CSE 326: Data Structures

Priority Queues
Leftist Heaps & Skew Heaps

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on behalf of James Fogarty

Autumn 2007

Outline

- Announcements
- Leftist Heaps
- Skew Heaps (if there's time)
  - Reading: Weiss, Ch. 6

Announcements

- Written HW #2 – out now, due Friday
- Project #1 due Wednesday at midnight
- Project #2 Phase A out now
  - Can work in pairs; start figuring out who you'd like to work with or whether you want to go alone
  - Let us know by Friday, Oct 12

New Heap Operation: Merge

Given two heaps, merge them into one heap

- first attempt: insert each element of the smaller heap into the larger.

  runtime:

- second attempt: concatenate binary heaps' arrays and run buildHeap.

  runtime:

Leftist Heaps

Idea:

Focus all heap maintenance work in one small part of the heap

Leftist heaps:

1. Most nodes are on the left
2. All the merging work is done on the right

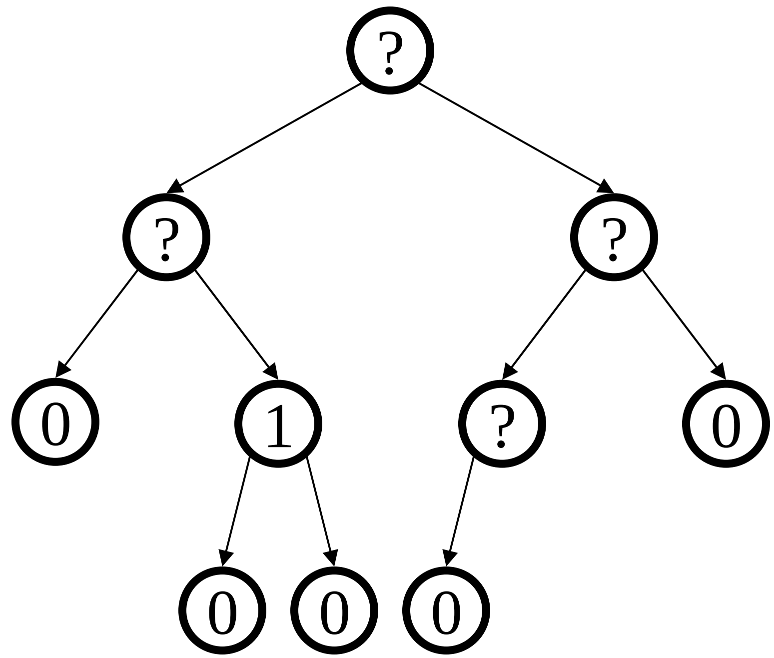
Definition: Null Path Length

null path length (npl) of a node x = the number of nodes between x and a null in its subtree

OR

npl(x) = min distance to a descendant with 0 or 1 children

- *npl*(null) = -1
- *npl*(leaf) = 0
- *npl*(single-child node) = 0



Equivalent definitions:

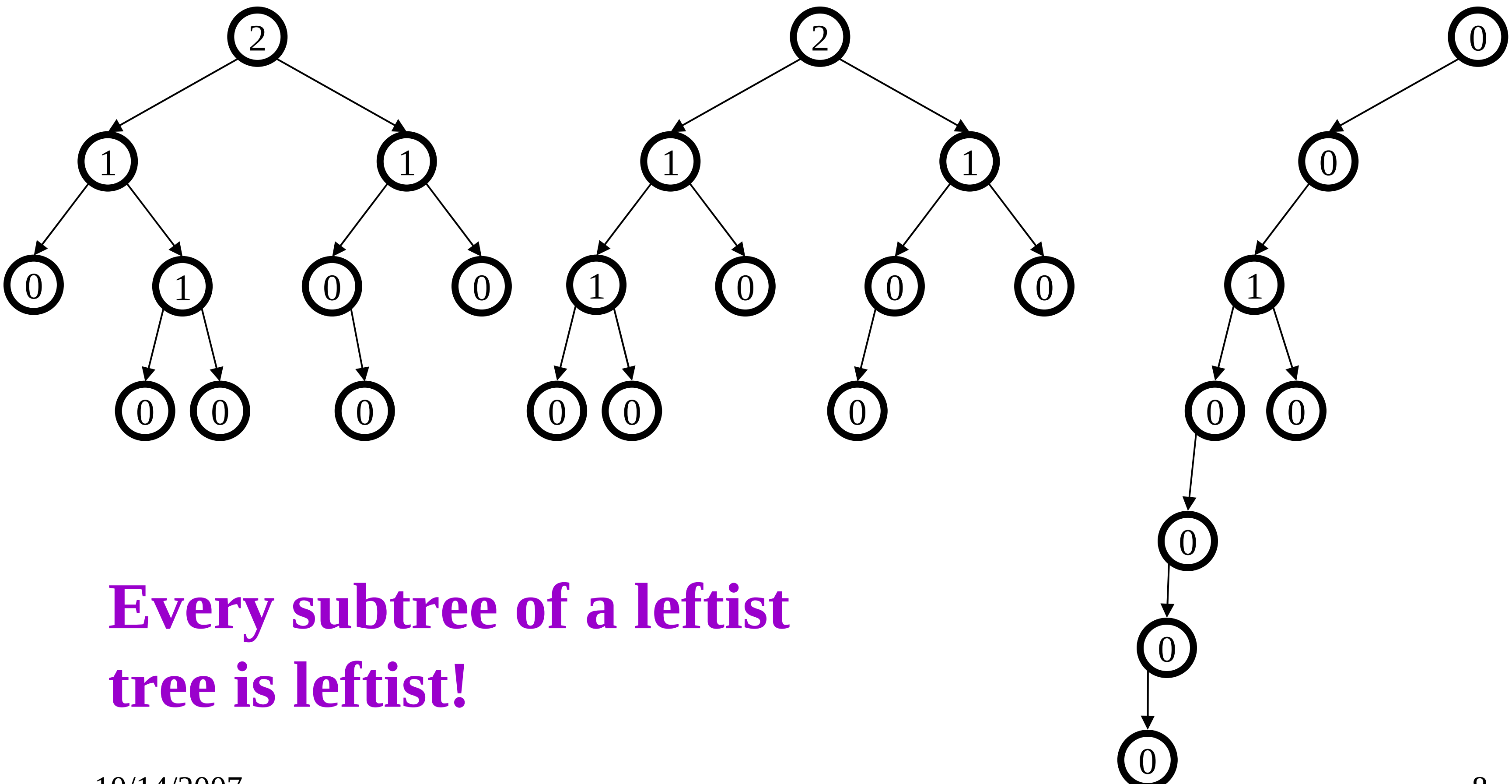
1. *npl(x)* is the height of largest complete subtree rooted at x
2. *npl(x)* = 1 + min{*npl*(left(x)), *npl*(right(x))}

Leftist Heap Properties

- Heap-order property
  - parent's priority value is ≤ to childrens' priority values
  - result: minimum element is at the root
- Leftist property
  - For every node x, $npl(\text{left}(x)) \geq npl(\text{right}(x))$
  - result: tree is at least as "heavy" on the left as the right

Are leftist trees…
complete?
balanced?

Are These Leftist?

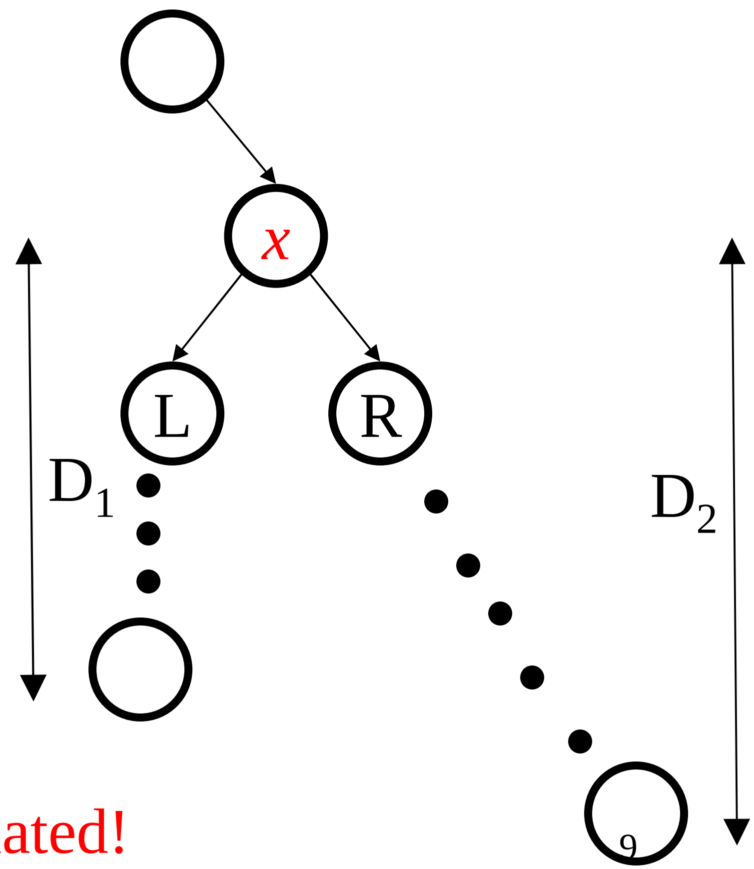


Every subtree of a leftist tree is leftist!

Right Path in a Leftist Tree is Short (#1)

<u>Claim</u>: The right path is as short as *any* in the tree.

<u>Proof</u>: (By contradiction)



Pick a shorter path: $D_1 < D_2$
Say it diverges from right path at x

$npl(L) \leq D_1$-1 because of the path of length D_1-1 to null

$npl(R) \geq D_2$-1 because every node on right path is leftist

Leftist property at x violated!

Right Path in a Leftist Tree is Short (#2)

<u>Claim</u>: If the right path has r nodes, then the tree has at least
2^r-1 nodes.

<u>Proof</u>: (By induction)

Base case : $r=1$. Tree has at least $2^1-1 = 1$ node

Inductive step : assume true for $r' < r$. Prove for tree with right path at least r.

1. Right subtree: right path of $r-1$ nodes
$\Rightarrow$ $2^{r-1}-1$ right subtree nodes (by induction)
2. Left subtree: also right path of length at least $r-1$ (by previous slide) $\Rightarrow$ $2^{r-1}-1$ left subtree nodes (by induction)

Total tree size: $(2^{r-1}-1) + (2^{r-1}-1) + 1 = 2^r-1$

Why do we have the leftist property?

Because it guarantees that:

- the *right path is really short* compared to the number of nodes in the tree
- A leftist tree of N nodes, has a right path of at most lg (N+1) nodes

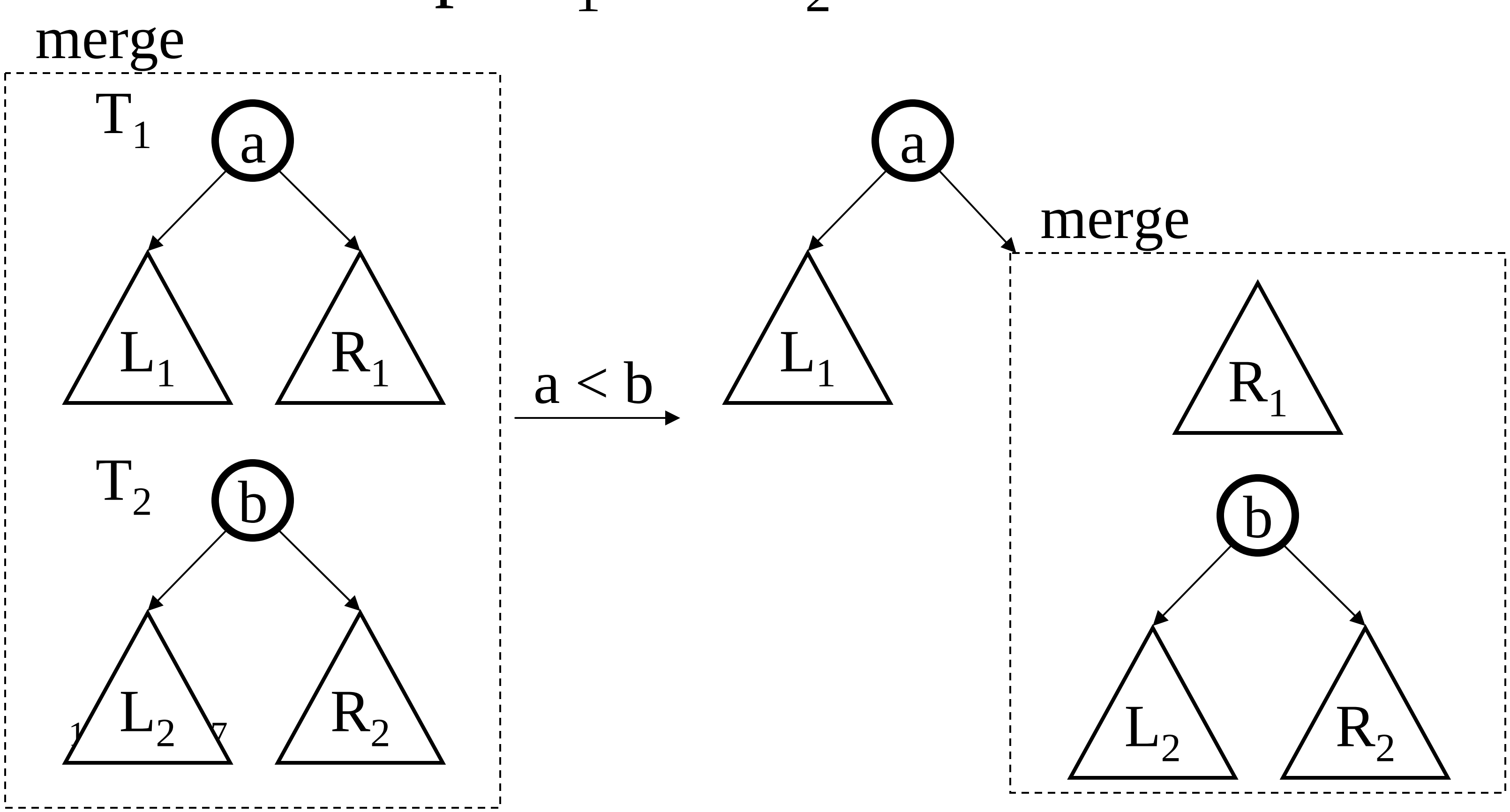
Idea – perform all work on the right path

Merge two heaps (basic idea)

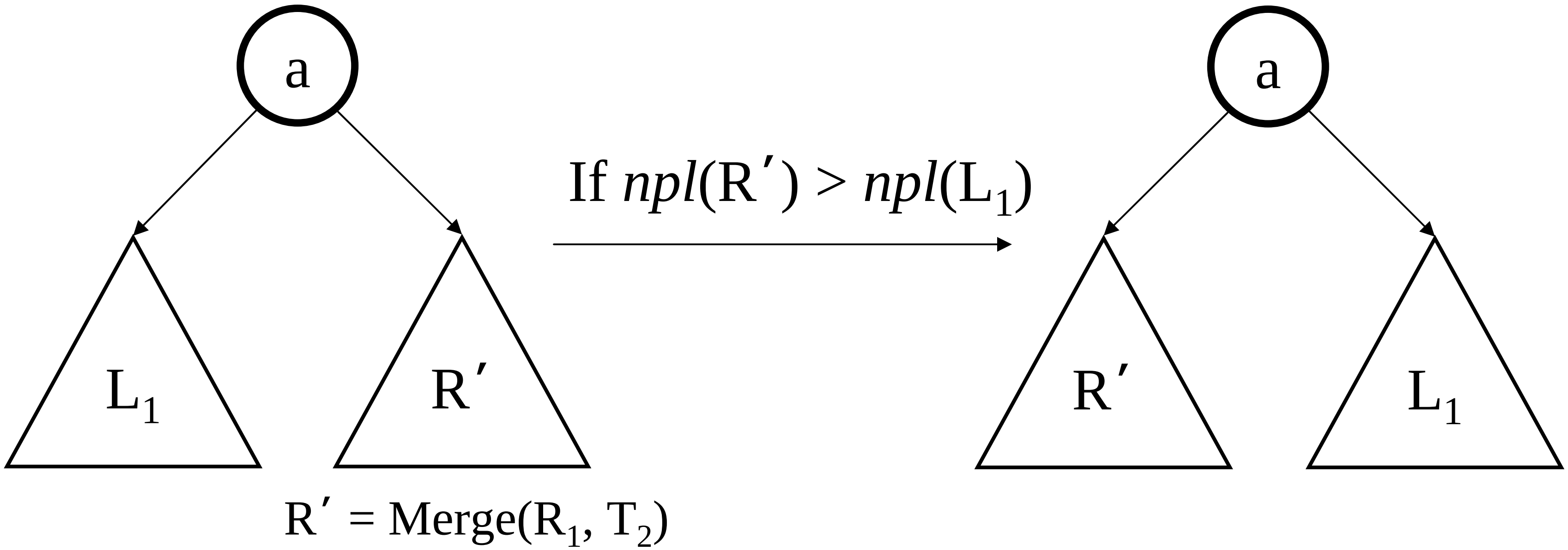
- Put the smaller root as the new root,
- Hang its left subtree on the left.
- <u>Recursively</u> merge its right subtree and the other tree.

Merging Two Leftist Heaps

- merge(T_1,T_2) returns one leftist heap containing all elements of the two (distinct) leftist heaps T_1 and T_2



Merge Continued



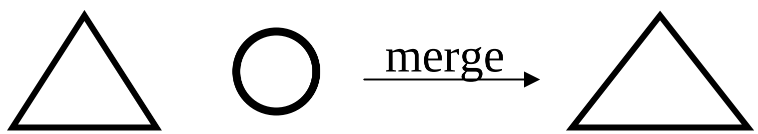
runtime:

Let's do an example, but first… Other Heap Operations

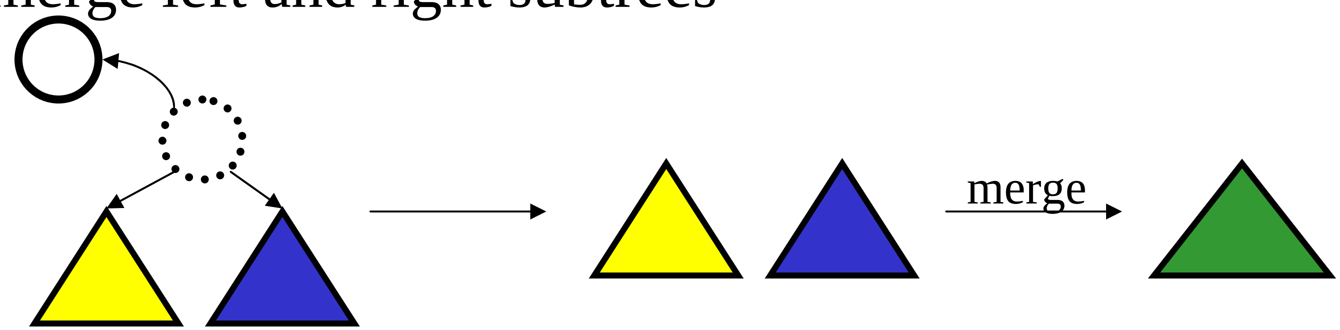
- insert ?
- deleteMin ?

Operations on Leftist Heaps

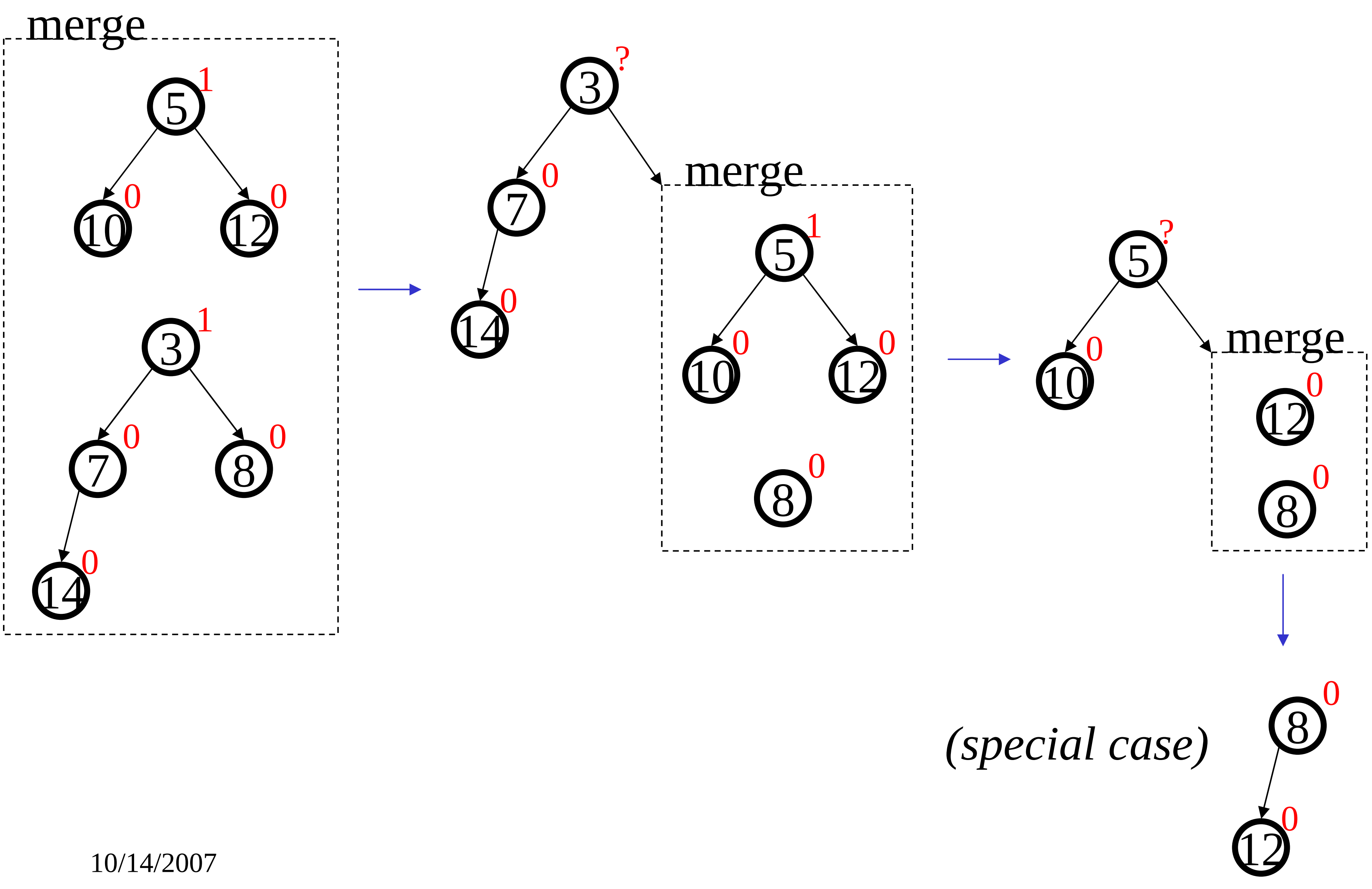
- merge with two trees of total size n: O(log n)
- insert with heap size n: O(log n)
  - pretend node is a size 1 leftist heap
  - insert by merging original heap with one node heap



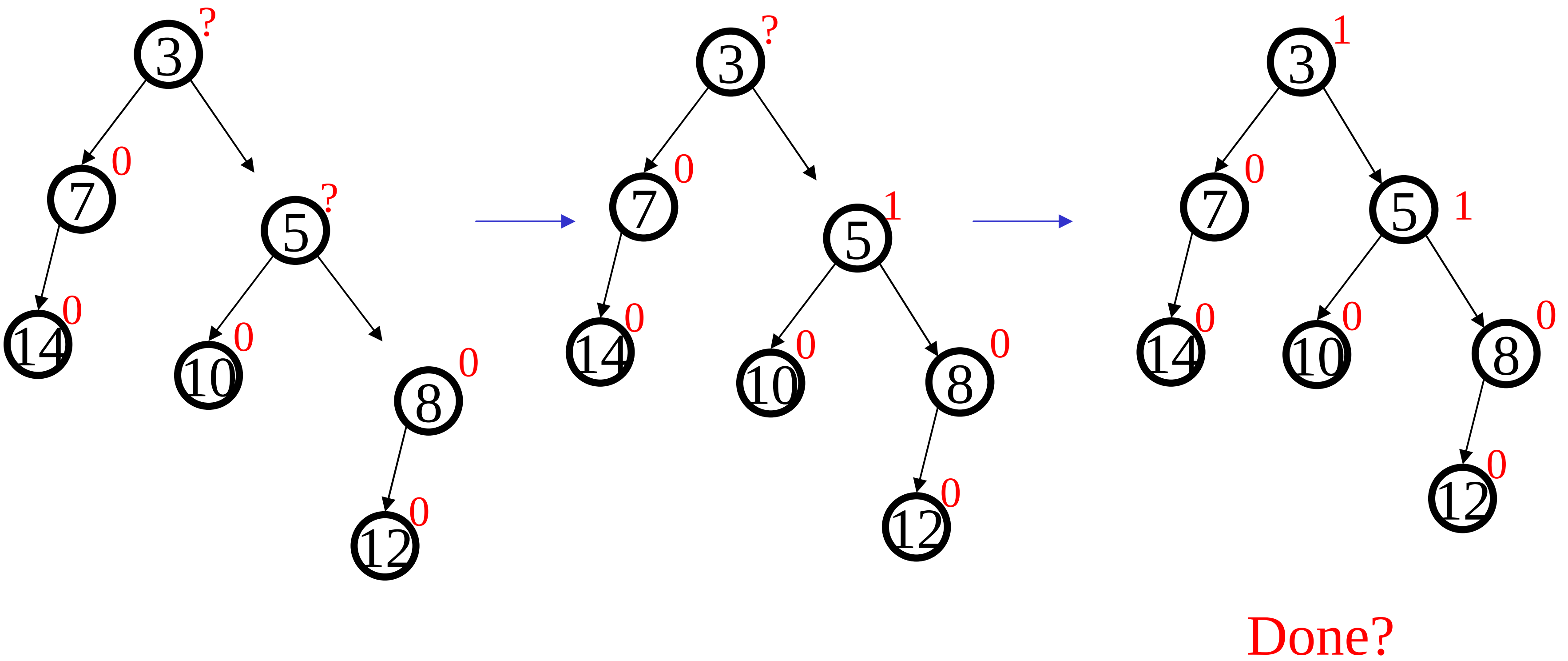
- deleteMin with heap size n: O(log n)
  - remove and return root
  - merge left and right subtrees



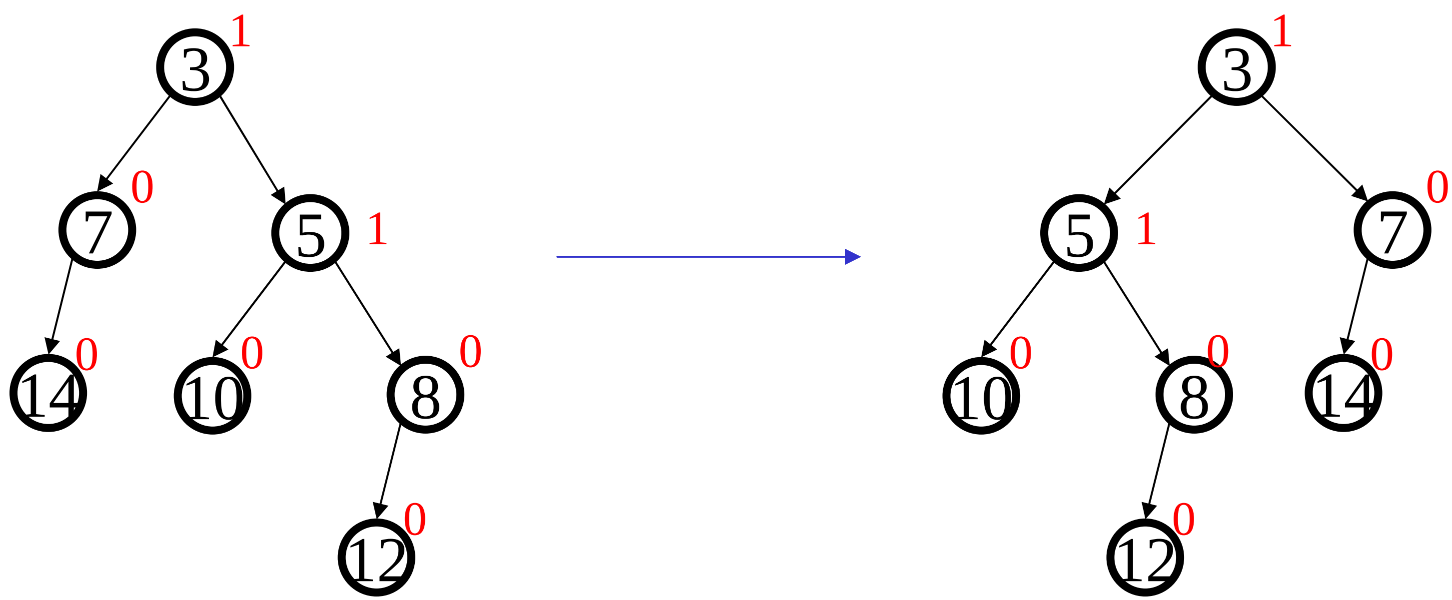
Leftest Merge Example



Sewing Up the Example



Finally…



Leftist Heaps: Summary

Good

-
-

Bad

-
-

Random Definition: Amortized Time

am·or·tized time:

Running time limit resulting from "writing off" expensive runs of an algorithm over multiple cheap runs of the algorithm, usually resulting in a lower <u>overall</u> running time than indicated by the worst possible case.

If M operations take total O(M log N) time,
amortized time per operation is O(log N)

Difference from **average time:**

Skew Heaps

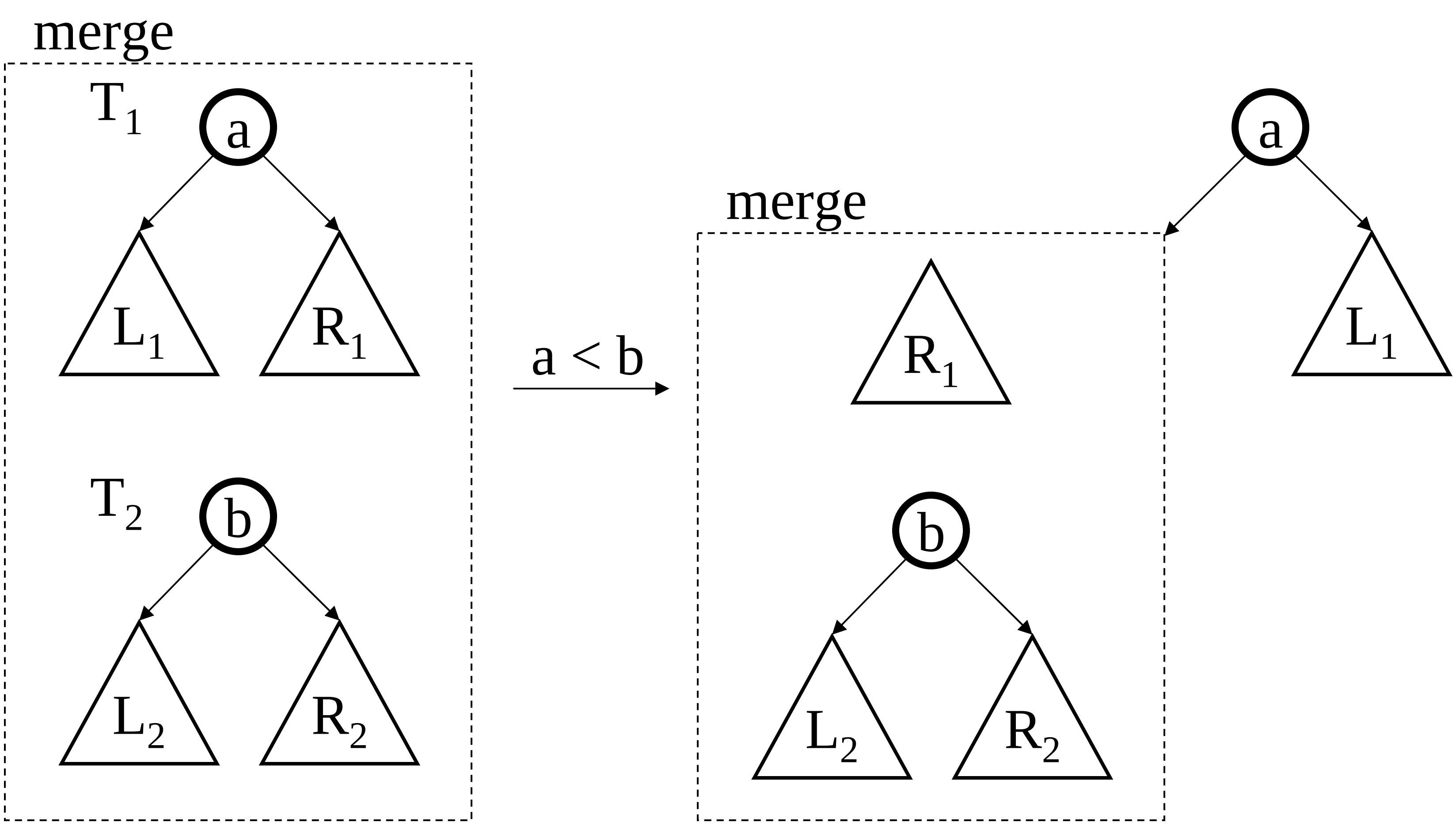
Problems with <u>leftist</u> heaps

- extra storage for npl
- extra complexity/logic to maintain and check npl
- right side is "often" heavy and requires a switch

Solution: <u>skew</u> heaps

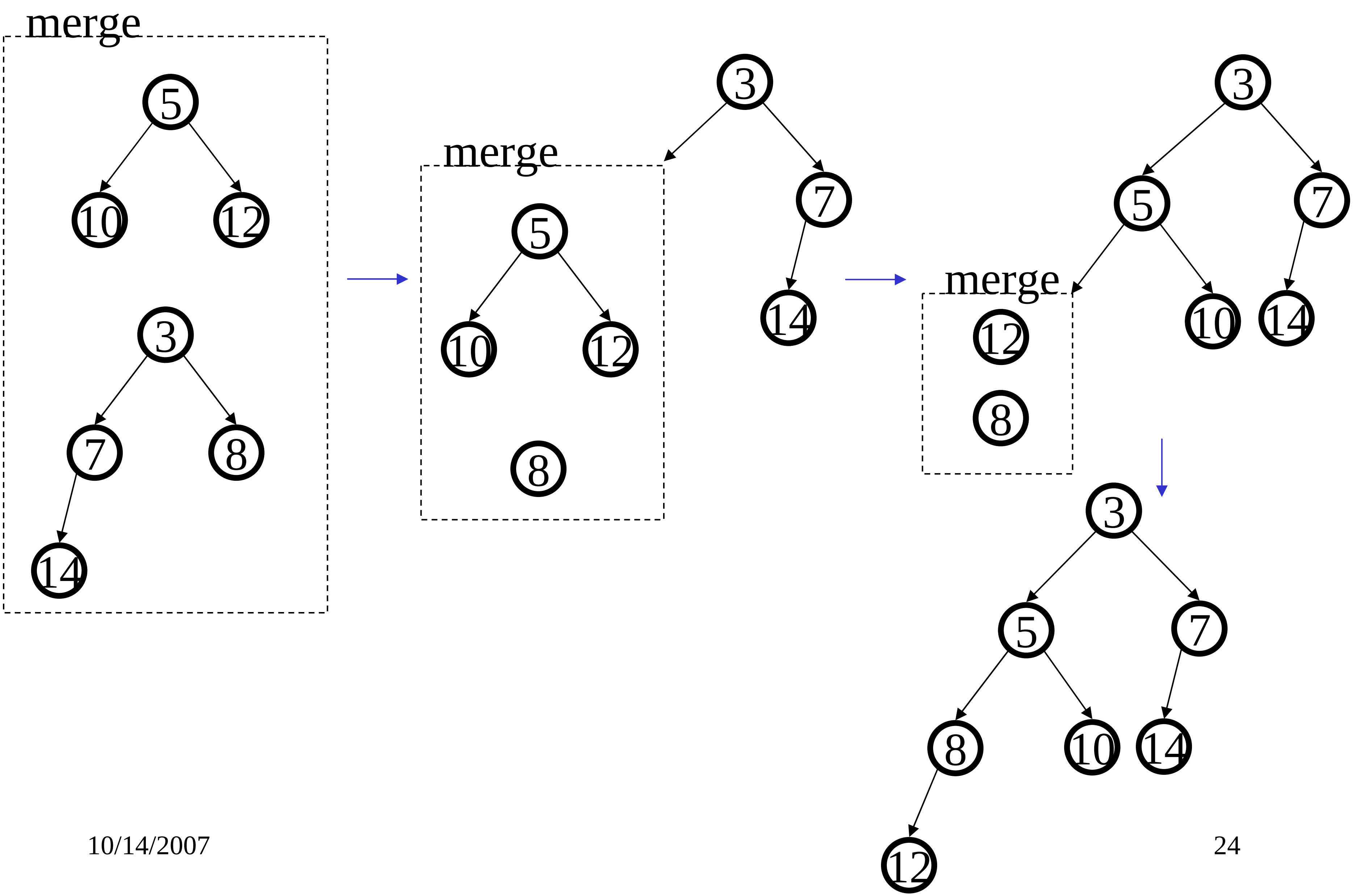
- "blindly" adjusting version of leftist heaps
- merge *always* switches children when fixing right path
- <u>amortized time</u> for: merge, insert, deleteMin = O(log n)
- however, <u>worst case time</u> for all three = O(n)

Merging Two Skew Heaps



Only one step per iteration, with children *always* switched

Example


merge
5
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merge
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merge
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12

Skew Heap Code

```
void merge(heap1, heap2) {
  case {
     heap1 == NULL: return heap2;
     heap2 == NULL: return heap1;
     heap1.findMin() < heap2.findMin():
          temp = heap1.right;
          heap1.right = heap1.left;
          heap1.left = merge(heap2, temp);
          return heap1;
     otherwise:
          return merge(heap2, heap1);
  }
}
```

Runtime Analysis: Worst-case and Amortized

- No worst case guarantee on right path length!
- All operations rely on merge

  $\Rightarrow$ worst case complexity of all ops =
- Probably won't get to amortized analysis in this course, but see Chapter 11 if curious.
- Result: M merges take time $M \log n$

  $\Rightarrow$ amortized complexity of all ops =

Comparing Heaps

- Binary Heaps
- Leftist Heaps
- d-Heaps
- Skew Heaps

Still room for improvement! (Where?)