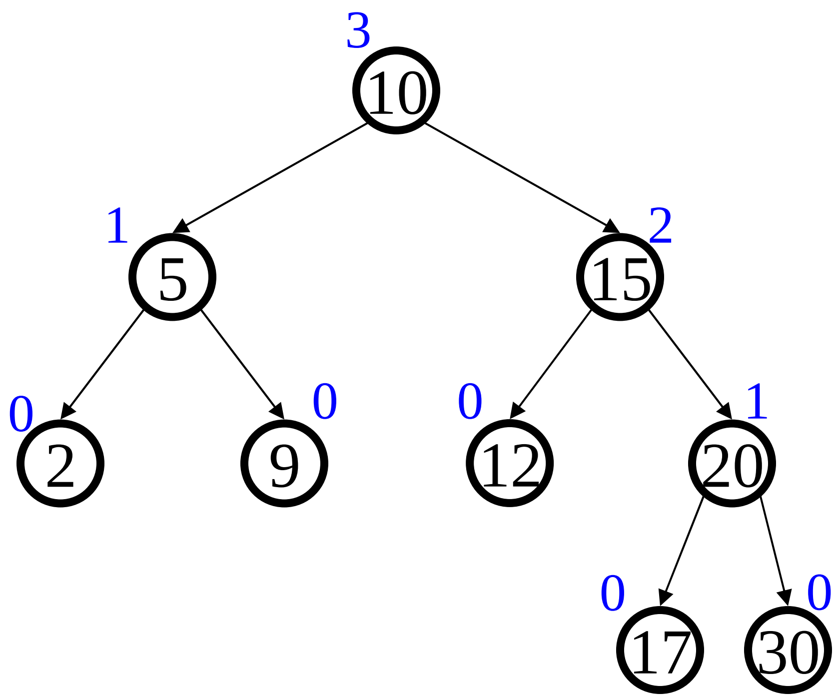
case #2: Perform double rotation and exit

Zig-zag

Both rotations keep the subtree height unchanged.
Hence only one rotation is sufficient!

Easy Insert

Insert(3)

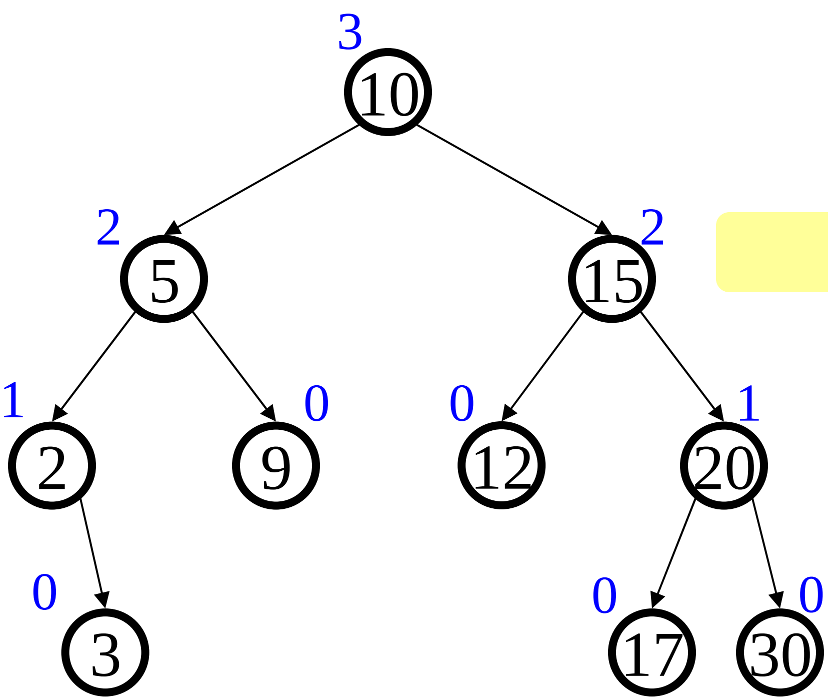


Unbalanced?

No

Hard Insert (Bad Case #1)

Insert(33)



Imbalance

Unbalanced?

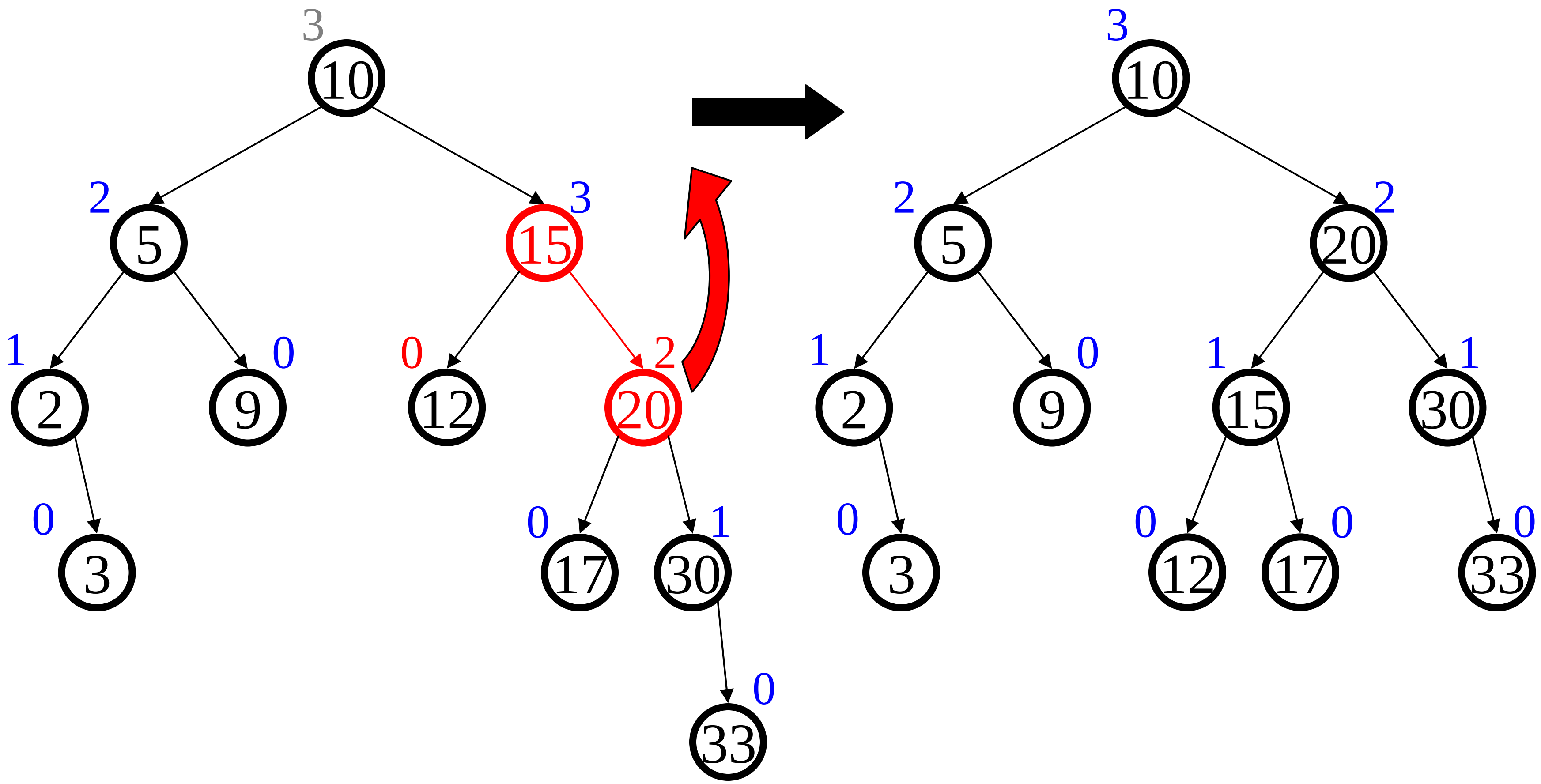
Yes, at 15

How to fix?

Single rotate

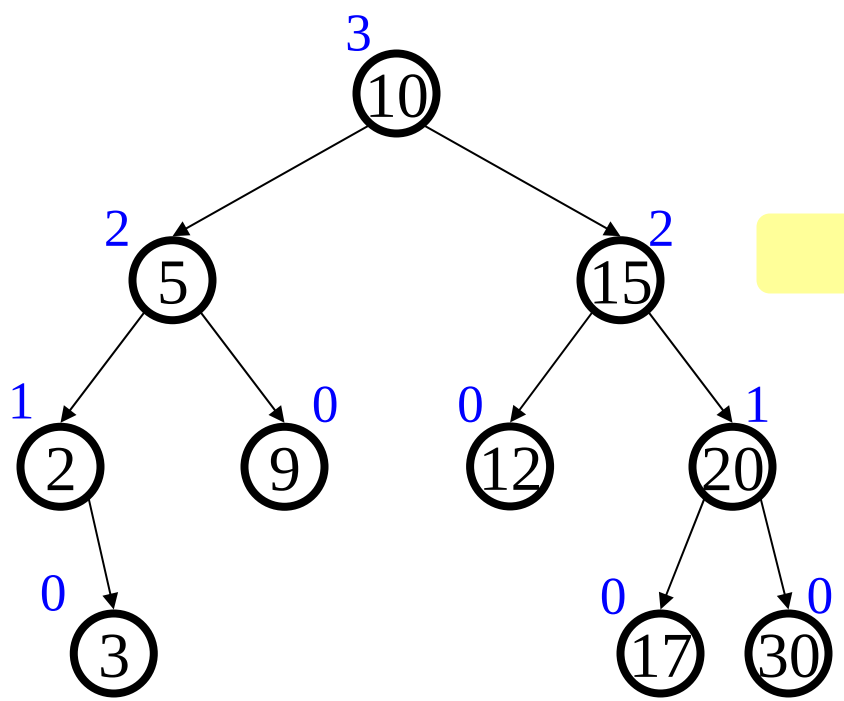
Zig-zig

Single Rotation



Hard Insert (Bad Case #2)

Insert(18)



Imbalance

Unbalanced?

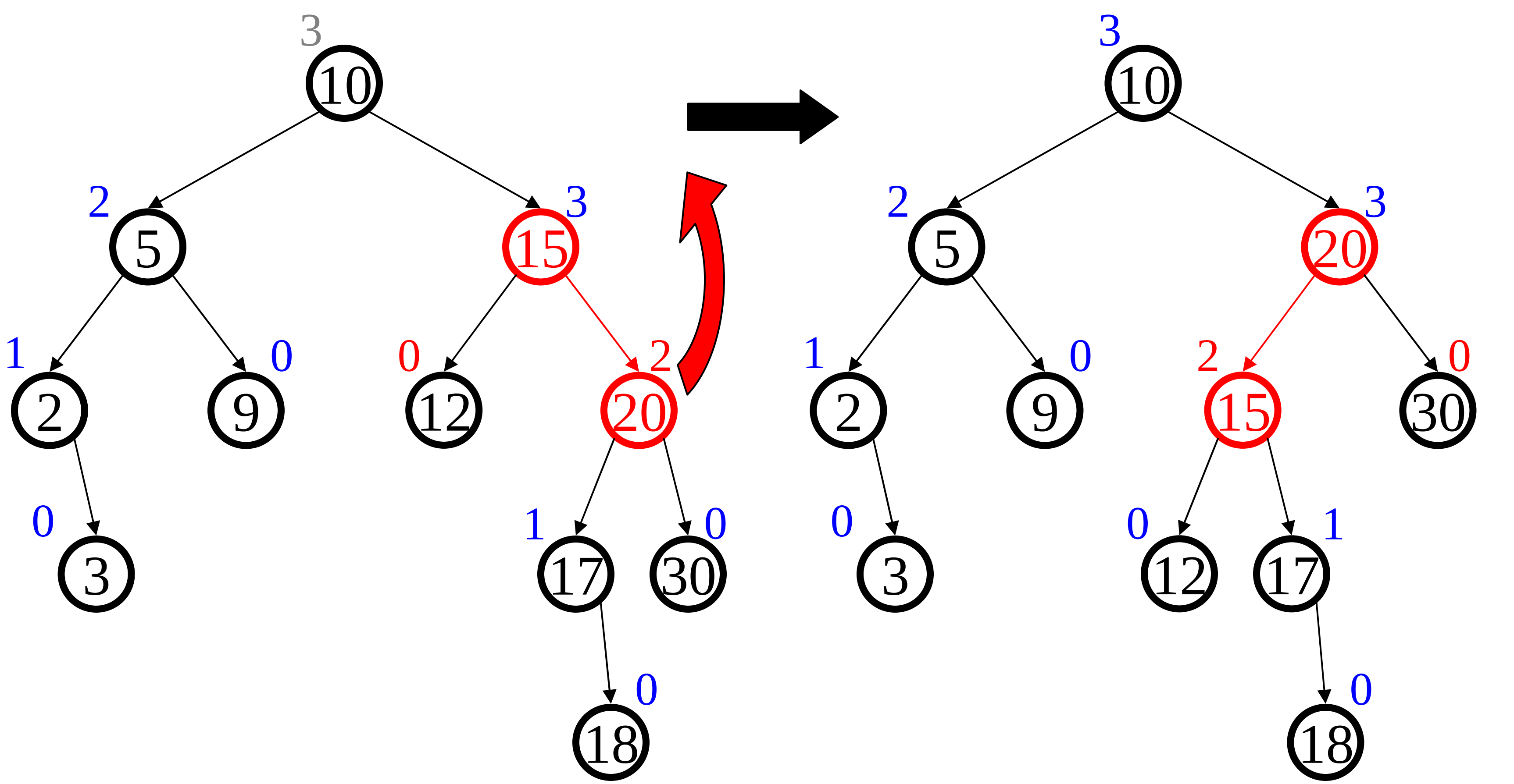
Yes, at 15

How to fix?

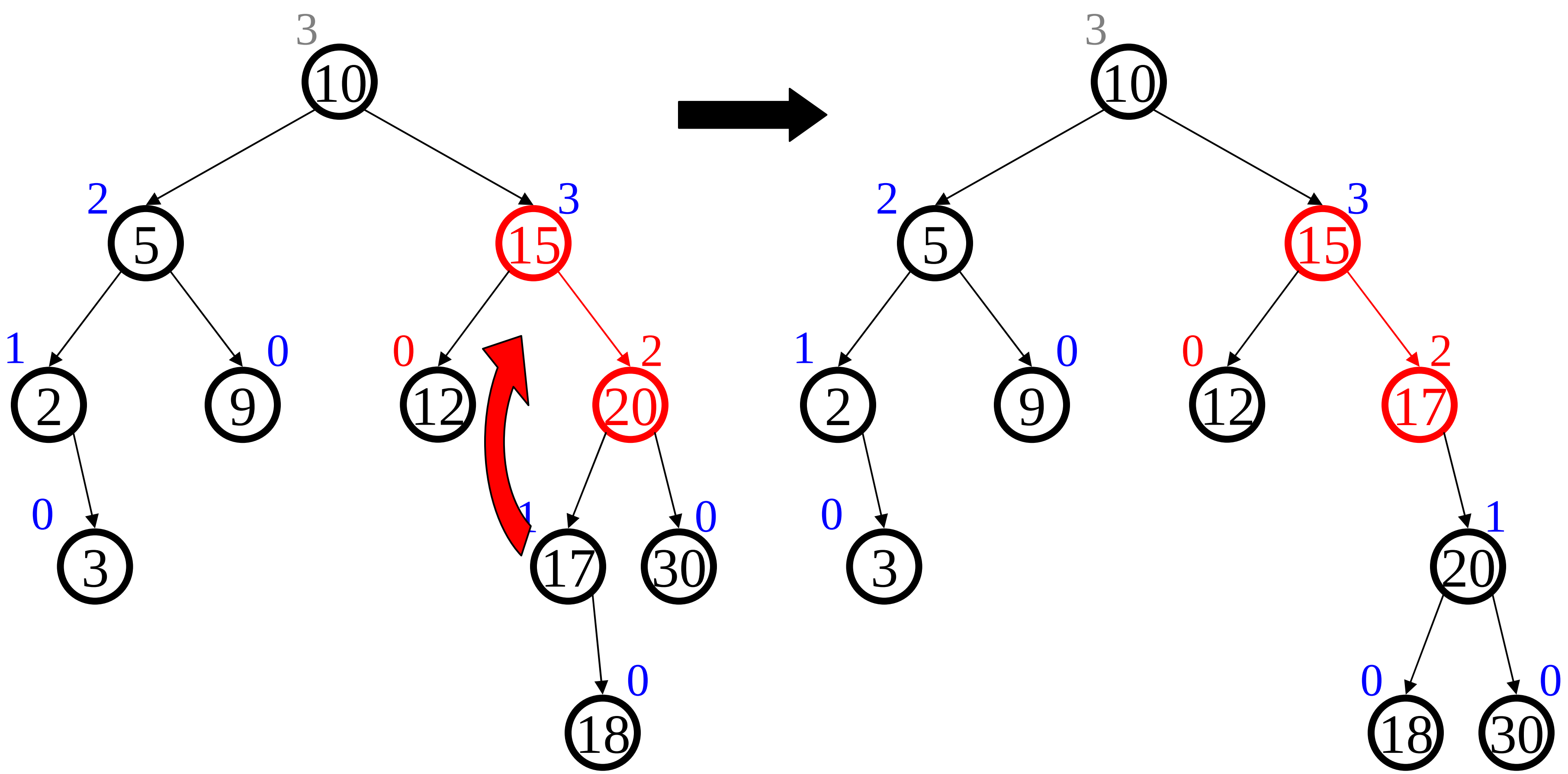
Double rotate

Zig-zag

Single Rotation (oops!)

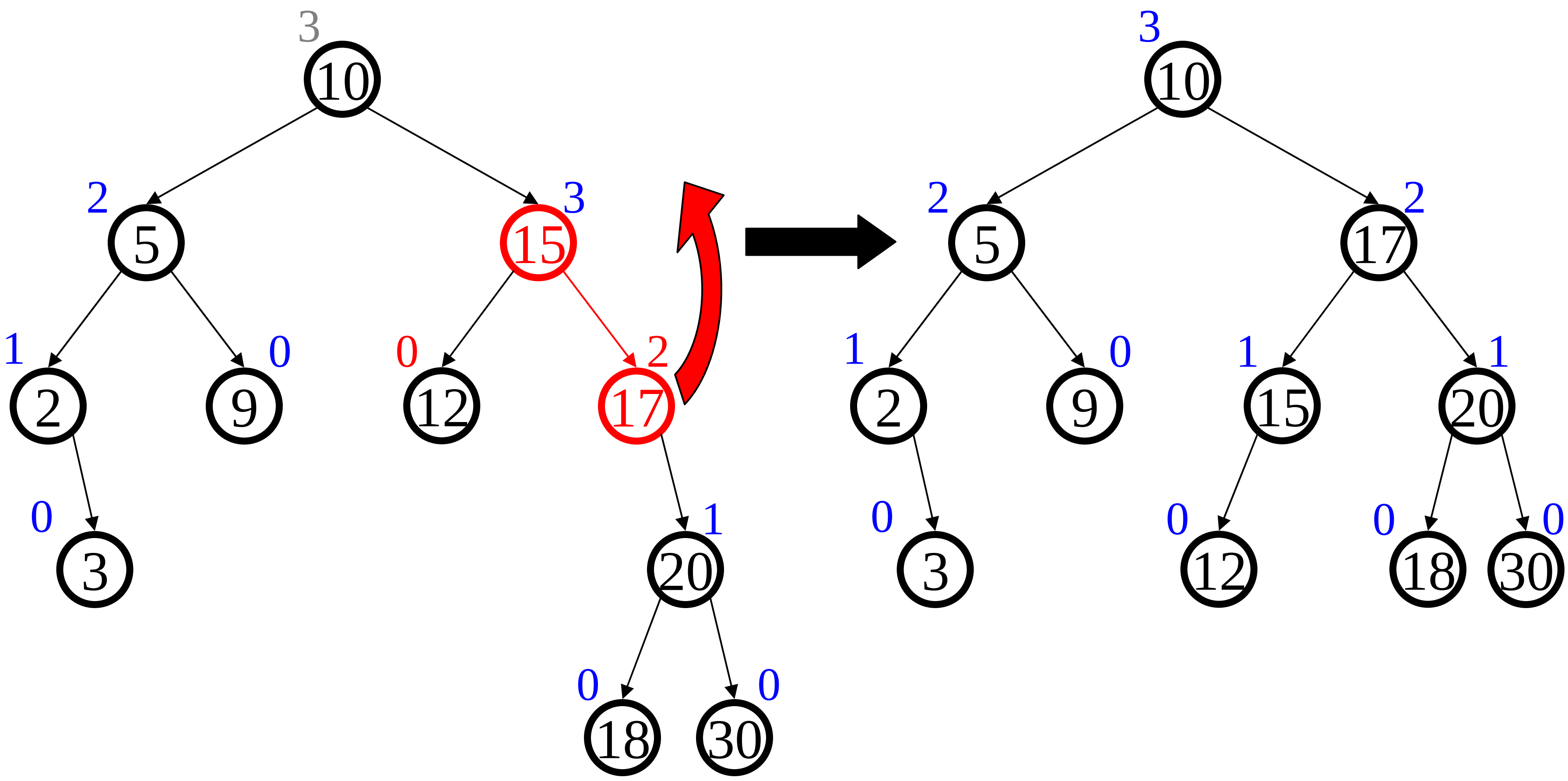


Double Rotation (Step #1)



Still unbalanced.
But like zig-zig tree!

Double Rotation (Step #2)



Insert into an AVL tree: 5, 8, 9, 4, 2, 7, 3, 1